Clustered Coloring of Graphs Excluding a Subgraph and a Minor

Chun-Hung Liu, David R. Wood

[2019+]

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G can be divided into connected groups forming H

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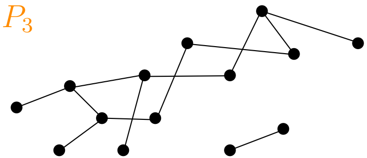

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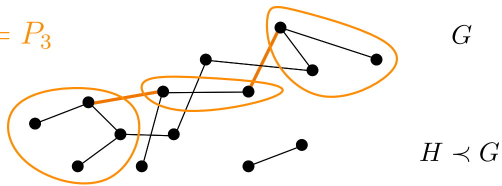

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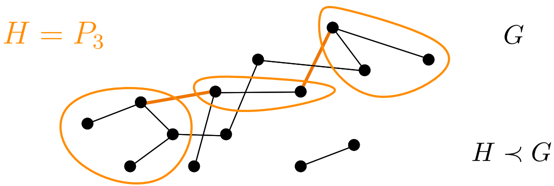

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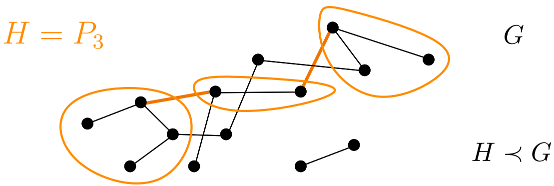

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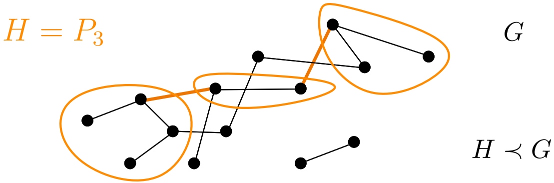

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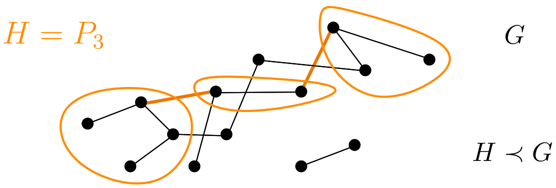

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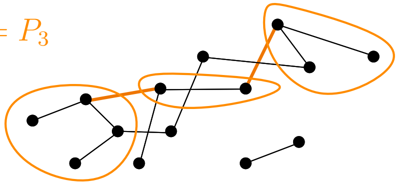

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$t \leqslant 5$ - OK

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[Norin,Song,Postle 2020]

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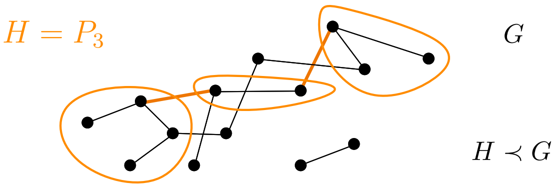

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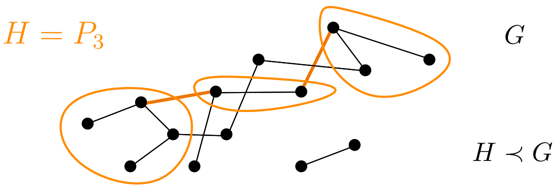

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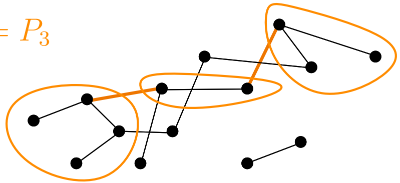

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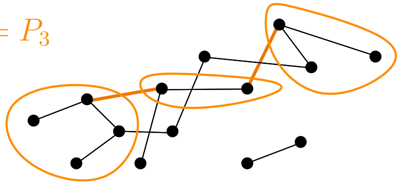

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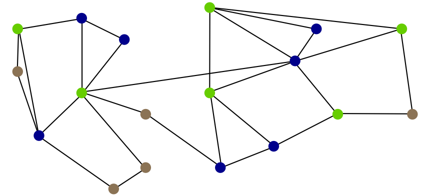

