The Catalan Matroid

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Based on F. Ardila
"The Catalan matroid (2002)"

Dyck paths

Let $\mathcal{P}_n$ be the set of paths of length $2n$, starting from $(0, 0)$ and with steps either $(1, 1)$ or $(1, -1)$.

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Definition

A path $p \in$P is a Dyck path, if it ends in $(2n, 0)$ and never passes below $y = 0$ line.

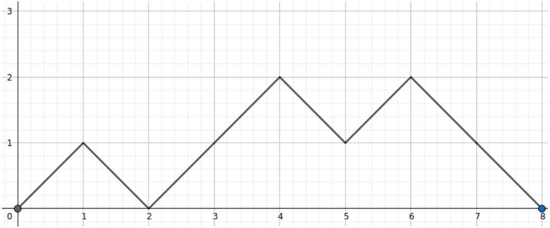

Figure: Dyck path of length 8

Catalan numbers

The well-known fact tells, that the number of Dyck paths of length $2n$ is equal to

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Matroid

The generalization of independence in vector spaces into combinatorial structures

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Definition

A matroid M is a pair of a ground set E and a family of independent sets $\mathcal{I}$ over E, such that:

- $\mathcal{I}$ is not empty,
- Every subset of an independent set is independent,
- For two independent sets A, B, such that $|A| > |B|$, we can find and element x in A and not in B, such that $B \cup \{x\}$ also is independent (augumentation property)

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It follows, that the finite vector space is a matroid.

Matroid - bases

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Definition (alternative and equivalent)

A matroid M is a pair of a ground set E and a family of basis sets $\mathcal{B}$ over E, such that:

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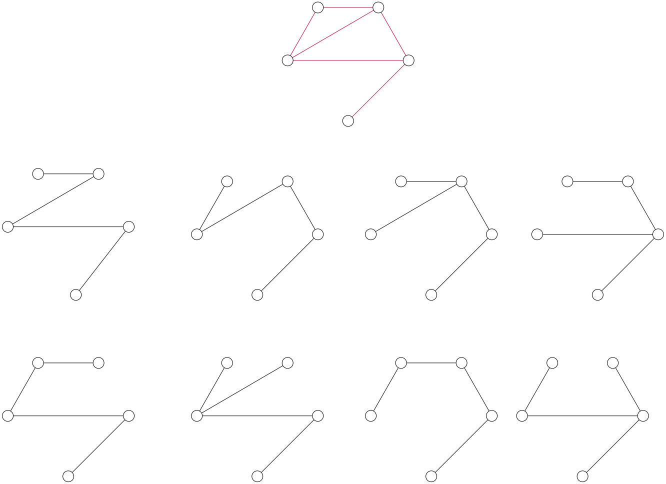
All bases will have the same size

Matroid example

The set of spanning trees of any graph forms a matroid



Column Matroid

Another important example – linearly independent subsets of columns for some matrix M over field $\mathbb{F}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The Catalan Matroid

Encode any path in $\mathcal{P}$ as a set of indices $\subseteq [2n]$, where it goes up.

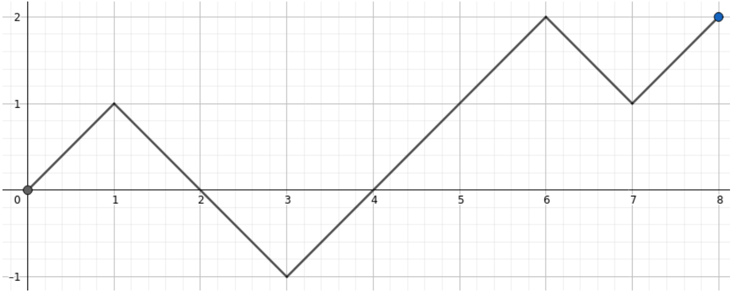


Figure: Encoding: $\{1, 4, 5, 6, 8\}$

The set of (the encodings of) Dyck paths is the set of basis of matroid (ground set $E = [2n]$):

- It's not empty: $\{1, 2, \ldots, n\}$ is a Dyck path.
- For any two Dyck paths A, B and any index of an up-step $a \in A \setminus B$ pick smallest index $b \in B \setminus A$ and exchange

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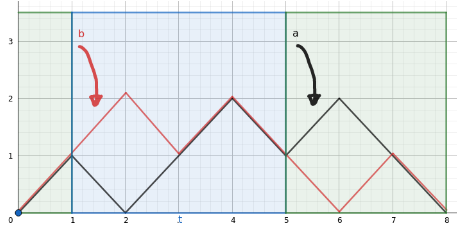
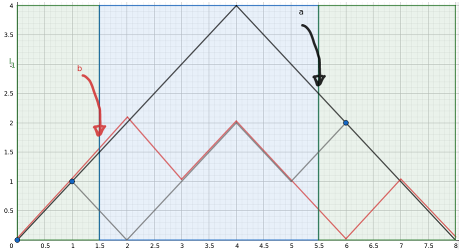
We'll have 2 cases

Case 1

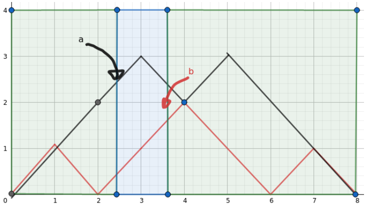
If $b < a$, then all points before b or after a don't change its height over $y = 0$ axis. Points inside that interval will raise by 2.

