Acyclic coloring of graphs with fixed maximum degree

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Acyclic coloring

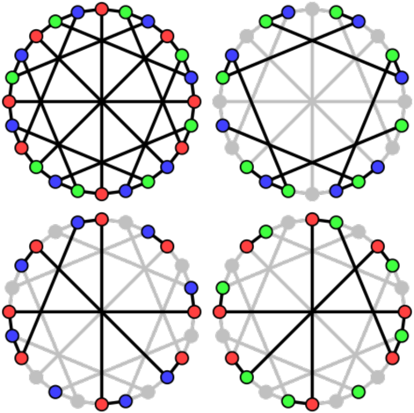
Definition 1.1

An acyclic k-coloring of a graph G is a proper vertex coloring of G, which uses at most k colors, such that the graph induced by the union of every two color classes is a forest

Definition 1.2

Acyclic chromatic number, denotes as $\chi_a(G)$, of a graph G is the fewest colors needed in any acyclic coloring of G.

Acyclic coloring



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Problem history

Acyclic coloring and acyclic chromatic number has been introduced by B. Grünbaum in 1973 in paper: „Acyclic colorings of planar graphs".

Conjecture 2.1

Every planar graph has an acyclic 5-coloring.

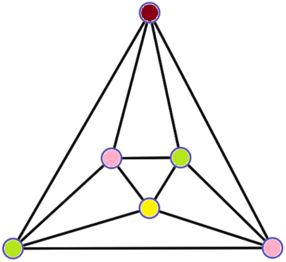
Conjecture 2.2

Acyclic chromatic number for graph with maximum degree Δ is equal to $\Delta + 1$.

Planar graphs

Conjecture 2.1

Every planar graph has an acyclic 5*-coloring.*

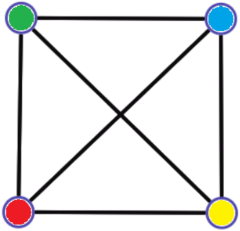


Conjecture was proved by O. V. Borodin in 1979 in paper titled: „On acyclic colorings of planar graphs".

Conjecture 2.2

Acyclic chromatic number for graph with maximum degree Δ is equal to $\Delta + 1$.

This is still an open problem for $\Delta \geqslant 5$.



Current knowledge

Current state of knowledge:

- $\Delta \leqslant 4$: proved by M. Burstein in 1979, „Every 4-valent graph has an acyclic 5-coloring"
- $\Delta = 5$: best known bound for $\chi_a(G)$ is 7
  - „Acyclic coloring of graphs of maximum degree five: nine colors are enough" by Fertin and Raspaud in 2008
  - „Acyclic vertex coloring of graphs of maximum degree 5" by Yadav, Varagani, Kothapalli, Venkaiah in 2011
  - „Graphs with maximum degree 5 are acyclically 7-colorable" by Kostochka and Stocker in 2011

Current knowledge

- $\Delta = 6$: best known bound for $\chi_a(G)$ is 9, there are some papers listed below where each one of them uses the same proofing technique that the first one
  - „Graphs with maximum degree 6 are acyclically 11-colorable" by Hocquard in 2011
  - „Acyclic vertex coloring of graphs of maximum degree six" by Zhao, Miao, Pang, Song in 2014
  - „Acyclic coloring of graphs with maximum degree at most six" by Wang and Miao in 2019
- $\Delta = 7$: best known bound for $\chi_a(G)$ is 12, article with proof has been published this year and also uses technique from Hocquard paper
  - „Acyclic coloring of graphs with maximum degree bounded" by Dieng, Hocquard, Naserasr in 2010
  - „Acyclic Coloring of Graphs with Maximum Degree 7" by Wang and Miao 2021

Current knowledge

- $\Delta \geqslant 8$: there are no exact results, the proved upper-bounds are very far from the B. Grünbaum conjecture. The best one is given by the formula:

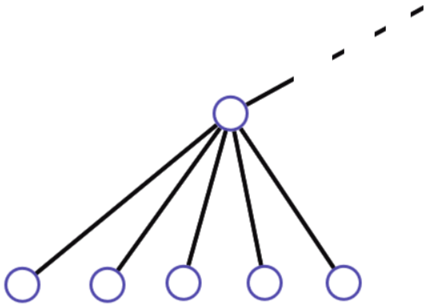
$$a(G) \leqslant \frac{\Delta^2-5\Delta}{2} + 2\lfloor \frac{\Delta-1}{2} \rfloor + 3$$

  and was presented in Dieng et al. paper mentioned in the previous slide: „Acyclic coloring of graphs with maximum degree bounded".

Hocquard proof for $\Delta = 6$

Definition 3.1

Good spanning tree of graph G with maximum degree Δ is a spanning tree T such that T contains a vertex adjacent to $\Delta - 1$ leaves.

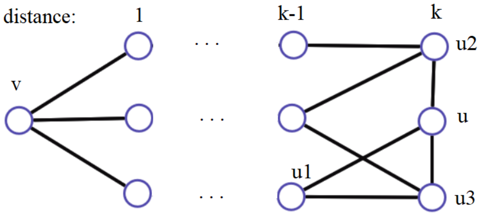


Hocquard proof for $\Delta = 6$

Lemma 3.1

Every regular connected graph admits a good spanning tree

Proof: Let G - Δ regular, connected graph
u is a vertex at maximum distance from v, distance k with smallest number of neighbours at distance $k-1$ from v.



Hocquard proof for $\Delta = 6$

Let $N(u) = \{u_1, u_2, ..., u_\Delta\}$ and define $G_1 = G \backslash (N(u) \backslash u_1)$

Construct good spaning tree:

1. Take any spanning tree T_1 of G_1
2. Add to T_1 the edges $(uu_2), ..., (uu_\Delta)$ that cover the vertices $u_2, ..., u_\Delta$

Problem:

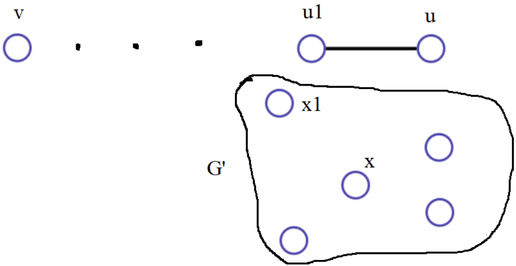
Is G connected?

Hocquard proof for $\Delta = 6$

Problem:

Is G connected?

By contradiction:
Pick G' - connected component of G_1 containing neither v nor u
Then every vertex of G' is at distance k from v in G.
Consider now neighbours of x from G' in G at distance $k-1$ form v.



Hocquard proof for $\Delta = 6$

Theorem 3.1

Every graph with maximum degree six is acyclically 11-colorable.

Proof idea:

- if $\delta(G) < 6$: we look for a post order of the spanning tree rooted in vertex that has $d(v) < 6$
- if $\delta(G) = 6$: in that case good spanning tree exists, we order vertices according to post order walk of that tree starting with vertex that has $\Delta - 1$ leaves adjacent to him.

Next according to the given ordering color each vertex.

Hocquard proof for $\Delta = 6$

Denote: $N_c(v)$ - colored neighborhood of v
$n_c(v)$ - number of vertices in $N_c(u)$ colored with the color c
$SC(N_c(u))$ - the set of colors used by vertices in $N_c(u)$

Observation

Let G be a graph of maximum degree 6 and let ϕ be a partial acyclic 11-coloring of G. Suppose v is an uncolored vertex of G. If all the neighbors of v use distinct colors, then by coloring v properly we extend ϕ. If a color c in $SC(N_c(v))$ appears $n_c(v) > 1$ times among the neighbors of v, then, in order to color v, we need to avoid at most $\lfloor \frac{5n_c(v)}{2} \rfloor$ distinct colors to prevent the creation of bicolored cycles going through v and the vertices colored with c.