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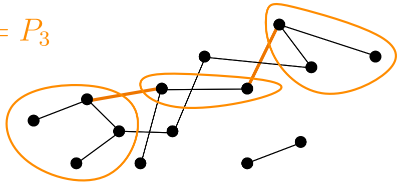

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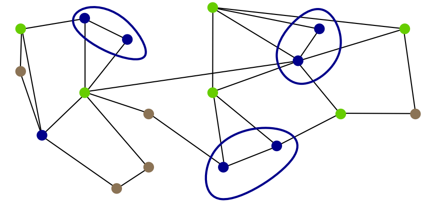

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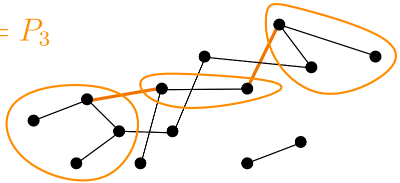

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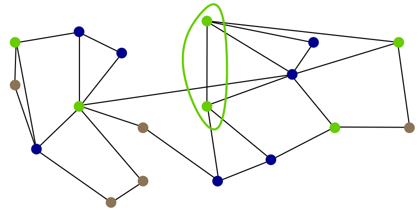

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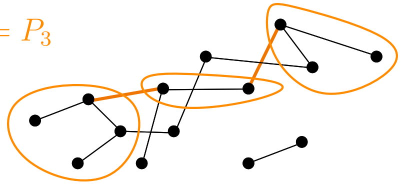

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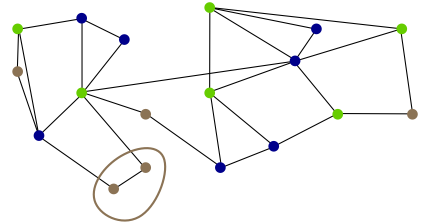

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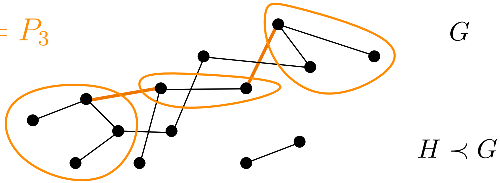

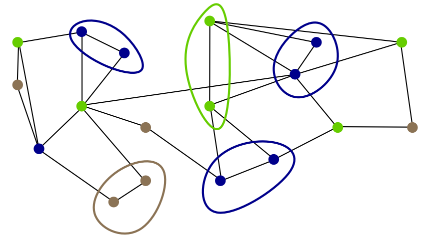

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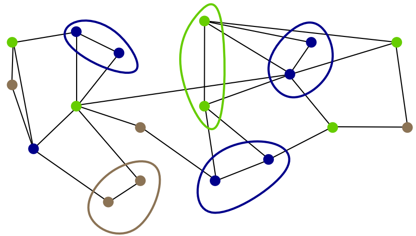

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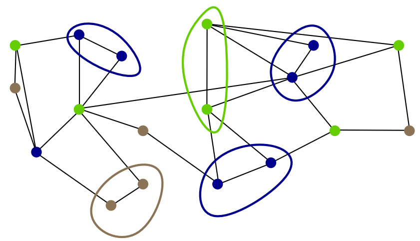

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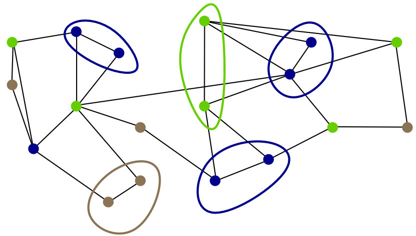

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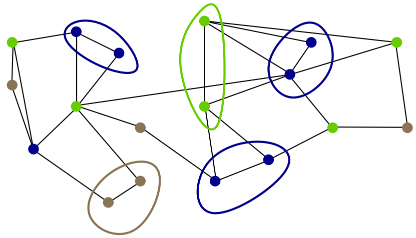

