

Perfect Numbers

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Presentation overview

1. Introduction
2. Even Perfect Numbers
3. Odd Perfect Numbers



Introduction

- A number is perfect if it is equal to the sum of its divisors, formally
 - $\sigma(n) = \sum_{x|n} x$
 - n is perfect iff $2 * n = \sigma(n)$
 - $2 * 6 = 6 + 3 + 2 + 1$
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Some perfect numbers

- 6,
- 28,
- 496,
- 8128
- 33550336,
- 8589869056,
- 137438691328,
- 2305843008139952128,
- 2658455991569831744654692615953842176,
- 191561942608236107294793378084303638130997321548169216

Definitions

- $k = \omega(N)$ - number of distinct prime factors of N .
- $N = \prod_{i=1}^k p_i^{\alpha_i}$




Basic properties

- $\sigma(nm) = \sigma(n)\sigma(m) \iff \gcd(n, m) = 1$
- $\sigma(p^m) = 1 + p + p^2 + \dots + p^m$
- $\sigma(N) = \prod_{i=1}^k (1 + p_i + \dots + p_i^{\alpha_i}) = \prod_{i=1}^k \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}$

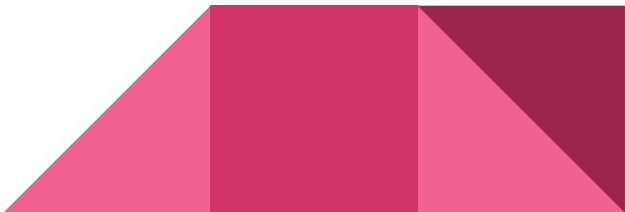
Euclid theorem

$$2^n - 1 \in \text{Primes} \Rightarrow 2^{n-1} * (2^n - 1) \in \text{Perfect}$$

- $\sigma(2^n - 1) = 2^n - 1 + 1$
 - $\sigma(2^{n-1}) = (2^n - 1) \div (2 - 1)$
 - $\sigma(N) = \sigma(2^{n-1})\sigma(2^n - 1)$
 - $\sigma(N) = (2^n - 1)2^n = 2N$
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Euler theorem

$2|n$ and $N \in \text{Perfect} \Rightarrow N = 2^{n-1}(2^n - 1)$ and $2^n - 1 \in \text{Primes}$

- $N = 2^{n-1}m$, $\sigma(N) = (2^n - 1)\sigma(m)$
 - $N \in \text{perfect} \Rightarrow 2^n m = (2^n - 1)\sigma(m)$
 - $\sigma(m) = \frac{2^n m}{2^n - 1} = m + \frac{m}{2^n - 1} = m + d$
 - $d \neq 1 \Rightarrow \sigma(m) = m + d + 1$
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Conclusion

The set of Mersennes' Primes is infinite iff.
the set of Even Perfect Numbers is infinite



Odd Perfect Numbers

- We don't know if they exist
- Theorems about them are in form
“Odd Perfect numbers have to satisfy...”



Lower bound

Why not just try to brute force method - check every odd number $\leq x$ to see if it's perfect.

- Odd perfect numbers are $\geq 10^{300}$

Brent and Cohen(1991)

- Odd perfect numbers are $\geq 10^{1500}$

Ochem and Rao (2012)



Multiplicative structure

- $N = \alpha^\beta m^2$, where $\alpha \equiv \beta \equiv 1 \pmod{4}$ and $\gcd(\alpha, m) = 1$
- Euler
- $N \equiv 1 \pmod{9}$ or $N \equiv 9 \pmod{36}$
Touchard (1953)



Cyclotomic polynomials

- $F_d(X) = \prod_{1 \leq k \leq n \text{ and } \gcd(k,n)=1} (X - e^{2i\pi \frac{k}{n}})$
- $p \in \text{Primes} \Rightarrow F_p(x) = 1 + x + x^2 + \dots + x^{p-1}$
- $x^n - 1 = \prod_{d|n} F_d(x)$
- $\sigma(N) = 2N = \prod_{i=0}^k \sigma(p_i^{\alpha_i}) = \prod_{i=0}^k \prod_{d|(\alpha_i+1), d>1} F_d(p_i)$

Prime factors

Odd perfect number has a prime factor $> 10^7$ -
Jenkins(2003)



Prime factors

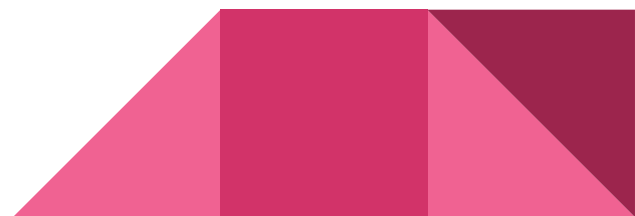
- $p_i | N \iff p_i | F_d(q)$
- Let $h(p, m)$ represent order of $p \pmod m$.
- $p^k || n \iff p^k | n \text{ and } p^{k+1} \nmid n$

Prime factors


Lemma 2.1. *It is true that $q|F_m(p)$ if and only if $m = q^b h(p; q)$. If $b > 0$, then $q||F_m(p)$. If $b = 0$, then $q \equiv 1 \pmod{m}$.*

It follows from Lemma [2.1](#) that, for r prime,

Lemma 2.2. *If $q|F_r(p)$, then either $r = q$ and $p \equiv 1 \pmod{q}$, so that $q||F_r(p)$, or $q \equiv 1 \pmod{r}$.*



Idea

- $F_r(p)$ is acceptable iff all of its prime factors $\leq 10^7$
 - Let's check all pairs (r, p) , where $r \leq \frac{10^7}{2}, p < 10^7$
 - After computer search 143 acceptable were found.
 - For each of those pairs author used a combinatorial proof
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Upper bound on prime factors

- $k = \omega(N)$ - the number of distinctive prime factors
- $N < 2^{4^k}$ - Nielsen (2003)



Upper bound on prime factors

- $\omega(N) \geq 9$ - Nielsen (2006)
- $N \not\equiv 3 \pmod{3} \Rightarrow \omega(N) \geq 11$ - McDaniel (1970)




Original theorem

$N \in \text{Perfect}$ and $\omega(N) = 8$, then $5|N$



References

1. Open Problem Garden - <http://www.openproblemgarden.org>
 2. J. Voight - On the nonexistence of odd perfect numbers
 3. J. Voight - Perfect Numbers: An elementary introduction
 4. Paul M. Jenkins - Odd Perfect Numbers have a prime factor exceeding 10^7
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Questions?

1. We know quite a lot about Even Perfect Numbers.
2. We don't know if Odd Perfect Numbers exist.
