Rodl Nibble

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Hypergraphs

- Hypergraphs are graphs with edges defined as subsets of vertices, i.e. edge can connect more than 2 vertices.
- Hypergraph is called r-uniform when all edges are subsets of exactly r vertices.

(n, k, r)-packings

- Family $\mathcal{F} \subseteq \binom{[n]}{k}$ is called a (n, k, r)-packing when no r vertices lie in more than one $S \in \mathcal{F}$, i.e. for every distinct $S_1, S_2 \in \mathcal{F}$ we have $|S_1 \cap S_2| < r$.
- Let $m(n, k, r)$ be maximal size of (n, k, r)-packing.
- Intuition - let G be complete r-uniform hypergraph on n vertices. Then $m(n, k, r)$ is maximal number of disjoint k-cliques that may be "packed" into G.

Bounds for maximal (n, k, r)-packings

Lemma

For any integers n, k, r such that $2 \leq r < k < n$ following inequality is true:

$$m(n,k,r) \leq \frac{\binom{n}{r}}{\binom{k}{r}}$$

Proof.

We can reformulate above inequality as:

$$m(n,k,r)\binom{k}{r} \leq \binom{n}{r}$$

and use simple counting argument - each $S \in \mathcal{F}$ has $\binom{k}{r}$ r-subsets of $[n]$. Summing them up for every $S \in \mathcal{F}$ we get family $\mathcal{F}_r$ of distinct r-subsets of $[n]$. Such family can have cardinality equal at most $\binom{n}{r}$. □

(n, k, r)-coverings

- Family $\mathcal{F} \subseteq \binom{[n]}{k}$ is called a (n, k, r)-covering when every r vertices lie in atleast one $S \in \mathcal{F}$.
- Let $M(n, k, r)$ be minimal size of (n, k, r)-covering.

Bounds for minimal (n, k, r)-coverings

Lemma

For any integers n, k, r such that $2 \leq r < k < n$ following inequality is true:

$$M(n, k, r) \geq \frac{\binom{n}{r}}{\binom{k}{r}}$$

Proof.

Analogous to proof for $m(n, k, r)$. □

(n, k, r)-tactical configurations

- Family $\mathcal{F} \subseteq \binom{[n]}{k}$ is called a (n, k, r)-tactical configuration when every r-set is contained in exactly one $S \in \mathcal{F}$.

Lemma

If (n, k, r)-tactical configuration exists then for every $0 \le i \le r-1$ we have

$$\binom{k-i}{r-i} \mid \binom{n-i}{r-i}$$

Proof.

Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be (n, k, r)-tactical configuration. For every i there are $\binom{n-i}{r-i}$ r-sets containing $[i]$. Each of them is contained in exactly one set $S \in \mathcal{F}$. On the other hand, every $S \in \mathcal{F}$ contains exactly $\binom{k-i}{r-i}$ r-sets containing $[i]$. Therefore we have following equality:

$$|\mathcal{F}|\binom{k-i}{r-i} = \binom{n-i}{r-i}$$

$\square$

Lemma

For fixed r, k such that $2 \leq r < k$ we have:

$$\lim_{n\to\infty} \frac{m(n,k,r)}{\binom{n}{r}/\binom{k}{r}} = 1 \Leftrightarrow \lim_{n\to\infty} \frac{M(n,k,r)}{\binom{n}{r}/\binom{k}{r}} = 1$$

Proof.