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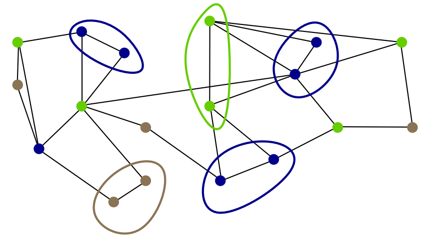

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and $\eta(s) = \lceil \frac{s-2}{2} \rceil$

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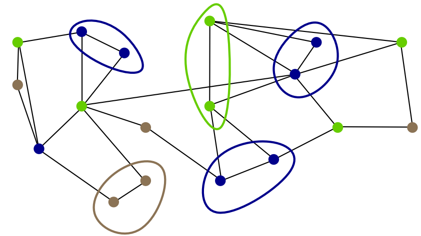

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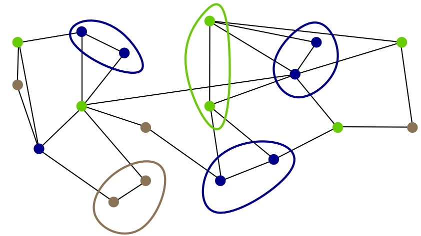

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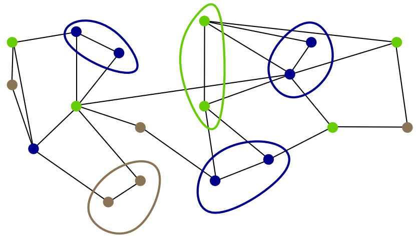

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Technical statement:

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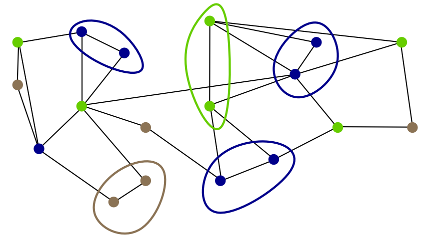

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Technical statement:

$$\forall_{s,t,H} \exists_{\eta=\eta(s,t,H)} \forall_G$$

G has no H-minor

G has no $K_{s,t}$-subgraph

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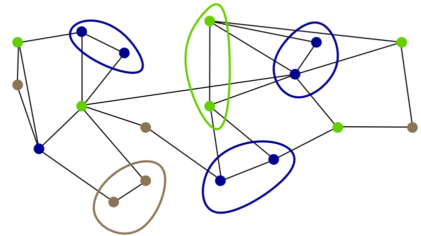

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$\forall_{s,t,H} \exists_{\eta=\eta(s,t,H)} \forall_G$ G has no H-minor
G has no $K_{s,t}$-subgraph

$\Rightarrow G$ is $(s+2)$-colorable with clustering η

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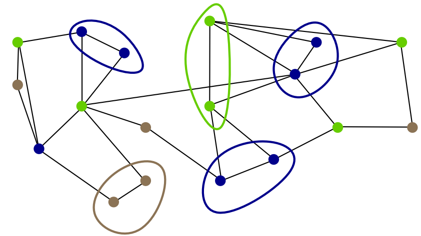

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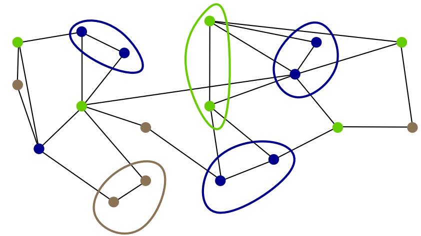
each monochromatic component is bounded by η

$\mathcal{G}$ - graph class

$\chi_c(\mathcal{G})$ = size of minimal η-clustered coloring

absolute $\eta = \eta(s)$



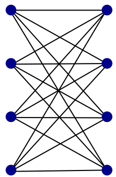
2-clustered 3 coloring

Hadwiger's clustered conjecture:

$\mathcal{G}_s = K_{s+1}$ minor free graphs

$\chi_c(\mathcal{G}_s) \leqslant s$

$\boxed{\chi_c(\mathcal{G}_s) \leqslant s+2}$



$K_{s,s,}$ has K_{s+1} minor

Technical statement: (optimal)

