Parking functions

Wojciech Buczek
Papers authors: Richard Kenyon, Mei Yin

Jagiellonian University

Definition

In our faculty, we have n hangers in the cloakroom. Let's say they are linear (non cycled).

Definition

In our faculty, we have n hangers in the cloakroom. Let's say they are linear (non cycled).

I like number 170, and I'm always trying to hang my jacket there - if, unfortunately, this number is taken by student that previously hang their jacket there, I'll try to hang my jacket **at the next available spot**, after 170.

Definition

In our faculty, we have n hangers in the cloakroom. Let's say they are linear (non cycled).

I like number 170, and I'm always trying to hang my jacket there - if, unfortunately, this number is taken by student that previously hang their jacket there, I'll try to hang my jacket **at the next available spot**, after 170.

Will I always find free spot?

Definition

In our faculty, we have n hangers in the cloakroom. Let's say they are linear (non cycled).

I like number 170, and I'm always trying to hang my jacket there - if, unfortunately, this number is taken by student that previously hang their jacket there, I'll try to hang my jacket **at the next available spot**, after 170.

Will I always find free spot?

Definition

In papers, we have students $\rightarrow$ **cars**, and hangers $\rightarrow$ **parking spots**.

Definition

In papers, we have students $\rightarrow$ **cars**, and hangers $\rightarrow$ **parking spots**.

Input is a number n and sequence $(\pi_1, \ldots, \pi_n)$. Car i prefers space π_i. If π_i is occupied by previous car, then car i takes next available space.

In papers, we have students → **cars**, and hangers → **parking spots**.

Input is a number n and sequence $(\pi_1, \ldots, \pi_n)$. Car i prefers space π_i. If π_i is occupied by previous car, then car i takes next available space.

We call $(\pi_1, \ldots, \pi_n)$ a **parking function**, if all cars can park.

Definition

In papers, we have students → **cars**, and hangers → **parking spots**.

Input is a number n and sequence $(\pi_1, \dots, \pi_n)$. Car i prefers space π_i. If π_i is occupied by previous car, then car i takes next available space.

We call $(\pi_1, \dots, \pi_n)$ a **parking function**, if all cars can park.

- $n = 2$: $11, 12, 21$

In papers, we have students $\rightarrow$ **cars**, and hangers $\rightarrow$ **parking spots**.

Input is a number n and sequence $(\pi_1, \ldots, \pi_n)$. Car i prefers space π_i. If π_i is occupied by previous car, then car i takes next available space.

We call $(\pi_1, \ldots, \pi_n)$ a **parking function**, if all cars can park.

- $n = 2$: $11, 12, 21$
- $n = 3$: $111, 112, 121, 211, 113, 131, 311, 122, 212, 221, 123, 132, 213, 231, 312, 321$

Properties of parking function

We have, that a parking function must have at least one $\pi_i = 1$.

Properties of parking function

We have, that a parking function must have at least one $\pi_i = 1$.

Similarly, at least two values $\leqslant 2$, three values $\leqslant 3$ etc.

We have, that a parking function must have at least one $\pi_i = 1$.

Similarly, at least two values $\leqslant 2$, three values $\leqslant 3$ etc.

π is a parking function $\implies \#\{k : a_k \leqslant i\} \geqslant i, \quad 1 \leqslant i \leqslant n$

We have, that a parking function must have at least one $\pi_i = 1$.

Similarly, at least two values $\leqslant 2$, three values $\leqslant 3$ etc.

π is a parking function $\implies \#\{k : a_k \leqslant i\} \geqslant i, \quad 1 \leqslant i \leqslant n$

$\impliedby$ is also true!

Properties of parking function

We have, that a parking function must have at least one $\pi_i = 1$.

Similarly, at least two values $\leqslant 2$, three values $\leqslant 3$ etc.

π is a parking function $\implies \#\{k : a_k \leqslant i\} \geqslant i, \quad 1 \leqslant i \leqslant n$

$\impliedby$ is also true!

Let's have sequence $\pi = (\pi_1, \ldots, \pi_n)$ for which $\#\{k : \pi_k \leqslant i\} \geqslant i$ and let's say π is not a parking function - take min p, such p-th car cannot park. We know, that there exists j, such that $\forall_i : i \geqslant j$, i-th place is taken and $\pi_p \geqslant j$ - take minimal j.

