Perfect Numbers

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Presentation overview

1. Introduction
2. Even Perfect Numbers
3. Odd Perfect Numbers

Introduction

- A number is perfect if it is equal to the sum of its divisors, formally
- $\sigma(n) = \sum_{x|n} x$
- n is perfect iff $2 * n = \sigma(n)$
- 2 * 6 = 6 + 3 + 2 + 1

Some perfect numbers

- 6,
- 28,
- 496,
- 8128
- 33550336,
- 8589869056,
- 137438691328,
- 2305843008139952128,
- 2658455991569831744654692615953842176,
- 191561942608236107294793378084303638130997321548169216

Definitions

- $k = \omega(N)$ - number of distinct prime factors of N.
- $N = \prod_{i=1}^{k} p_i^{\alpha_i}$

Basic properties

- $\sigma(nm) = \sigma(n)\sigma(n) \iff gcd(n, m) = 1$
- $\sigma(p^m) = 1 + p + p^2 + \ldots + p^m$
- $\sigma(N) = \prod_{i=1}^{k} (1 + p_i + \ldots + p_i^{\alpha_i}) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}$

Euclid theorem

$$2^n - 1 \in Primes \Rightarrow 2^{n-1} * (2^n - 1) \in Perfect$$

- $\sigma(2^n - 1) = 2^n - 1 + 1$
- $\sigma(2^{n-1}) = (2^n - 1) \div (2 - 1)$
- $\sigma(N) = \sigma(2^{n-1})\sigma(2^n - 1)$
- $\sigma(N) = (2^n - 1)2^n = 2N$

Euler theorem

$$2|n \; and \; N \in Perfect \Rightarrow N = 2^{n-1}(2^n - 1) \; and \; 2^n - 1 \in Primes$$

- $N = 2^{n-1}m, \; \sigma(N) = (2^n - 1)\sigma(m)$
- $N \in perfect \Rightarrow 2^n m = (2^n - 1)\sigma(m)$
- $\sigma(m) = \frac{2^n m}{2^n - 1} = m + \frac{m}{2^n - 1} = m + d$
- $d \neq 1 \Rightarrow \sigma(m) = m + d + 1$

Conclusion

The set of Mersennes' Primes is infinite iff.

the set of Even Perfect Numbers is infinite

Odd Perfect Numbers

- We don’t know if they exist
- Theorems about them are in form

  “Odd Perfect numbers have to satisfy...”

Lower bound

Why not just try to brute force method - check every odd number <= x to see if it's perfect.

- Odd perfect numbers are >= 10^300
  Brent and Cohen(1991)
- Odd perfect numbers are >= 10^1500
  Ochem and Rao (2012)

Multiplicative structure

- $N = \alpha^{\beta} m^2, where\ \alpha \equiv \beta \equiv 1\ (mod\ 4)\ and\ gcd(\alpha, m) = 1$

  - Euler

- $N \equiv 1\ (mod\ 9)\ or\ N \equiv 9\ (mod\ 36)$

  Touchard (1953)

Cyclotomic polynomials

- $F_d(X) = \prod_{1<=k<=n \; and \; gcd(k,n)=1} (X - e^{2i\pi\frac{k}{n}})$
- $p \in Primes \Rightarrow F_p(x) = 1 + x + x^2 + \ldots + x^{p-1}$
- $x^n - 1 = \prod_{d|n} F_d(x)$
- $\sigma(N) = 2N = \prod_{i=0}^{k} \sigma(p_i^{\alpha_i}) = \prod_{i=0}^{k} \prod_{d|(\alpha_i+1),d>1} F_d(p_i)$

Prime factors

Odd perfect number has a prime factor > 10^7 - Jenkins(2003)

Prime factors

- $p_i | N \iff p_i | F_d(q)$
- Let $h(p, m)$ represent order of p mod m.
- $p^k || n \iff p^k | n \; and \; p^{k+1} \not| n$

Prime factors

Lemma 2.1. *It is true that* $q|F_m(p)$ *if and only if* $m = q^b h(p;q)$*. If* $b > 0$*, then* $q\|F_m(p)$*. If* $b = 0$*, then* $q \equiv 1 \pmod{m}$*.*

It follows from Lemma 2.1 that, for r prime,

Lemma 2.2. *If* $q|F_r(p)$*, then either* $r = q$ *and* $p \equiv 1 \pmod{q}$*, so that* $q\|F_r(p)$*, or* $q \equiv 1 \pmod{r}$*.*

Idea

- $F_r(p)$ is acceptable iff all of its prime factors <= 10^7
- Let's check all pairs (r, p), where $r <= \frac{10^7}{2}, p < 10^7$
- After computer search 143 acceptable were found.
- For each of those pairs author used a combinatorial proof

Upper bound on prime factors

- $k = \omega(N)$ - the number of distinctive prime factors
- $N < 2^{4^k}$ - Nielsen (2003)

Upper bound on prime factors

- $\omega(N) \geq 9$ - Nielsen (2006)
- $N \not| 3 \Rightarrow \omega(N) \geq 11$ - McDaniel (1970)

Original theorem

$$N \in Perfect\, and\, \omega(N) = 8,\ then\, 5|N$$

References

1. Open Problem Garden - http://www.openproblemgarden.org
2. J. Voight - On the nonexistence of odd perfect numbers
3. J. Voight - Perfect Numbers: An elementary introduction
4. Paul M. Jenkins -Odd Perfect Numbers have a prime factor exceeding 10^7

Questions?

1. We know quite a lot about Even Perfect Numbers.
2. We don't know if Odd Perfect Numbers exist.