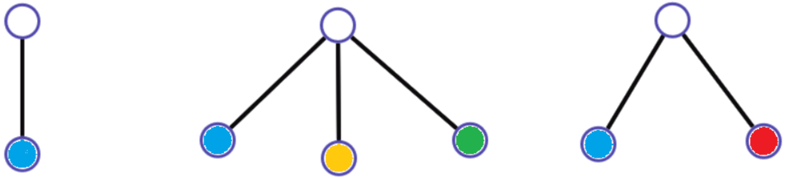
Lemma 3.2

Let G be a graph of maximum degree 6 and let ϕ be a partial acyclic 11-coloring of G. Then, for any uncolored vertex u such that $\#cn(u) \leqslant 3$, there exists a color for u that allows us to extend ϕ.

Proof:
Let's just consider possible cases

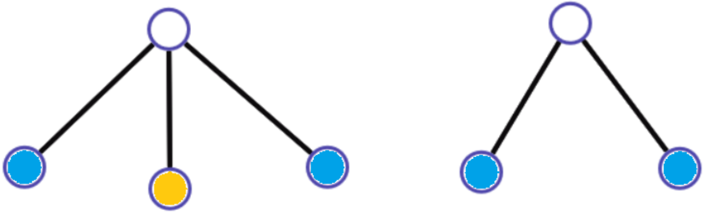
Hocquard proof for $\Delta = 6$

- Case 1: Each colored neighbour has different color



Hocquard proof for $\Delta = 6$

- Case 2: At least two neighbours have the same color

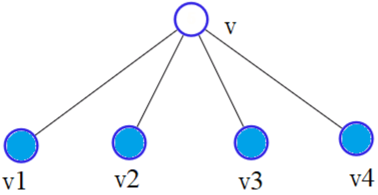


Hocquard proof for $\Delta = 6$

Lemma 3.3

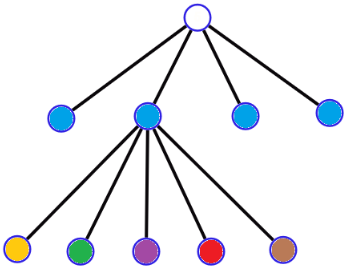
Let G be a graph of maximum degree 6 and let ϕ be a partial acyclic 11-coloring of G. Then, for any uncolored vertex u such that $\#cn(u) = 4$, there exists a color for u that allows us to extend ϕ.

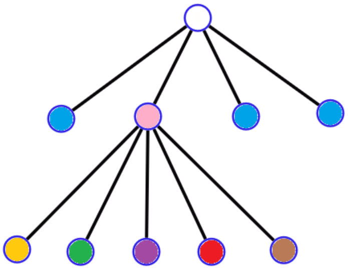
Let's consider the case from proof where uncolored vertex has 4 neighbours in the same color:



Hocquard proof for $\Delta = 6$

Case 1: One of the colored neighbours has 5 colored neighbours, each by different color:





Hocquard proof for $\Delta = 6$

Case 2: Each colored neighbour has no more than 4 colors used in his neighborhood. We can use earlier observation to calculate number of colors we need to avoid:

$$(\lfloor \frac{4 * 4}{2} \rfloor) = 8$$

and one more color to for maintaining proper coloring.

Then, two choices remain to pick a color for uncolored vertex.

Hocquard proof for $\Delta = 6$

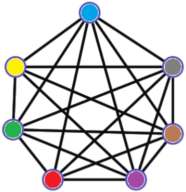
Lemma 3.4

Let G be a graph of maximum degree 6 and let ϕ be a partial acyclic 11-coloring of G. Then, for any uncolored vertex u such that $\#cn(u) = 5$, there exists a color for u that allows us to extend ϕ.

Open questions for $\Delta = 6$

Question 1

Exhibit a graph G with $\Delta(G) = 6$ and $\chi_a(G) > 7$ (if such a graph exists).



Question 2

Prove that every graph with $\Delta(G) = 6$ has $\chi_a(G) < 9$.