We have, that a parking function must have at least one $\pi_i = 1$.

Similarly, at least two values $\leqslant 2$, three values $\leqslant 3$ etc.

π is a parking function $\implies \#\{k : a_k \leqslant i\} \geqslant i, \quad 1 \leqslant i \leqslant n$

$\impliedby$ is also true!

Let's have sequence $\pi = (\pi_1, \ldots, \pi_n)$ for which $\#\{k : \pi_k \leqslant i\} \geqslant i$ and let's say π is not a parking function - take min p, such p-th car cannot park. We know, that there exists j, such that $\forall_i : i \geqslant j$, i-th place is taken and $\pi_p \geqslant j$ - take minimal j.

That means, that we have $> n - j + 1 + 1$ cars, that want to park in or after place j. So for place $j - 1$, the condition from assumption fails, because we have $n - (n - j + 1 + 1) = j - 2$ cars that can park on or before place $j - 1$.

In another wording:

$\pi = (\pi_1, \dots, \pi_n)$

LEMMA

π is a parking function if and only if increasing rearrangement b of π satisfy $b_i \leqslant i$.

In another wording:

$\pi = (\pi_1, \ldots, \pi_n)$

LEMMA

π is a parking function if and only if increasing rearrangement b of π satisfy $b_i \leqslant i$.

COROLLARY

Every permutation of a parking function is a parking function.

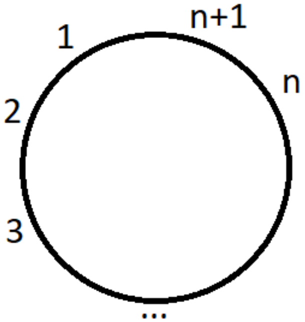
Theorem (Pyke, 1959; Konheim and Weiss, 1966)

Let $f(n)$ be the number of parking functions of size n. Then $f(n) = (n+1)^{n-1}$

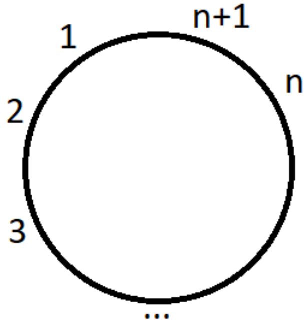
Theorem (Pyke, 1959; Konheim and Weiss, 1966)

Let $f(n)$ be the number of parking functions of size n. Then $f(n) = (n+1)^{n-1}$

Proof:
Add an additional space $n+1$ and make parking a cycle.

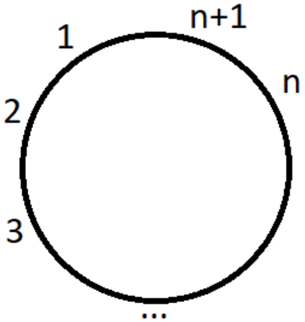


Proof:
Add an additional space $n+1$ and make parking a cycle.



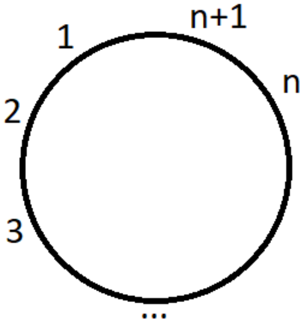
Now all cars can park, and always one space will remain. Notice, that π is a parking function if and only if the empty space is $n+1$.

Proof:
Add an additional space $n+1$ and make parking a cycle.



Now all cars can park, and always one space will remain. Notice, that π is a parking function if and only if the empty space is $n+1$. Furthermore, for $\pi = (\pi_1, \dots, \pi_n)$ if car i parks at p_i, then for $\pi' = (\pi_1 + k, \dots, \pi_n + k)$, car i parks at p_{i+k}.

Proof:
Add an additional space $n+1$ and make parking a cycle.