$\forall_{s,t,H} \exists_{\eta=\eta(s,t,H)} \forall_G$ G has no H-minor

G has no $K_{s,t}$-subgraph

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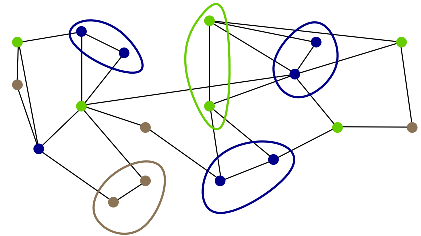

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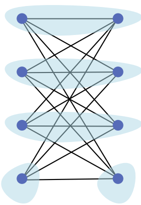

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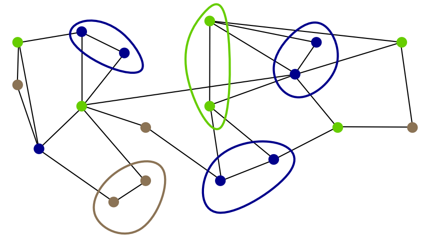

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We will prove

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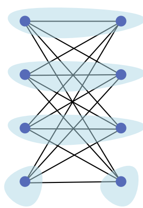

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planar

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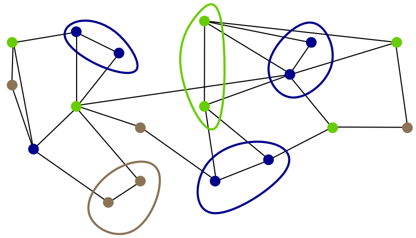

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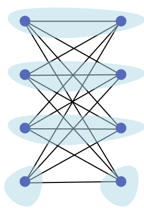

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~~planar~~

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~~G has no H-minor~~

G has no $K_{s,t}$-subgraph

$(s+1)$

$\Rightarrow G$ is $\cancel{(s+2)}$-colorable with clustering η

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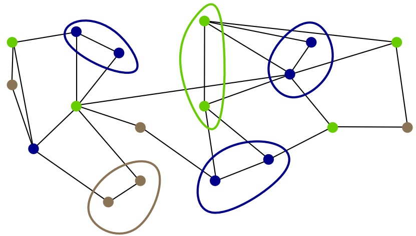

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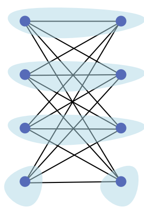

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We will prove

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$$\forall_{s,t,\overset{\omega}{\cancel{H}}} \exists_{\eta=\eta(s,t,\overset{\omega}{\cancel{H}})} \forall_G$$

tw$(G) \leqslant \omega$

~~G has no H-minor~~

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~~planar~~

$(s+1)$ choosable

$\Rightarrow G$ is $\cancel{(s+2)}$-~~colorable~~ with clustering η

$\forall_{s,t,\omega} \exists_{\eta=\eta(s,t,\omega)} \forall_G$

- $\text{tw}(G) \leqslant \omega$
- G has no $K_{s,t}$-subgraph

$\Rightarrow G$ is $(s+1)$-choosable with clustering η

$\forall_{s,t,\omega} \exists_{\eta=\eta(s,t,\omega)} \forall_G$

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(A, B) is a Separation if $A \cup B = G$ and $E(A) \cap E(B) = \emptyset$.

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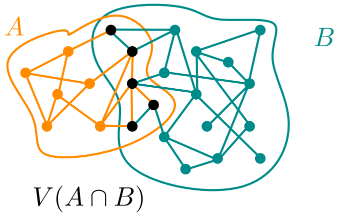

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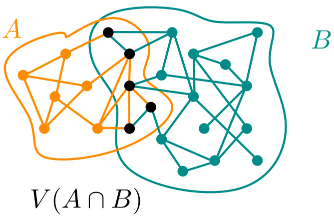
$\mathcal{T}$ - set of some separation of order $< \theta$



