The Hats game. On max degree and diameter.

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Introduction

In the hat guessing game, the sages, located at graph vertices, try to guess colors of their own hats. They can see the colors of hats on sages at the adjacent vertices only. The sages act as a team using the deterministic strategy, fixed at the beginning. If at least one of them guesses a color of his own hat correctly, we say that the sages win.

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The maximum number of possible colors, for which the sages can guarantee the win, is called the *hat guessing number* of graph G and denoted $HG(G)$.

Computation of the hat guessing number for an arbitrary graph is a hard problem. Currently it is solved only for few classes of graphs: for complete graphs, trees, cycles, etc.

Notations

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We will denote a function that is equal to a constant m as $\star m$.

Definition

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We also consider a generalized hat guessing game $\mathcal{G} = \langle G, h, g \rangle$ in which multiple guesses are allowed. The sages win if for at least one of sages the color of his hat matches with one of his guesses.

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It is clear that game $\langle G, h \rangle$ is the same as generalized game $\langle G, h, \star 1 \rangle$.

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Sum of games

Definition.

Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ be two graphs, $S \subseteq G_1$ be a clique, and $v \in V_2$. Set $G = (V, E)$ to be clique join of graphs G_1 and G_2 with respect to S and v. We say that G is a **sum of graphs** G_1, G_2 **with respect to** S **and** v and denote it by $G = G_1 +_{S,v} G_2$.

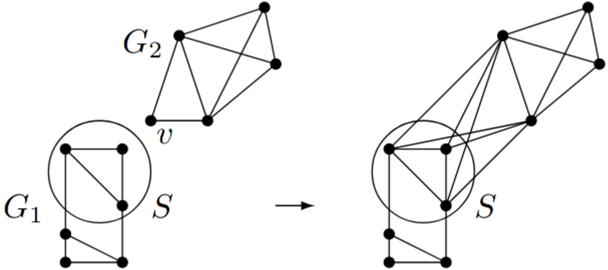

Figure 1: Game $G_1 +_{S,v} G_2$

Sum of games

Definition.

We say that function f is a **gluing** of functions f_1 and f_2 and denote it by $f = f_1 +_{S,v} f_2$, if

$$f(u) = \begin{cases} f_1(u) & u \in V_1 \backslash S \\ f_2(u) & u \in V_2 \backslash \{v\} \\ f_1(u) \cdot f_2(v) & u \in S \end{cases}$$

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Let $\mathcal{G}_1 = \langle G_1, h_1 \rangle, \mathcal{G}_2 = \langle G_2, h_2 \rangle$ be two games. A **sum of games** $\mathcal{G}_1, \mathcal{G}_2$ with respect to S and v is a game $\mathcal{G} = \langle G_1 +_{S,v} G_2, h_1 +_{S,v} h_2 \rangle$.

The sum of generalized hat guessing games is defined similarly.

Sum of games

Theorem 2.1.

Let $\mathcal{G}_1 = \langle G_1, h_1, g_1 \rangle$, $\mathcal{G}_2 = \langle G_2, h_2, g_2 \rangle$ be two winning games, $S \subseteq G_1$ be a clique, and $v \in V_2$. Then the game $\mathcal{G} = \mathcal{G}_1 +_{S,v} \mathcal{G}_2$ is also winning.

Product of games

Definition.

Let $\mathcal{G}_1 = \langle G_1, h_1 \rangle$, $\mathcal{G}_2 = \langle G_2, h_2 \rangle$ be two games, and let one vertex in G_1 and one vertex in G_2 are marked A. A **product of games** $\mathcal{G}_1, \mathcal{G}_2$ **with respect to vertex** A is just $\mathcal{G}_1 +_{\{A\},A} \mathcal{G}_2$. We will denote it by $\mathcal{G} = \mathcal{G}_1 \times_A \mathcal{G}_2$.

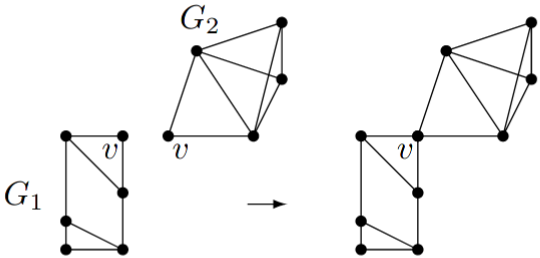

Figure 2: Game $G_1 \times_v G_2$

Product of games

Corollary 2.1.1.