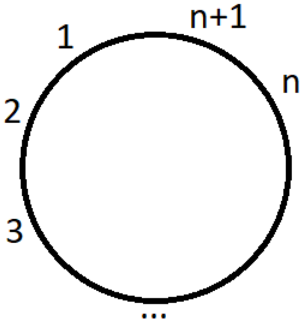


Now all cars can park, and always one space will remain. Notice, that π is a parking function if and only if the empty space is $n+1$. Furthermore, for $\pi = (\pi_1, \dots, \pi_n)$ if car i parks at p_i, then for $\pi' = (\pi_1 + k, \dots, \pi_n + k)$, car i parks at p_{i+k}. So, exactly one of the vectors $(\pi + p, \pi + p, \dots, \pi + p)$ (mod $n+1$) is a parking function, so:

$$f(n) = \frac{(n+1)^n}{n+1} = (n+1)^{n-1}$$

Proof:

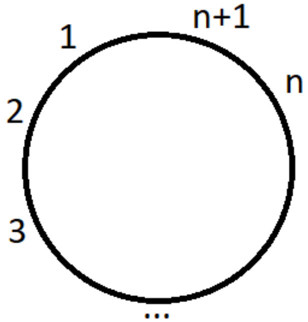
Add an additional space $n+1$ and make parking a cycle.



Now all cars can park, and always one space will remain. Notice, that π is a parking function if and only if the empty space is $n+1$. Furthermore, for $\pi = (\pi_1, \ldots, \pi_n)$ if car i parks at p_i, then for $\pi' = (\pi_1 + k, \ldots, \pi_n + k)$, car i parks at p_{i+k}. So, exactly one of the vectors $(\pi + p, \pi + p, \ldots, \pi + p)$ (mod $n+1$) is a parking function, so:

$$f(n) = \frac{(n+1)^n}{n+1} = (n+1)^{n-1}$$

Proof:
Add an additional space $n+1$ and make parking a cycle.



Now all cars can park, and always one space will remain. Notice, that π is a parking function if and only if the empty space is $n+1$. Furthermore, for $\pi = (\pi_1, \ldots, \pi_n)$ if car i parks at p_i, then for $\pi' = (\pi_1 + k, \ldots, \pi_n + k)$, car i parks at p_{i+k}. So, exactly one of the vectors $(\pi + p, \pi + p, \ldots, \pi + p)$ $\pmod{n+1}$ is a parking function, so:

$$f(n) = \frac{(n+1)^n}{n+1} = (n+1)^{n-1}$$

Based on this proof, we can easily randomize parking functions:

1. Pick random $\pi = (\pi_1, \dots, \pi_n)$
2. If π is parking function, return π.
3. Otherwise, $\pi := \pi + (1, \dots, 1) \pmod{n+1}$, and go to 2.

Theorem (Pyke, 1959; Konheim and Weiss, 1966)

Let $f(n)$ be the number of parking functions of size n. Then $f(n) = (n+1)^{n-1}$

Counting parking functions

Theorem (Pyke, 1959; Konheim and Weiss, 1966)

Let $f(n)$ be the number of parking functions of size n. Then $f(n) = (n+1)^{n-1}$

Theorem (Sylvester-Borchardt-Cayley)

Number of forests based on n vertices is equal to $f(n)$

Theorem (Pyke, 1959; Konheim and Weiss, 1966)

Let $f(n)$ be the number of parking functions of size n. Then $f(n) = (n+1)^{n-1}$

Theorem (Sylvester-Borchardt-Cayley)

Number of forests based on n vertices is equal to $f(n)$

Theorem

Let $f(n,k)$ be the number of parking functions with n cars and k parking slots. Then $f(n,k) = \frac{n-m+1}{n+1}(n+1)^m = (n-m+1)(n+1)^{m-1}$

THEOREM (PYKE, 1959; KONHEIM AND WEISS, 1966)

Let $f(n)$ be the number of parking functions of size n. Then $f(n) = (n+1)^{n-1}$

THEOREM (SYLVESTER-BORCHARDT-CAYLEY)

Number of forests based on n vertices is equal to $f(n)$

THEOREM

Let $f(n,k)$ be the number of parking functions with n cars and k parking slots. Then $f(n,k) = \frac{n-m+1}{n+1}(n+1)^m = (n-m+1)(n+1)^{m-1}$

THEOREM

Number of forests based on $n+1$ vertices having $n+1-m$ distinct tress such that specified set of $n+1-m$ vertices are roots is equal to $f(n+1, n+1-m)$.

$PF(m,n)$ - parking functions with m cars and n slots.