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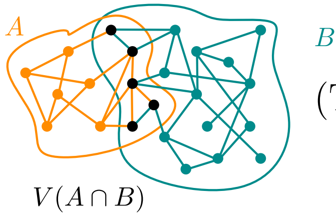

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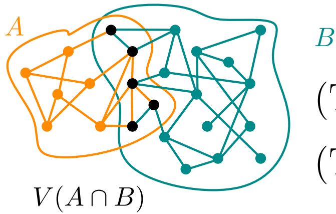

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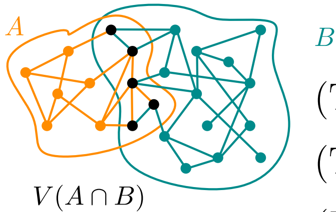

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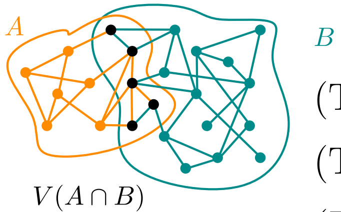

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Example: C fixed cycle in G

$\mathcal{T} = \{(A, B) : \mathrm{ord} = 1, C \subset B\}$

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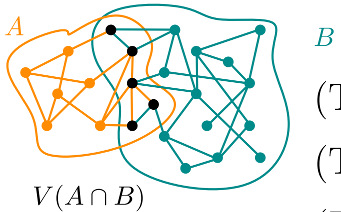

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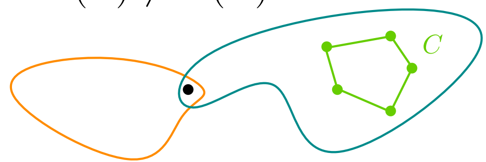

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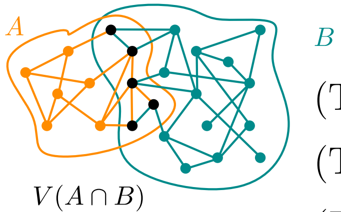

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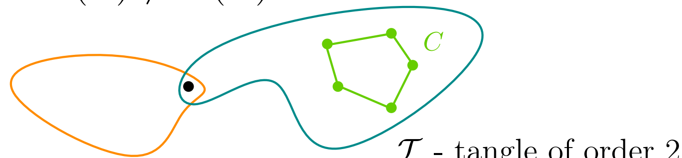

$\forall_{s,t,\omega} \exists_{\eta=\eta(s,t,\omega)} \forall_G$

- $\operatorname{tw}(G) \leqslant \omega$
- G has no $K_{s,t}$-subgraph

$\Rightarrow G$ is $(s+1)$-choosable with clustering η

???

Advanced example: $G = \boxplus_k$

$\mathcal{T} = \{(A,B) : \text{ord} < k, \text{full row} \subset B\}$

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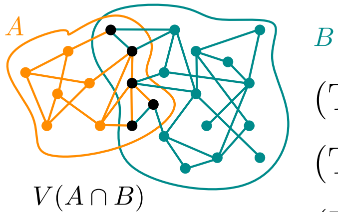

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Example: C fixed cycle in G

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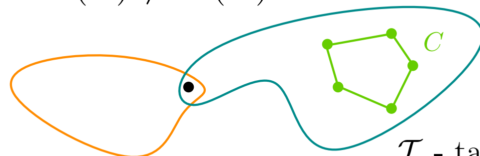


$\mathcal{T}$ - tangle of order 2

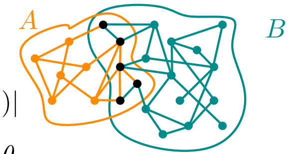
$$\forall_{s,t,\omega} \exists_{\eta=\eta(s,t,\omega)} \forall_G$$

- $\mathrm{tw}(G) \leqslant \omega$
- G has no $K_{s,t}$-subgraph

$\Rightarrow G$ is $(s+1)$-choosable with clustering η

???

