Colouring locally sparse graphs with the first moment method

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Theorem (Johansson, 1996)

The list chromatic number of a triangle free graph with maximum degree Δ doesn't exceed $O(\Delta/ln\Delta)$ as $\Delta \to \infty$

Theorem (Molloy, 2019)

The list chromatic number of a triangle free graph with maximum degree Δ doesn't exceed $(1 + o(1))\Delta/ln\Delta$ as $\Delta \to \infty$

Theorem 1.

Let G be a n-vertex graph of maximum degree Δ such that every graph induced by a neighbourhood in G has average degree at most $d \leq \frac{\Delta}{6} - 1$. Write $\rho := \frac{\Delta}{d+1}$ and let $\ell \geq (d+1)(ln\rho)$. Then for every list assignment $L : V(G) \to 2^{\mathbb{N}}$ with

$$|L(v)| \geq \left(1 + \frac{2}{ln\rho}\right) \frac{deg(v)}{W\left(\frac{deg(v)}{\ell}\right)}$$

for every vertex $v \in V(G)$, there are at least ℓ^n proper L-colorings of G.

Where W i s W-Lambert function $z \mapsto W(z)$, defined as the reciprocal of $z \mapsto ze^z$.
We use well known fact $e^{W(z)} = z/W(z)$.
We also note that $W(z) = ln\, z - ln\, ln\, z + o(1)$ as $z \to \infty$. By fixing $d := 0$ Theorem 1. implies Molloy's result for triangle-free graph.

Lemma 1.

Let H be a graph, and L a list assignment of H such that $|L(v)| \geq deg(v) + 1$ for every $v \in V(H)$. If $\boldsymbol{c}$ is a uniform random proper L-coloring of H, then given some color $x \in \bigcup_{v \in V(H)} L(v)$, the probability that $\boldsymbol{c}(v) \neq x$ for all $v \in V(H)$ is at least

$$\prod_{v \in V(H)} \left(1 - \frac{1}{|L(v)| - deg(v)}\right)$$

Proof.
Sample a uniform proper L-coloring $\boldsymbol{c}$ of H. Let $v \in V(H)$. For every L-coloring c' of $H' = H \setminus v$, we have

$$\mathbb{P}[\boldsymbol{c}(v) = x | \, \boldsymbol{c}|_{H'} = c'] \leq \frac{1}{|L(v)| - deg(v)}$$

Overview of the proof

For a graph G with a list assignment L and vertex $v \in V(G)$. Set $G' = G \setminus v$

- Uniformly sample a proper coloring of G'
- Mark neighbours of v with short lists due to coloring on $G' \setminus N(v)$
- Uncolor unmarked vertices in $N(v)$
- Choose a proper re-coloring of those vertices uniformly

We then prove that the expected size of the list of available colors at v is large enough.

Proof of theorem 1

Fix $\rho := \frac{\Delta}{d+1} \geq 6, t := (d+1)(ln\rho + 1)$, and $\ell \geq (d+1)(ln\rho)^3$.
For every $v \in V(G)$, let $k(v) := |L(v)| \geq \left(1 + \frac{2}{ln\rho}\right) \frac{deg(v)}{W\left(\frac{deg(v)}{\ell}\right)}$.
Note that $k(v) \geq \frac{deg(v)}{W\left(\frac{deg(v)}{\ell}\right)} = \ell^{W(deg(v)/\ell)} \geq \ell$.
Let L be a list assignment of G with $|L(v)| \geq k(v)$ for every $v \in V(G)$.

For every proper L-coloring c of subgraph H of G and vertex $v \in V(G)$, let

$$L_c(v) := L(v) \setminus c(N_H(v))$$

$$\ell_c(v) := |L_c(v)|$$

Let $\mathscr{C}(H)$ - set of proper L-colorings of H.
We want to show by induction that for every $H \subseteq G$ and $v \in V(H)$ we have

$$|\mathscr{C}(H)| \geq |\mathscr{C}(H \setminus v)|$$

Equivalently $\mathbb{E}[\ell_c(v)] \geq \ell$ for a uniform random coloring $c \in \mathscr{C}(H \setminus v)$

Suppose $|V(H)| \geq 2$. Let $H' := H \setminus v$ and $H_0 := H'\ N(v)$.
Let $\boldsymbol{c}$ be drawn uniformly from $\mathscr{C}(H')$. Set $\boldsymbol{c}_0 = \boldsymbol{c}|_{H_0}$
For $u \in N_H(v)$ denote d_u to be the degree of u in $H[N_H(v)]$ and set $t_u := (d_u + 1)(ln\rho + 1)$.
We know that

$$\mathbb{P}[\ell_{\boldsymbol{c}}(u) \leq t_u] = \frac{|\{c' \in \mathscr{C}(H') : \ell_{c'} \leq t_u\}|}{|\mathscr{C}(H')} \leq \frac{t_u|\mathscr{C}(H' \setminus v)|}{\ell|\mathscr{C}(H' \setminus v)|} \leq \frac{t_u}{\ell} \quad (1)$$