Lemma

$\pi = (\pi_1, \ldots, \pi_m)$ with non decreasing rearrangement $(\lambda_1, \ldots, \lambda_m)$. Then $\pi \in PF(m,n)$ if and only if

$$\lambda_i \leqslant n - m + i, \quad 1 \leqslant i \leqslant m$$

Let $\pi \in PF(m, n)$.

Let $\pi \in PF(m, n)$.

Specification is $r(\pi) = (r_1, \ldots, r_n)$, where $r_k = \#\{i : \pi_i = k\}$.

Let $\pi \in PF(m, n)$.

Specification is $r(\pi) = (r_1, \ldots, r_n)$, where $r_k = \#\{i : \pi_i = k\}$.

Order permutation is a permutation $\sigma(pi) \in \mathfrak{S}_m$, where:

$$\sigma_i = \#\{j : \pi_j < \pi_i, \text{ or } \pi_j = \pi_i \text{ and } j \leqslant i\}$$

Let $\pi \in PF(m, n)$.

Specification is $r(\pi) = (r_1, \dots, r_n)$, where $r_k = \#\{i : \pi_i = k\}$.

Order permutation is a permutation $\sigma(pi) \in \mathfrak{S}_m$, where:

$$\sigma_i = \#\{j : \pi_j < \pi_i, \text{ or } \pi_j = \pi_i \text{ and } j \leqslant i\}$$

σ_i is the position of π_i in the non decreasing rearrangement of π.

Let $\pi \in PF(m, n)$.

Specification is $r(\pi) = (r_1, \ldots, r_n)$, where $r_k = \#\{i : \pi_i = k\}$.

Order permutation is a permutation $\sigma(pi) \in \mathfrak{S}_m$, where:

$$\sigma_i = \#\{j : \pi_j < \pi_i, \text{ or } \pi_j = \pi_i \text{ and } j \leqslant i\}$$

σ_i is the position of π_i in the non decreasing rearrangement of π. For $\pi = (4, 2, 2, 4)$, $\sigma(\pi) = 3124$.

Let $\pi \in PF(m, n)$.

Specification is $r(\pi) = (r_1, \ldots, r_n)$, where $r_k = \#\{i : \pi_i = k\}$.

Order permutation is a permutation $\sigma(pi) \in \mathfrak{S}_m$, where:

$$\sigma_i = \#\{j : \pi_j < \pi_i, \text{ or } \pi_j = \pi_i \text{ and } j \leqslant i\}$$

σ_i is the position of π_i in the non decreasing rearrangement of π. For $\pi = (4, 2, 2, 4)$, $\sigma(\pi) = 3124$.

We can easily recover a parking function π by replacing i in $\sigma(\pi)$ with the i-th smallest term in the sequence $1^{r_1} \ldots n^{r_n}$.

Not every par of a length n vector r and permutation σ can be the specification and the order permutation of a parking function with m cars and n spots.

Not every par of a length n vector r and permutation σ can be the specification and the order permutation of a parking function with m cars and n spots.

Vector r should satisfy the balance condition, equivalent to the sequence $1^{r_1} \dots n^{r_n}$ satisfy *General case non decreasing rearrangement lemma*.

Let $k_i(\pi)$ for $i = 1, \ldots, n - m$ (and $k_0(\pi) = 0$ and $k_{n-m+1}(\pi) = n + 1$) represent the $n - m$ parking spots that are never attempted.

Let $k_i(\pi)$ for $i = 1, \ldots, n - m$ (and $k_0(\pi) = 0$ and $k_{n-m+1}(\pi) = n + 1$) represent the $n - m$ parking spots that are never attempted.

$$\sum_{s=1}^{k_i} r_s = k_i - i, \quad \forall 1 \leqslant i \leqslant n - m \tag{1}$$

$$\sum_{s=1}^{j} r_s > j - i - 1, \quad \forall k_i < j < k_{i+1},\ 0 \leqslant i \leqslant n - m \tag{2}$$

$$\sum_{s=1}^{n} r_s = m \tag{3}$$

Let $k_i(\pi)$ for $i = 1, \ldots, n - m$ (and $k_0(\pi) = 0$ and $k_{n-m+1}(\pi) = n + 1$) represent the $n - m$ parking spots that are never attempted.

$$\sum_{s=1}^{k_i} r_s = k_i - i, \quad \forall 1 \leqslant i \leqslant n - m \tag{1}$$

$$\sum_{s=1}^{j} r_s > j - i - 1, \quad \forall k_i < j < k_{i+1},\ 0 \leqslant i \leqslant n - m \tag{2}$$

$$\sum_{s=1}^{n} r_s = m \tag{3}$$

r is a specification of a parking function if there exists $n - m$ numbers $k_1, \ldots, k_{n-m}$ with $0 := k_0 < k_1 < \ldots < k_{n-m} = n + 1$, satisfying the above.