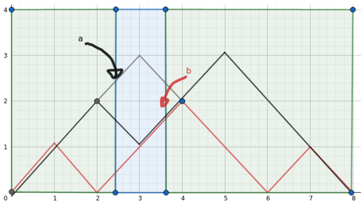




Case 2

If $b > a$, then any point of B before b is not higher than its corresponding point in A. Moreover, height of points in (a, b) in B is less by at least 2 than corresponding points in A, so A can fall down on that segment.





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Set $S \subseteq E$ of matroid $(E, \mathcal{B})$ is a spanning set, iff $S \supseteq B$ for some $B \in \mathcal{B}$

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Spanning sets in that matroid are paths, that never go below $y = 0$ line.

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For $S \subseteq E$ in matroid $M = (S, \mathcal{I})$, the rank $r(S)$ is the size of the largest independent set being subset of S.

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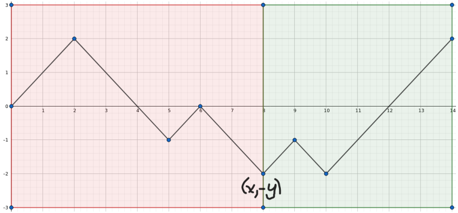

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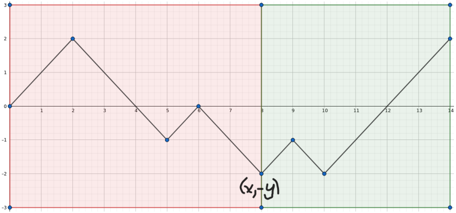

We can pick all $\frac{x-y}{2}$ up-steps from left part. The right part is a shifted spanning set, so we can select a basis of size $\frac{2n-x}{2}$ from it. Combining, we obtain independent set of size $n - \frac{y}{2}$.

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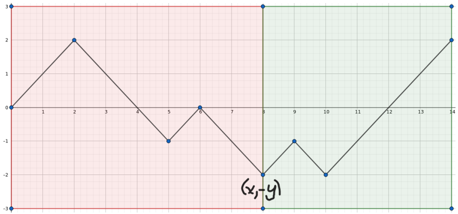

To show that we can't find larger independent subset note, that we cannot select more than $\frac{2n-x}{2}$ up steps from the right part.

Duality

Definition (Dual matroid)

Given matroid $M = (E, \mathcal{B})$ define $M^* = (E, \overline{\mathcal{B}})$, where $\overline{\mathcal{B}} = \{\overline{B} : E \setminus \overline{B} \in \mathcal{B}\}$. M^* is a matroid, called dual matroid.

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Bases of dual of the Catalan matroid $\mathcal{C}_n$ is the set of mirrored Dyck paths along the x axis. If we bijective map the dual basis $\overline{B} = \{c_1, \ldots, c_n\}$ into $B = \{2n+1-c_n, \ldots, 2n+1-c_1\}$, then we receive back the Dyck path. Therefore, the Catalan Matroid is self-dual.

Shifted matroid

Any basis B of The Catalan Matroid $\mathbf{C}_n = (E, \mathcal{B})$ can be viewed as a set positive integers $a_1, \ldots a_n$, such that $a_1 \leq s_1,\ a_2 \leq s_2,\ \ldots,\ a_n \leq s_n$, where $s_1 = 1, s_2 = 3, \ldots, s_n = 2n - 1$

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We can generalize this matroid by providing different non-empty set $S = \{s_1 < \cdots < s_n\}$. It can be shown that such construction will yield a matroid. We'll call it the shifted matroid $\mathbf{SM}(s_1, \dots, s_n)$

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Note, that for any basis $B = a_1, \ldots a_n$ we always exchange it to another basis, by removing arbitrary a_i and adding $x < a_i, x \notin B$

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Lemma (Klivans)

If the above property holds for loop-less matroid M, then $M \cong SM(s_1, \dots, s_n)$ for some $s_1 \dots s_n$

Young tableau

A Young diagram with written numbers from 1 to n in it, such that numbers in all squares are smaller than each number to its southeast.