Given $c \in \mathscr{C}(H')$ and $c_0 = c|_{H_0}$
Let $L_c^0(v) = L(v) \setminus \{c(u) : u \in N(v), \ell_{c_0}(u) \leq t_u\}$. Denote $\boldsymbol{k}(v) = |L_{\boldsymbol{c}}^0(v)|$

$$\begin{aligned}\mathbb{E}[\boldsymbol{k}(v)] &\geq k(v) - \sum_{u \in N(v)} \mathbb{P}[\ell_{\boldsymbol{c}_0}(u) \leq t_u] \geq k(v) - \frac{t deg(v)}{\ell} \\ &\geq \left(1 + \frac{2}{ln\rho}\right) \frac{deg(v)}{W\left(\frac{deg(v)}{\ell}\right)} - \frac{1}{ln\rho} \frac{deg(v)}{ln\rho - 1} \\ &\geq \left(1 + \frac{1}{ln\rho}\right) \frac{deg(v)}{W\left(\frac{deg(v)}{\ell}\right)}\end{aligned} \tag{2}$$

We use $ln\rho - 1 \geq W(\rho/(ln\rho)^3) \geq W(deg(v)/\ell)$, since $\rho \geq 6$.

For a color $x \in L(v)$, let $\mathcal{N}_c(x) = \{u \in N(v) : x \in L_c(u), \ell_{c_0}(u) \geq t_u$. Using the fact that $\frac{\ell_{c_0}(u)}{\ell_{c_0}(u)-d_u-1} \leq \frac{t_u}{t_u-d_u-1} = 1 + \frac{1}{ln\rho}$ if $u \in \mathcal{N}_c(x)$ (and also $\ell_{c_0}(u) \geq t_u$)

$$\begin{aligned}\sum_{x\in L(v)} \sum_{u\in\mathcal{N}_c(x)} \frac{1}{\ell_{c_0}(u) - d_u - 1} &\leq \sum_{x\in L(v)} \sum_{u\in\mathcal{N}_c(x)} \left(1 + \frac{1}{ln\rho}\right) \frac{1}{\ell_{c_0}(u)} \\ &\leq \left(1 + \frac{1}{ln\rho}\right) \sum_{\substack{u\in N(v) \\ \ell_{c_0}(u)\geq t_u}} \sum_{y\in L_c(u)} \frac{1}{\ell_{c_0}(u)} \\ &\leq \left(1 + \frac{1}{ln\rho}\right) deg(v)\end{aligned} \tag{3}$$

Let $c_0 \in \mathscr{C}(H_0)$ be a possible realisation of $\boldsymbol{c}|_{H_0}$.
Let $X_0 = \{u \in N(v) : \ell_{c_0}(u) \geq t_u\}$ and let $H_1 := H' \setminus X_0$.
Observe that for every $u \in X_0$ and every extension c_1 of x_0 to H_1 we have

$$\ell_{c_1}(u) - deg_{H[X_0]}(u) \geq \ell_{c_0}(u) \geq t_u - d_u > 1$$

Now let c_1 be a possible realisation of $\boldsymbol{c}|_{H_1}$ given $\boldsymbol{c}_0 = c_0$.
Assuming $\boldsymbol{c}|_{H_1} = c_1$, then $L^0_{\boldsymbol{c}}(v) = L_{c_1}(v)$ and for every color $x \in L_{c_1}(v)$, $\mathscr{N}_{\boldsymbol{c}}(x) = \mathscr{N}_{c_1}(x)$.

Using Lemma 1. on $H[X_0]$ and list assignment L_{c_1}, for every color $x \in L_{c_1}(v)$ we obtain

$$\begin{aligned}
\mathbb{P}[x \in L_{\mathbf{c}}(v) \mid \mathbf{c}|_{H_1} = c_1] &= \mathbb{P}[\mathbf{c}(v) \neq x, \text{for every } u \in X_0 \mid \mathbf{c}|_{H_1} = c_1] \\
&\geq \prod_{u \in \mathcal{N}_{c_1}(x)} \left(1 - \frac{1}{\ell_{c_1}(u) - deg_{H[X_0]}(u)}\right) \\
&\geq \prod_{u \in \mathcal{N}_{c_1}(x)} \left(1 - \frac{1}{\ell_{c_0}(u) - d_u}\right) \\
&= \mathbb{E}\left[\prod_{u \in \mathcal{N}_{\mathbf{c}}(x)} \left(1 - \frac{1}{\ell_{\mathbf{c}_0}(u) - d_u}\right) \middle| \mathbf{c}|_{H_1} = c_1\right]
\end{aligned}$$