$\mathrm{ord}(A, B)$
$:= |V(A \cap B)|$



$\mathcal{T}$ - ...order $< \theta$

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(T2) $A_1 \cup A_2 \cup A_3 \neq G$

(T3) $V(A) \neq V(G)$

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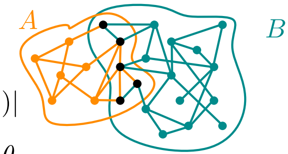
$\Rightarrow G$ is $(s+1)$-choosable with clustering η

???



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(T3) $V(A) \neq V(G)$

- $\mathrm{tw}(G) \leqslant \omega$

$:=$ no tangle of order $\omega + 2$

$\forall_{s,t,\omega} \exists_{\eta=\eta(s,t,\omega)} \forall_G$

- $\operatorname{tw}(G) \leqslant \omega$ (:= no tangle of order $\omega + 2$)
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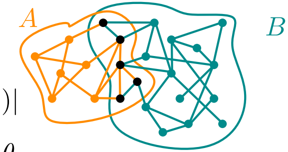

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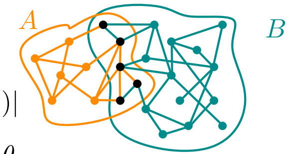

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$X \subset V(G)$

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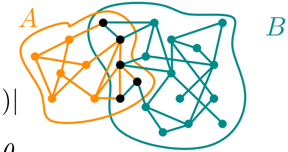

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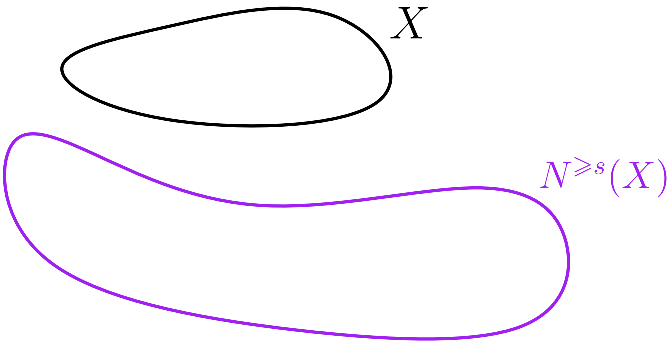

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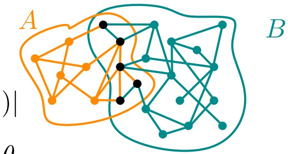


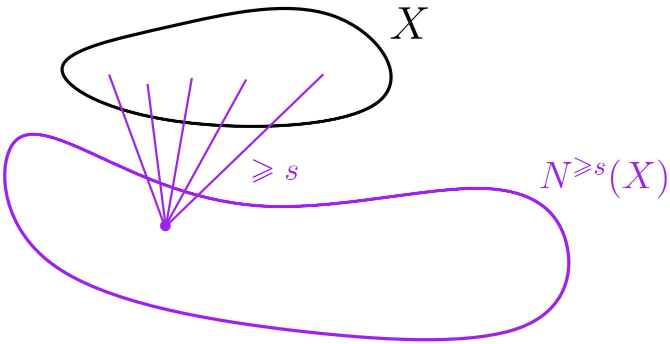
$\mathcal{T}$ - ...order $< \theta$

(T1) $(A, B) \in \mathcal{T}$ or $(B, A) \in \mathcal{T}$

(T2) $A_1 \cup A_2 \cup A_3 \neq G$

(T3) $V(A) \neq V(G)$

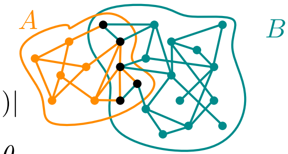
$X \subset V(G)$



$\forall_{s,t,\omega} \exists_{\eta=\eta(s,t,\omega)} \forall_G$

- $\mathrm{tw}(G) \leqslant \omega$ (:= no tangle of order $\omega + 2$)
- $\boxed{G \text{ has no } K_{s,t}\text{-subgraph}}$