1	2	5	7	11
3	6	10	12	
4	9	15	18	
8	14	16		
13				
17				

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Namely, $s_1 = 1$ and $s_i = s_{i-1} + \lambda'_{i-1}$, where λ'_{i-1} is the number of cells in the $(i-1)$-st column.

Representability

Definition

Matroid M is representable over field $\mathbb{F}$ iff it is isomorphic to some column matroid over $\mathbb{F}^d$

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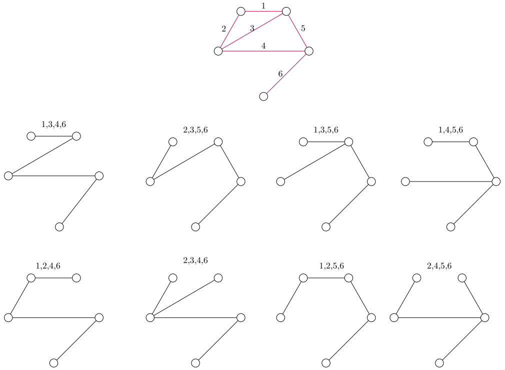

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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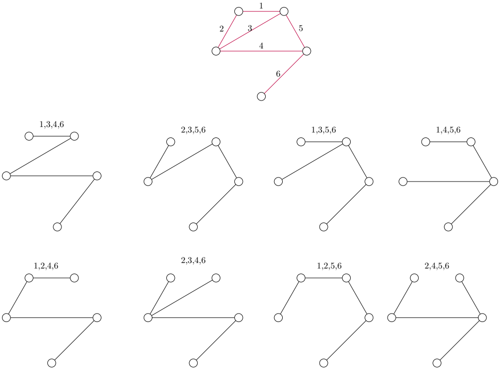

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Is Catalan Matroid representable over some field?

Representability over $\mathbb{Q}$

Lemma

The Catalan matroid is representable over $\mathbb{Q}$

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$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 & \dots & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & a_{n5} & a_{n6} & \dots & a_{n,2n-1} & 0 \end{bmatrix}$$

Where $\{a_{ij}\}$ is arbirtary set of generic integers.

Representability over finite fields

Lemma

The Catalan matroid is not representable over $\mathbb{F}_q$ *if* $q \leq n - 2$

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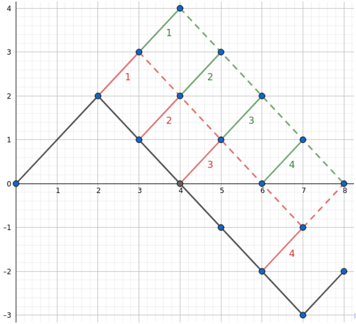
Sketch of proof:

- Uniform matroid $\mathbf{U}_{2,k}$ (bases - subsets of size 2 of set $[n]$) is representable over $\mathbb{F}_q$ iff $q \geq k - 1$



Representability over finite fields

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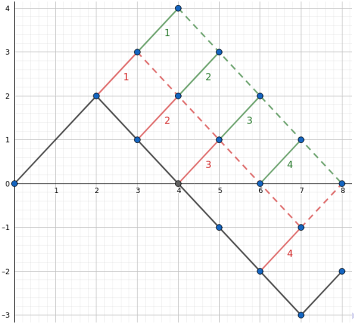
The Catalan matroid is not representable over $\mathbb{F}_q$ if $q \leq n - 2$

Sketch of proof:

- Uniform matroid $\mathbf{U}_{2,k}$ (bases - subsets of size 2 of set $[n]$) is representable over $\mathbb{F}_q$ iff $q \geq k - 1$
- C_n has $\mathbf{U}_{2,n}$ as a minor: empty set maps to $\{1, 2, \ldots, n-2, 2n\}$



Tutte Polynomial

Definition

$T_M(q,t) = \sum_{A \subseteq E} (q-1)^{r(E)-r(A)} (t-1)^{|A|-r(A)}$

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Dually, $i(B)$ is the number of elements from B, that, when added to dual basis B^* of dual matroid, are the smallest element in the dual circuit.

Lemma

$T_M(q,t) = \sum_{B \in \mathcal{B}} q^{i(B)} t^{e(B)}$

Internal and external activities

Lemma

$e(B)$ is equal to number of non-zero positions, where the Dyck path touches X axis
$i(B)$ is equal to number of up-steps before the first down-step of a Dyck path.

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Because Tutte polynomial is symmetrical over q and t, those number are equidistributed.

Tutte Polynomial formula

$$\sum_{n\geq 0} T_{C_n}(q,t)x^n = \frac{1+(qt-q-t)xC(x)}{1-qtx+(qt-q-t)xC(x)}$$

Where $C(x) = \frac{1-\sqrt{1-4x}}{2}$ is the Catalan numbers generating function.