Lemma

The set $\mathfrak{K}(m, n)$ of 'good' pairs is in one-to-one correspondence with $PF(m, n)$

Lemma

The set $\mathfrak{K}(m, n)$ of 'good' pairs is in one-to-one correspondence with $\mathfrak{F}(n+1, n+1-m)$

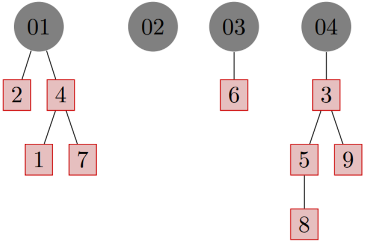
Lemma

The set $\mathfrak{K}(m, n)$ of 'good' pairs is in one-to-one correspondence with $\mathfrak{F}(n+1, n+1-m)$

Lemma

The set $\mathfrak{K}(m, n)$ of 'good' pairs is in one-to-one correspondence with $\mathfrak{F}(n+1, n+1-m)$

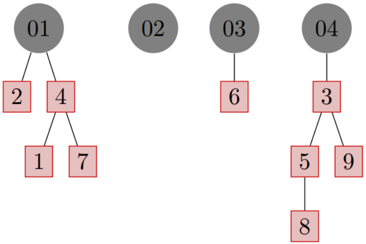
Connect all roots to new vertex V, create BFS order starting from V and remove vertex V from order.



Lemma

The set $\mathfrak{K}(m, n)$ of 'good' pairs is in one-to-one correspondence with $\mathfrak{F}(n+1, n+1-m)$

Connect all roots to new vertex V, create BFS order starting from V and remove vertex V from order.

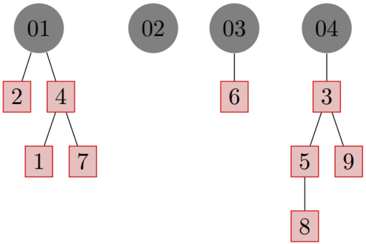


BFS: $01, 02, 03, 04, 2, 4, 6, 3, 1, 7, 5, 9, 8$.
$\sigma = 246317598$ (remove root verticles)

Lemma

The set $\mathfrak{K}(m, n)$ of 'good' pairs is in one-to-one correspondence with $\mathfrak{F}(n+1, n+1-m)$

Connect all roots to new vertex V, create BFS order starting from V and remove vertex V from order.

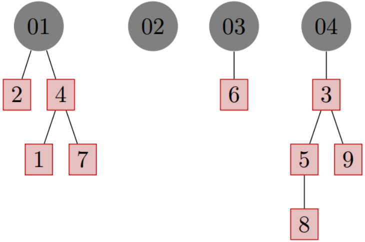


BFS: $01, 02, 03, 04, 2, 4, 6, 3, 1, 7, 5, 9, 8$.
$\sigma = 246317598$ (remove root verticles) r_i - number of successors of v_i, $r = (2{,}0{,}1{,}1{,}0{,}2{,}0{,}2{,}0{,}0{,}1{,}0)$.

Lemma

The set $\mathfrak{K}(m,n)$ of 'good' pairs is in one-to-one correspondence with $\mathfrak{F}(n+1, n+1-m)$

Connect all roots to new vertex V, create BFS order starting from V and remove vertex V from order.



BFS: $01, 02, 03, 04, 2, 4, 6, 3, 1, 7, 5, 9, 8$.
$\sigma = 246317598$ (remove root verticles) r_i - number of successors of v_i, $r = (2,0,1,1,0,2,0,2,0,0,1,0)$.