From the law of total expectation

$$\mathbb{E}[\ell_{\mathbf{c}}(v)] \geq \mathbb{E}\left[\sum_{x \in L^{\circ}_{\mathbf{c}}(v)} \prod_{u \in \mathcal{N}_{\mathbf{c}}(x)} \left(1 - \frac{1}{\ell_{\mathbf{c}_{\circ}}(u) - d_u}\right)\right]$$

Using the fact that $1 - \frac{1}{x} \geq e^{-1/(x-1)}$ for every $x > 1$, Jensen's inequality and the convexity of function $x \mapsto xe^{-a/x}$ for $a > 0$ on the domain $(0, +\infty)$, we get

$$\begin{aligned}
\mathbb{E}[\ell_c(v)] &\geq \mathbb{E}\left[\sum_{x \in L_{\mathbf{c}}^{0}(v)} exp\left(\prod_{u \in \mathcal{N}_{\mathbf{c}}(x)} -\frac{1}{\ell_{\mathbf{c}_0}(u) - d_u - 1}\right)\right] \\
&\geq \mathbb{E}\left[\boldsymbol{k}(v) e^{-(1+1/ln\rho) deg(v)/\boldsymbol{k}(v)}\right] \\
&\geq \mathbb{E}\left[\boldsymbol{k}(v)\right] e^{-(1+1/ln\rho) deg(v)/\mathbb{E}[\boldsymbol{k}(v)]} \\
&\geq \left(1 + \frac{1}{ln\rho}\right) \frac{deg(v)}{W\left(\frac{deg(v)}{\ell}\right)} e^{-W\left(\frac{deg(v)}{\ell}\right)} \geq \ell
\end{aligned}$$

Theorem 2 (Bernshteyn, Brazelton, Cao, Kang; 2021

For every $\epsilon > 0$, there exists Δ_0 such that the following holds. Let G be and n-vertex triangle-free graph of maximum degree $\Delta \geq \Delta_0$ and with m edges. Then, for every $q \geq (1+\epsilon)\Delta/ln\Delta$, the number of proper q-colorings of G is at least $(1-1/q)^m((1-\delta)q)^n$, where $\delta = \frac{4}{q}e^{\Delta/q}$.

They also show this bound to be asymptotically sharp for Δ-regular graphs.
For $q = (1+o(1))\Delta/ln\Delta$, the theorem gives $e^{(\frac{1}{2}-o(1)n\ ln\Delta}$ q-colorings of G. Theorem 1. with $\ell = \sqrt{\Delta}$ gives the same bound for a number of $2q$-colorings.

Theorem 3

Let G be an n-vertex graph of maximum degree Δ such taht every graph induced by a negihbourhood in G has avarege degree at most $d \leq \frac{\Delta}{6} - 1$. Let $f := \Delta/(d+1)$ and suppose $L : V(G) \to 2^{\mathbb{N}}$ is a list assignment with $|L(v)| \geq \left(1 + \frac{1}{ln\rho}\right) q(v)$, where

$$q(v) \geq \left(1 + \frac{1}{ln\rho}\right) \frac{deg(v)}{W\left(\frac{deg(v)}{(d+1)(ln\rho)^3}\right)}$$

for every vertex $v \in V(G)$. Then there are at least $(q/\sqrt{\Delta})^n$ proper L-colorings of G, where D is the geometric mean of the degrees in G, and q is the geometric mean of $\{q(v)\}_{v \in V(G)}$

Proof.
Following the proof of Theorem 1. we show

$$\mathbb{E}[\ell_{\boldsymbol{c}}(v)] \geq \mathbb{E}[\boldsymbol{k}(v)]e^{-(1+1/(ln\ \rho)deg_H(v)/\mathbb{E}[\boldsymbol{k}(v)]} \geq q(v)e^{-(1+1/(ln\ \rho)deg_H(v)/q(v)}$$

We order vertices of G $v_1, \ldots, v_n$ such that $(q(v_i))_{i=1}^n$ is non-decreasing.
Let $H_i = G[v_1, \ldots, v_{i-1}]$

$$
\begin{aligned}
|\mathscr{C}(G)| &\geq \prod_{i=1}^{n} q(v_i) e^{-(1+1/ln\ \rho) deg_{H_i}(v_i)/q(v_i)} \\
&= q^n exp\left(-(1+\frac{1}{ln\ \rho})\sum_{uv \in E(G)} min\left\{\frac{1}{q(u)}, \frac{1}{q(v)}\right\}\right) \\
&\geq q^n exp\left(-(1+\frac{1}{ln\ \rho})\sum_{uv \in E(G)} \left(\frac{1}{2q(u)}, \frac{1}{2q(v)}\right)\right) \\
&= q^n exp\left(-(1+\frac{1}{ln\ \rho})\sum_{i=1}^{n} \frac{deg(v_i)}{2q(v_i)}\right) \\
&\geq q^n exp\left(-\frac{1}{2}\sum_{i=1}^{n} ln\ deg(v_i)\right) \\
&= q^n D^{-n/2}
\end{aligned}
$$