Let $\mathcal{G}_1 = \langle G_1, h_1 \rangle$ and $\mathcal{G}_2 = \langle G_2, h_2 \rangle$ be two games such that $V(G_1) \cap V(G_2) = \{A\}$. If the sages win in games $\mathcal{G}_1$ and $\mathcal{G}_2$, then they win also in game $\mathcal{G} = \mathcal{G}_1 \times_A \mathcal{G}_2$.

Substitution

Definition.

By the **substitution of graph G_1 to graph G_2 on the place of vertex v** we call the graph $G_1 \cup (G_2 \backslash \{v\})$ with adding of all edges that connect each vertex of G_1 with each neighbor of v.

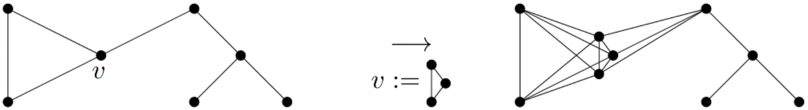

Figure 3: A substitution.

Corollary 2.1.2.

Let the sages win in games $\mathcal{G}_1 = \langle G_1, h_1 \rangle$ and $\mathcal{G}_2 = \langle G_2, h_2 \rangle$, where G_1 is a complete graph. Let $v \in V(G_2)$ be an arbitrary vertex and G be the graph of substitution G_1 on place v. Then the game $\mathcal{G} = \langle G, h \rangle$ is winning, where

$$h(u) = \begin{cases} h_2(u) & u \in G_2 \\ h_1(u) \cdot h_2(v) & u \in G_1 \end{cases}$$

HATS on complete graphs

Theorem 2.4.

The HATS game $\langle K_n, h, g \rangle$ is winning if and only if

$$\sum_{v \in V} \frac{g(v)}{h(v)} \geqslant 1.$$

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We say that a game $\langle K_n, h, g \rangle$ is **precise** if $\sum_{v \in V} \frac{g(v)}{h(v)} = 1$.

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Independence polynomials

Let $G = \langle V, E \rangle$ be a graph. For the set of variables $\mathbf{x} = (x_v)_{v \in V}$ we define *independence polynomials* of G as

$$P_G(\mathbf{x}) = \sum_{\substack{I \subseteq V \\ \text{I-independent set}}} \prod_{v \in I} x_v.$$

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The *monovariate signed independence polynomial* $U_G(x)$ is obtained by plugging $-x$ for each variable x_v of P_G.

Fractional hat chromatic number

Definition.

We define **fractional hat chromatic number** $\hat{\mu}(G)$ as

$$\hat{\mu}(G) = sup\left\{\frac{h}{g} \mid \langle G, \star h, \star g\rangle \text{ is a winning game}\right\}.$$

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$$\hat{\mu}(G) = sup\left\{\frac{h}{g} \mid \langle G, \star h, \star g\rangle \text{ is a winning game}\right\}.$$

Lemma.

For chordal graphs G $\hat{\mu}(G) = 1/r$, where r is the smallest positive root of $U_G(x)$.

Maximal games

Lemma.

$\langle G, h, g \rangle$ is losing whenever $Z_G(\mathbf{r}) > 0$, where $\mathbf{r} = (g_v/h_v)_{v \in V}$.

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Definition.

We say that a game on an arbitrary graph G is **maximal** if:

- $Z_G(\mathbf{r}) = 0$, where $\mathbf{r} = (g_v/h_v)_{v \in V}$,
- $Z_G(\mathbf{x}) > 0$, for every $\mathbf{0} \leqslant \mathbf{x} < \mathbf{r}$.

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The maximal game can be winning or losing, but if we increase the hatness function (or decrease the number of guesses), the game becomes losing due to positivity of Z_G.

Complete graphs

Definition.

We say that a game on an arbitrary graph G is **maximal** if:

- $Z_G(\mathbf{r}) = 0$, where $\mathbf{r} = (g_v/h_v)_{v \in V}$,
- $Z_G(\mathbf{x}) > 0$, for every $\mathbf{0} \leqslant \mathbf{x} < \mathbf{r}$.

Definition.

We say that a game $\langle K_n, h, g \rangle$ is **precise** if $\sum\limits_{v \in V} g_v/h_v = 1$.

Let G be a complete graph and the game $\langle G, h, g \rangle$ be precise. Then $Z_G(\mathbf{x}) = 1 - \sum x_v$ and the game is maximal.

Sum of maximal games

Theorem 3.1.