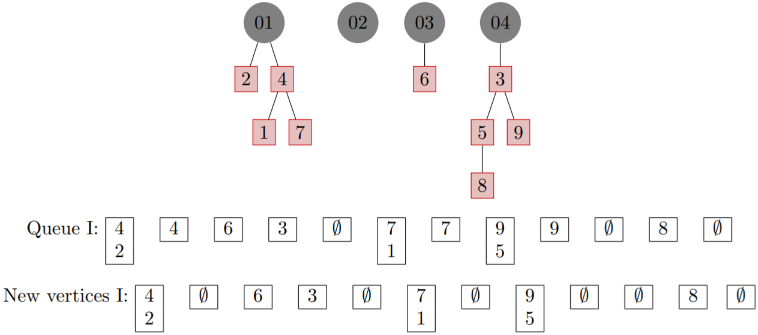
We need to show that r is balanced.

We will use a queue, where at each time step we will

1. read in the successors of the next v_i (if the queue is empty)
2. remove the top element of the queue then read in the successors of the next v_i.

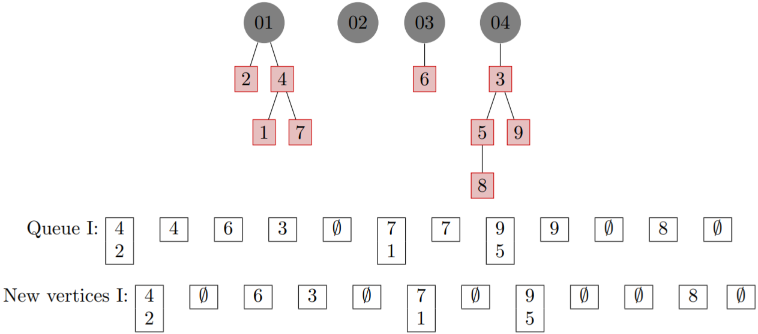
We will use a queue, where at each time step we will

1. read in the successors of the next v_i (if the queue is empty)
2. remove the top element of the queue then read in the successors of the next v_i.



We will use a queue, where at each time step we will

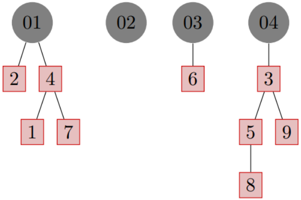
1. read in the successors of the next v_i (if the queue is empty)
2. remove the top element of the queue then read in the successors of the next v_i.



The queue length at time k coincides with the number of cars that attempt to park at spot k,

We will use a queue, where at each time step we will

1. read in the successors of the next v_i (if the queue is empty)
2. remove the top element of the queue then read in the successors of the next v_i.



Queue I:	4, 2	4	6	3	$\emptyset$	7, 1	7	9, 5	9	$\emptyset$	8	$\emptyset$
New vertices I:	4, 2	$\emptyset$	6	3	$\emptyset$	7, 1	$\emptyset$	9, 5	$\emptyset$	$\emptyset$	8	$\emptyset$

The queue length at time k coincides with the number of cars that attempt to park at spot k, and the number of new verticles in the queue at time k coincides with the number of cars r_k whose first preference is spot k.

index	=	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12
Queue	=	$(4,2)$,	(4),	(6),	(3),	$\varnothing$,	$(7,1)$,	(7),	$(9,5)$,	(9),	$\varnothing$,	(8),	$\varnothing$
New verticles	=	$(4,2)$,	$\varnothing$,	(6),	(3),	$\varnothing$,	$(7,1)$,	$\varnothing$,	$(9,5)$,	$\varnothing$,	$\varnothing$,	(8),	$\varnothing$
r	=	2,	0,	1,	1,	0,	2,	0,	2,	0,	1,	0,	0

$$\begin{array}{lcllllllllllll}
\text{index} & = & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12 \\
\text{Queue} & = & (4,2), & (4), & (6), & (3), & \varnothing, & (7,1), & (7), & (9,5), & (9), & \varnothing, & (8), & \varnothing \\
\text{New verticles} & = & (4,2), & \varnothing, & (6), & (3), & \varnothing, & (7,1), & \varnothing, & (9,5), & \varnothing, & \varnothing, & (8), & \varnothing \\
\text{r} & = & 2, & 0, & 1, & 1, & 0, & 2, & 0, & 2, & 0, & 1, & 0, & 0
\end{array}$$