($\Rightarrow$) Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be (n,k,r)-packing of size $(1-o(1))\frac{\binom{n}{r}}{\binom{k}{r}}$. $\mathcal{F}$ covers $|\mathcal{F}|\binom{k}{r} = (1-o(1))\binom{n}{r}$ r-sets. We can transform $\mathcal{F}$ to (n,k,r)-covering $\mathcal{F}'$ by adding $o(1)\binom{n}{r}$ k-sets containing each uncovered r-set. Then we have

$$|\mathcal{F}'| = |\mathcal{F}| + o(1)\binom{n}{r} = (1-o(1))\frac{\binom{n}{r}}{\binom{k}{r}} + o(1)\binom{n}{r} = (1+o(1))\frac{\binom{n}{r}}{\binom{k}{r}}$$

□

Erdos-Hanani conjecture, 1963

Conjecture

For fixed r, k *such that* $2 \leq r < k$ *we have:*

$$\lim_{n\to\infty} \frac{m(n,k,r)}{\binom{n}{r}/\binom{k}{r}} = \lim_{n\to\infty} \frac{M(n,k,r)}{\binom{n}{r}/\binom{k}{r}} = 1$$

Hypergraph covers

- Let H be an r-uniform hypergraph on n vertices. Cover of H is a set of edges whose union contains all vertices, i.e. it is $(n, r, 1)$-covering whose sets are edges of H.

Pippenger theorem, 1989

Theorem

For every integer $t \geq 2$ and reals $\kappa \geq 1$, $\alpha > 0$ there are $\gamma = \gamma(t, \kappa, \alpha) > 0$ and $d_0 = d_0(t, \kappa, \alpha)$ such that for every $n \geq D \geq d_0$ the following holds. Every t-uniform hypergraph $H = (V, E)$ on a set V of n vertices which all vertices have positive degrees and which satisfies the following conditions:

- *For all vertices $v \in V$ but at most γ of them, $d(v) = (1 \pm \gamma)D$.*
- *For all $v \in V$, $d(v) \leq \kappa D$.*
- *For any two distinct $v, w \in V$, $d(v, w) < \gamma D$.*

contains a cover of at most $(1 + \alpha)\frac{n}{t}$ edges.

Idea of proof of Pippenger theorem

- Fixing small $\epsilon > 0$ one shows that a random set of roughly $\epsilon n/t$ has with high probability only some $O(\epsilon^2 n)$ vertices covered more than once and hence covers at least $\epsilon n - O(\epsilon^2 n)$ vertices.
- Moreover, after deleting the vertices covered, the induced hypergraph on the remaining vertices still satisfies the properties described in three points (for some other values of n, γ, κ and D.
- Therefore, one can choose again a random set of this hypergraph, covering roughly an ϵ-fraction of its vertices with nearly no overlaps.
- Proceeding in this way for a large number of times we are finally left with at most ϵn uncovered vertices, and then we can cover them trivially.

Proof of Erdos-Hanani conjecture

Theorem

For fixed r, k such that $2 \leq r < k$ we have:

$$\lim_{n\to\infty} \frac{M(n,k,r)}{\binom{n}{r}/\binom{k}{r}} = 1$$

Proof.

Let $t := \binom{k}{r}$ and H be t-uniform hypergraph satisfying:

$$V(H) = \binom{[n]}{r} \quad E(H) = \left\{ \binom{F}{r} : F \in \binom{[n]}{k} \right\}$$

Each vertex of H has degree $D = \binom{n-r}{k-r}$. □

Proof of Erdos-Hanani conjecture

Proof.

Let $\kappa = 1$ and fix $\alpha > 0$. Then we have $\gamma = \gamma(t, \kappa, \alpha)$.
Every two distinct vertices lie in at most
$\binom{n-r-1}{k-r-1} = \frac{k-r}{n-r}\binom{n-r}{k-r} = \frac{k-r}{n-r}D$ common edges. From some point d_0, for every $n \geq D \geq d_0$: $\frac{k-r}{n-r} \leq \gamma$ what gives that $\frac{k-r}{n-r}D \leq \gamma D$.
Therefore, conditions in Pippenger theorem are satisfied for $(t, \kappa, \alpha) = (\binom{k}{r}, 1, \alpha)$ what implies that H has cover of size $(1+\alpha)\frac{\binom{n}{r}}{\binom{k}{r}}$. At the end, we can see that covers of H are exactly $(n, k, r)-$ coverings what finishes the proof. □