Let $\mathcal{G}_1 = \langle G_1, h_1, g_1 \rangle$ and $\mathcal{G}_2 = \langle G_2, h_2, g_2 \rangle$ be two maximal games, $S \subseteq G_1$ be a clique, and $v \in V_2$. Then the game $\mathcal{G} = \mathcal{G}_1 +_{S,v} \mathcal{G}_2$ is also maximal.

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Corollary 3.1.1.

Let the game $\mathcal{G} = \langle G, \star h \rangle$ be obtained by a sequence of sum operations from a set of precise winning HATS games on complete graphs. Then $HG(G) = h$.

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Proof.

By induction based on Theorem 3.1 $\mathcal{G}$ is maximal. $\mathcal{G}$ is winning by Theorem 2.1 on sum of games and therefore $HG(G) \geqslant h$.
It follows from maximality condition that $1/h$ is the smallest positive root of $U_G(x)$. G is a chordal graph, therefore $h = \hat{\mu}(G)$.

$$HG(G) \leqslant \hat{\mu}(G) = h \leqslant HG(G)$$

Sum of maximal games

Corollary 3.1.2.

Let the game $\mathcal{G} = \langle G, h, g \rangle$ be obtained by a sequence of sum operations from a set of precise winning games on complete graphs. Let h/g be a constant function, $h/g = h_0 \in \mathbb{Q}$. Then $\hat{\mu}(G) = h_0$.

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HG and maximal degree - first example

It is well known that $HG(G) \leqslant e\Delta(G)$ (Lovász Local Lemma), but no examples of graphs for which $HG(G) > \Delta(G) + 1$ are known.
We start from a nice concrete graph.

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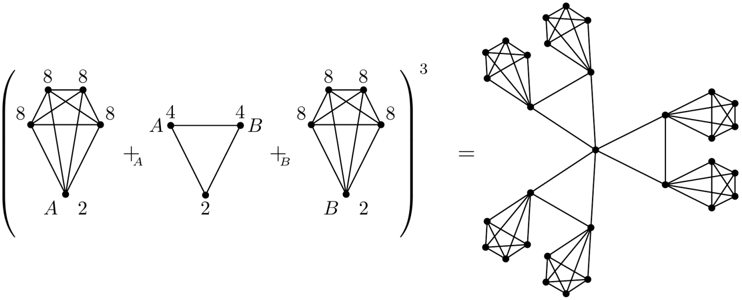

Figure 4: Graph G for which $\Delta(G) = 6$ and $\mathrm{HG}(G) = 8$.

HG and maximal degree - first example

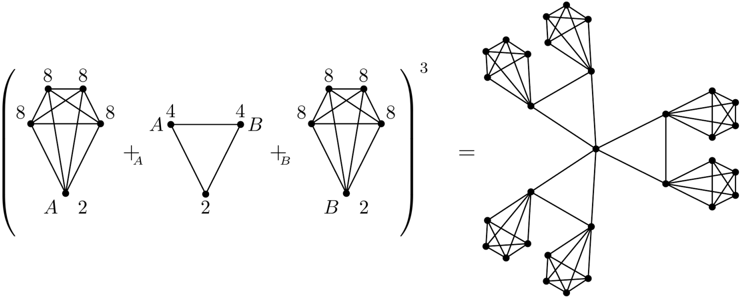

Figure 4: Graph G for which $\Delta(G) = 6$ and $\mathrm{HG}(G) = 8$.

The games on the left are winning precise games by Theorem 2.4 (the value of hatness function is written near each vertex). We combine these graphs by theorem 2.1.1 on game product, and multiply three copies of the obtained graph. We obtain graph G for which the game $\langle G, \star 8 \rangle$ is winning. By corollary 3.1.1 $HG(G) = 8$.

HG and maximal degree - first example

Lemma 4.2.

1. For any positive integer k there exists a graph G such that $HG(G) = \Delta(G) + k$.
2. There exists a sequence of graphs G_n such that $\Delta(G_n) \to +\infty$ and $\lim_{n \to +\infty} HG(G_n)/\Delta(G_n) = 8/7$.

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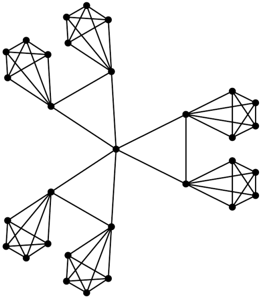
Proof.



We take the previous graph and substitute the games $\langle K_n, \star n\rangle$ in place of each vertex.