We let $k_1 = 5$, $k_2 = 10$, $k_3 = 12$.

index	=	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12
Queue	=	$(4,2)$,	(4),	(6),	(3),	$\varnothing$,	$(7,1)$,	(7),	$(9,5)$,	(9),	$\varnothing$,	(8),	$\varnothing$
New verticles	=	$(4,2)$,	$\varnothing$,	(6),	(3),	$\varnothing$,	$(7,1)$,	$\varnothing$,	$(9,5)$,	$\varnothing$,	$\varnothing$,	(8),	$\varnothing$
r	=	2,	0,	1,	1,	0,	2,	0,	2,	0,	1,	0,	0

We let $k_1 = 5$, $k_2 = 10$, $k_3 = 12$.

Recall:

$$\sum_{s=1}^{k_i} r_s = k_i - i, \quad \forall 1 \leqslant i \leqslant n - m \tag{4}$$

$$\sum_{s=1}^{j} r_s > j - i - 1, \quad \forall k_i < j < k_{i+1}, \ 0 \leqslant i \leqslant n - m \tag{5}$$

$$\sum_{s=1}^{n} r_s = m \tag{6}$$

index	=	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12
Queue	=	$(4,2)$,	(4),	(6),	(3),	$\varnothing$,	$(7,1)$,	(7),	$(9,5)$,	(9),	$\varnothing$,	(8),	$\varnothing$
New verticles	=	$(4,2)$,	$\varnothing$,	(6),	(3),	$\varnothing$,	$(7,1)$,	$\varnothing$,	$(9,5)$,	$\varnothing$,	$\varnothing$,	(8),	$\varnothing$
r	=	2,	0,	1,	1,	0,	2,	0,	2,	0,	1,	0,	0

We let $k_1 = 5,\ k_2 = 10,\ k_3 = 12.$

Recall:

$$\sum_{s=1}^{k_i} r_s = k_i - i, \quad \forall 1 \leqslant i \leqslant n - m \tag{4}$$

$$\sum_{s=1}^{j} r_s > j - i - 1, \quad \forall k_i < j < k_{i+1},\ 0 \leqslant i \leqslant n - m \tag{5}$$

$$\sum_{s=1}^{n} r_s = m \tag{6}$$

r is a specification of a parking function if there exists $n-m$ numbers $k_1, \ldots, n-m$ with $0 := k_0 < k_1 < \ldots < k_{n-m} = n+1$, satisfying the above.

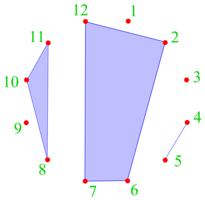
Noncrossing partitions

A **noncrossing partition** of $\{1, 2, \dots, n\}$ is a partition $\{B_1, \dots, B_k\}$ of $\{1, \dots, n\}$ such that:

$$a < b < c < d \;\wedge\; a, c \in B_i \;\wedge\; b, d \in B_j \implies i = j$$

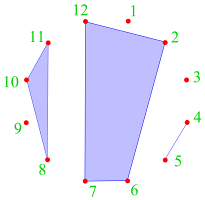
A **noncrossing partition** of $\{1, 2, \ldots, n\}$ is a partition $\{B_1, \ldots, B_k\}$ of $\{1, \ldots, n\}$ such that:

$$a < b < c < d \ \wedge \ a, c \in B_i \ \wedge \ b, d \in B_j \implies i = j$$



A **noncrossing partition** of $\{1, 2, \ldots, n\}$ is a partition $\{B_1, \ldots, B_k\}$ of $\{1, \ldots, n\}$ such that:

$$a < b < c < d \;\wedge\; a, c \in B_i \;\wedge\; b, d \in B_j \implies i = j$$



Theorem (H.W. Becker, 1948-49)

The number of noncrossing partitions of $\{1, 2, \ldots, n\}$ is the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

A **maximal chain** $\mathfrak{m}$ of noncrossing partitions of $\{1, \ldots, n+1\}$ is a sequence $\pi_0, \pi_1, \ldots, \pi_n$ of noncrossing partitions of $\{1, \ldots, n+1\}$ such that π_i is obtained from π_{i-1} by merging two blocks into one (π_0 has $n+1$ blocks, so π_i has exactly $n+1-i$ blocks).

