Lonely runner conjecture

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Definition

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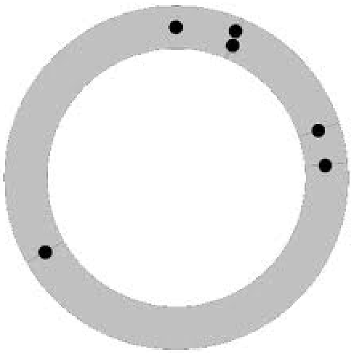
If there are k runners on the track with distinct speeds, a runner r_i becomes *lonely* at some given time if none of the other $k-1$ runners are within a distance of $1/k$ of r_i at that time.

Lonely runner

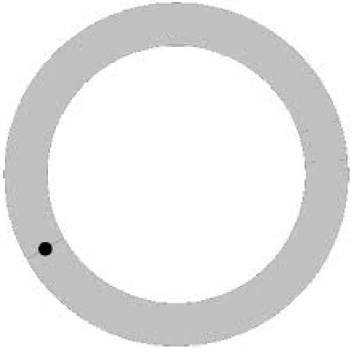


Conjecture

Let k be an arbitrary natural number, and consider k runners with distinct, fixed, integer speeds traveling along a circle of unit circumference. Then each runner becomes lonely at some time.

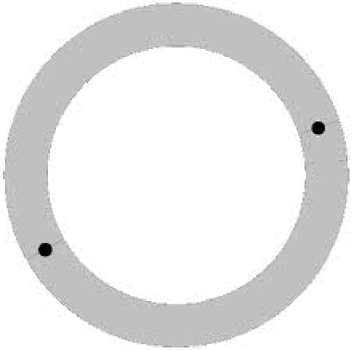
k=1

For k = 1
any t



For k = 2

$t = \frac{1}{2(v_1 - v_0)}$



Open

Still unsolved for $k \geq 8$

Lonely runner

To give a precise statement, let T = [0, 1) denote the circle (the onedimensional torus). For a real number x, let $\{x\}$ be the fractional part of x (the position of x on the circle), and let $\|x\|$ denote the distance of x to the nearest integer (the circular distance from x to zero).
Notice that ‖x - y‖ is just the length of the shortest circular arc determined by the points x and y on the circle. It is not difficult to see that the following statement is equivalent to the Lonely Runner Conjecture.

Conjecture 1. For every integer $k \geq 1$ and for every set of positive integers $\{d_1, d_2, ..., d_k\}$ there exists a real number t such that $\|td_i\| \geq \frac{1}{k+1}$ for all i = 1, 2, ... , k.

Let $D = \{d_1, d_2, \dots, d_k\}$ be a set of k positive integers. Consider the quantity

$$\kappa(D) = \sup_{x \in \mathbb{T}} \min_{d_i \in D} \|x d_i\|$$

and the related function $\kappa(k) = \inf \kappa(D)$, where the infimum is taken over all k-element sets of positive integers. So, the Lonely Runner Conjecture states that $\kappa(k) \geqslant \frac{1}{k+1}$.

Lonely runner

The trivial bound is $\kappa(k) > \frac{1}{2k}$, as the sets $\{x \in T : \|xd_i\| < \frac{1}{2k}\}$ simply cannot cover the whole circle (since each of them is a union of d_i open arcs of length $\frac{1}{kd_i}$ each)

THEOREM 1. *Let k be a fixed positive integer and let $\varepsilon > 0$ be fixed real number. Let $D \subseteq \{1, 2, \ldots, n\}$ be a k-element subset chosen uniformly at random. Then the probability that $\kappa(D) \geqslant \frac{1}{2} - \varepsilon$ tends to 1 with $n \to \infty$.*

The proof uses elementary Fourier analytic technique for subsets of $\mathbb{Z}_p$.

Lonely runner

For a given set D, consider a graph $G(D)$ whose vertices are positive integers, with two vertices a and b joined by an edge if and only if $|a-b| \in D$. Let $\chi(D)$ denote the chromatic number of this graph. It is not hard to see that $\chi(D) \leqslant |D|+1$.

To see a connection to parameter $\kappa(D)$, put $N = \lceil \kappa(D)^{-1} \rceil$ and split the circle into N intervals $I_i = [(i-1)/N, i/N)$, $i = 1, 2, \ldots, N$ (cf. [**15**]). Let

t be a real number such that $min_{d \in D} \|dt\| = \kappa(D)$. Then define a colouring $c : \mathbb{N} \to \{1, 2, \ldots, N\}$ by $c(a) = i$ if and only if $\{ta\} \in I_i$. If $c(a) = c(b)$ then $\{ta\}$ and $\{tb\}$ are in the same interval I_i. Hence $\|ta - tb\| < 1/N \leqslant \kappa(D)$, and therefore $|a - b|$ is not in D. This means that c is a proper colouring of a graph $G(D)$. So, we have a relation

$$\chi(D) \leqslant \left\lceil \frac{1}{\kappa(D)} \right\rceil.$$

Now, by Theorem 1 we get that $\chi(D) \leqslant 3$ for almost every graph $G(D)$.

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