- $\forall_{v\in V}\ h(v) = 8n$
- $HG(G) = 8n$
- $\Delta(G) = 7n - 1$

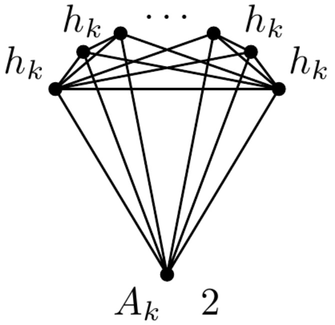
□

HG and maximal degree - second example

Theorem 4.3.

There exists a sequence of graphs G_n such that $\Delta(G_n) \to +\infty$ and $HG(G_n)/\Delta(G_n) = 4/3$.

HG and maximal degree - second example



Precise game T_k:

- 2^{n-k} top vertices
- $h(k) = 2^{n-k+1}$
- bottom vertex A_k
- $h(A_k) = 2$

HG and maximal degree - second example

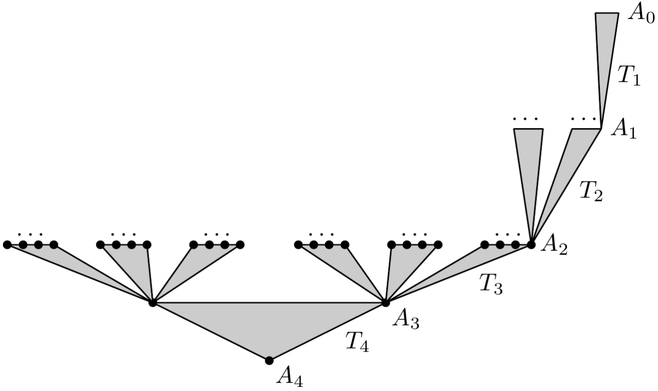


Figure 6: Scary example $\tilde{G}_n$, $n = 5$

HG and maximal degree - second example

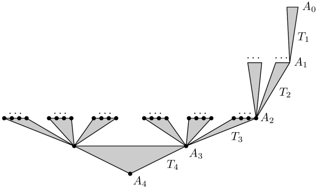


Figure 6: Scary example $\tilde{G}_n$, $n = 5$

- $deg\ A_{n-1} = 2$
- $h(A_{n-1}) = 2$
- $deg\ A_k = k \cdot 2^{n-k} + 2^{n-k-1}$

Maximum degree is reached for $k = 1$.

- $max\ deg\ A_k = 2^{n-1} + 2^{n-2} = 3/4 \cdot 2^n$
- $h(A_k) = 2^{n-k} \cdot 2^k = 2^n$

HG and maximal degree - second example

Let graph G_n be a product of n copies of graph $\tilde{G}_n$ by vertex A_{n-1}. Then:

- $deg\ A_{n-1} = 2n$,
- $h(A_{n-1}) = 2^n$.

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Let graph G_n be a product of n copies of graph $\tilde{G}_n$ by vertex A_{n-1}. Then:

- $deg\ A_{n-1} = 2n$,
- $h(A_{n-1}) = 2^n$.

The degrees and hatnesses of the other vertices remain unchanged. Since the hatness function is constant on the obtained graph, by corollary 3.1.1 $HG(G_n) = 2^n$ and the following equality holds.

$$\frac{HG(G_n)}{\Delta(G_n)} = \frac{2^n}{\frac{3}{4}2^n} = \frac{4}{3}$$

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HG and diameter

Theorem 6.2.

For any odd d and even $h_0 > 3$ there exists graph G with diameter d and $HG(G) = h_0$.

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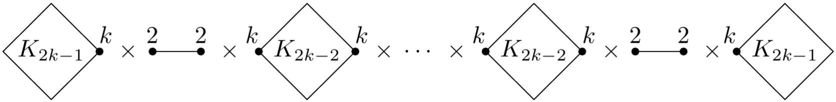

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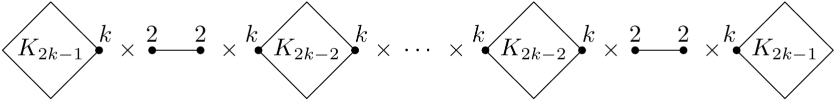

Hatnesses of the other vertices are equal $2k$. These games are precise winning by Theorem 2.4. We glue pairs of vertices to the left and to the right of each "×" sign and multiply their hatnesses. By Corollary 3.1.1 the obtained game $\langle G_n, \star 2k\rangle$ is maximal winning and $HG(G_n) = 2k$. Moreover, diameter of the graph equals $2n - 1$.