Noncrossing partitions

A **maximal chain** $\mathfrak{m}$ of noncrossing partitions of $\{1, \ldots, n+1\}$ is a sequence $\pi_0, \pi_1, \ldots, \pi_n$ of noncrossing partitions of $\{1, \ldots, n+1\}$ such that π_i is obtained from π_{i-1} by merging two blocks into one (π_0 has $n+1$ blocks, so π_i has exactly $n+1-i$ blocks).

$$1 - 2 - 3 - 4 - 5 \qquad 1 - 25 - 3 - 4 \qquad 1 - 25 - 34 \qquad 125 - 34 \qquad 12345$$

Noncrossing partitions

A **maximal chain** $\mathfrak{m}$ of noncrossing partitions of $\{1, \ldots, n+1\}$ is a sequence $\pi_0, \pi_1, \ldots, \pi_n$ of noncrossing partitions of $\{1, \ldots, n+1\}$ such that π_i is obtained from π_{i-1} by merging two blocks into one (π_0 has $n+1$ blocks, so π_i has exactly $n+1-i$ blocks).

$$1-2-3-4-5 \qquad 1-25-3-4 \qquad 1-25-34 \qquad 125-34 \qquad 12345$$

Define:
min B - least element of B
$j < B : j < k \quad \forall k \in B$

A **maximal chain** $\mathfrak{m}$ of noncrossing partitions of $\{1, \ldots, n+1\}$ is a sequence $\pi_0, \pi_1, \ldots, \pi_n$ of noncrossing partitions of $\{1, \ldots, n+1\}$ such that π_i is obtained from π_{i-1} by merging two blocks into one (π_0 has $n+1$ blocks, so π_i has exactly $n+1-i$ blocks).

$$1-2-3-4-5 \qquad 1-25-3-4 \qquad 1-25-34 \qquad 125-34 \qquad 12345$$

Define:
$\min B$ - least element of B
$j < B : j < k \quad \forall k \in B$

Suppose π is obtained from π_{i-1} by merging together blocks B and B', with $\min B < \min B'$.
Define:
$\Lambda_i(\mathfrak{m}) = \max\{j \in B : j < B'\}$
$\Lambda(\mathfrak{m}) = (\Lambda_1(\mathfrak{m}), \ldots, \Lambda_n(\mathfrak{m}))$

A **maximal chain** $\mathfrak{m}$ of noncrossing partitions of $\{1, \ldots, n+1\}$ is a sequence $\pi_0, \pi_1, \ldots, \pi_n$ of noncrossing partitions of $\{1, \ldots, n+1\}$ such that π_i is obtained from π_{i-1} by merging two blocks into one (π_0 has $n+1$ blocks, so π_i has exactly $n+1-i$ blocks).

$$1-2-3-4-5 \qquad 1-25-3-4 \qquad 1-25-34 \qquad 125-34 \qquad 12345$$

Define:
$\min B$ - least element of B
$j < B : j < k \quad \forall k \in B$

Suppose π is obtained from π_{i-1} by merging together blocks B and B', with $\min B < \min B'$.
Define:
$\Lambda_i(\mathfrak{m}) = \max\{j \in B : j < B'\}$
$\Lambda(\mathfrak{m}) = (\Lambda_1(\mathfrak{m}), \ldots, \Lambda_n(\mathfrak{m}))$

$$\Lambda(m) = (2, 3, 1, 2)$$

Theorem

Λ is a bijection between the maximal chains of noncrossing partitions of $\{1, \ldots, n+1\}$ and parking functions of length n.

Theorem

Λ is a bijection between the maximal chains of noncrossing partitions of $\{1, \ldots, n+1\}$ and parking functions of length n.

Corollary

The number of maximal chains of noncrossing partitions of $\{1, \ldots, n+1\}$ is: $(n+1)^{n-1}$

Thanks!

Image sources:
https://mmaxtreme.pl/32-kurtki-zimowe-letnie-wiosenne-i-przeciwdeszczowe
https://arxiv.org/pdf/2103.17180.pdf
https://www.math.miami.edu/~armstrong/Talks/NCPF_Bielefeld.pdf
https://klein.mit.edu/~rstan/transparencies/parking.pdf