$\Rightarrow G$ is $(s+1)$-choosable with clustering η



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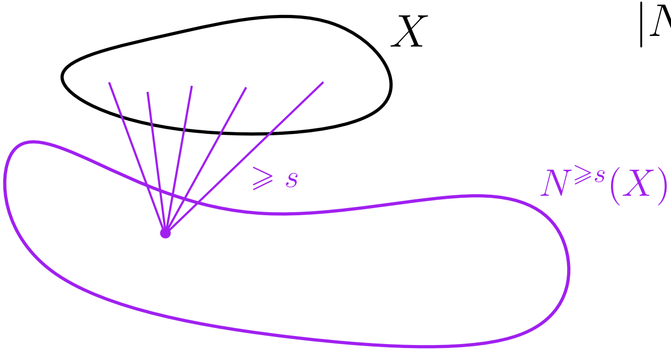
$X \subset V(G)$

$$|N^{\geqslant s}(X)| \leqslant (t-1) \cdot \binom{|X|}{s} + 1$$



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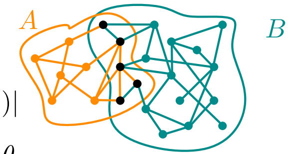

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(T3) $V(A) \neq V(G)$

$$|N^{\geqslant s}(X)| \leqslant f(|X|, s, t)$$

We start with $|L(v)| = s + 1$

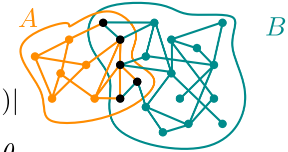
$\forall_{s,t,\omega} \exists_{\eta=\eta(s,t,\omega)} \forall_G$

- $\mathrm{tw}(G) \leqslant \omega$ (:= no tangle of order $\omega + 2$)
- G has no $K_{s,t}$-subgraph

$\Rightarrow G$ is $(s+1)$-choosable with clustering η

We start with $|L(v)| = s + 1$

Iteratively we enlarge colored set Y (until not too big)



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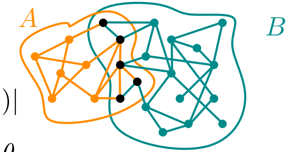
Iteratively we enlarge colored set Y (until not too big)

Invariant:



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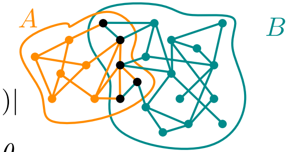
- $\mathrm{tw}(G) \leqslant \omega$ (:= no tangle of order $\omega + 2$)
- G has no $K_{s,t}$-subgraph

$\Rightarrow G$ is $(s+1)$-choosable with clustering η

$\mathrm{ord}(A, B) := |V(A \cap B)|$



$\mathcal{T}$ - ...order $< \theta$

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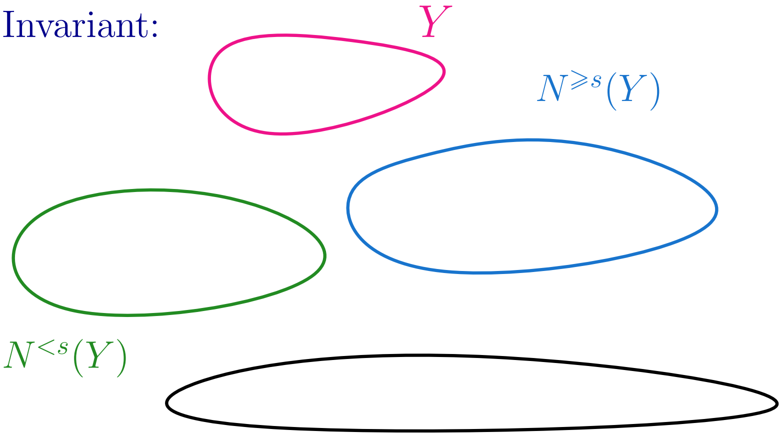
Iteratively we enlarge colored set Y (until not too big)

Invariant:



$$\forall_{s,t,\omega}\exists_{\eta=\eta(s,t,\omega)}\forall_G$$

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$\mathrm{ord}(A,B)$
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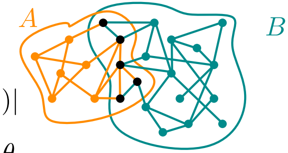

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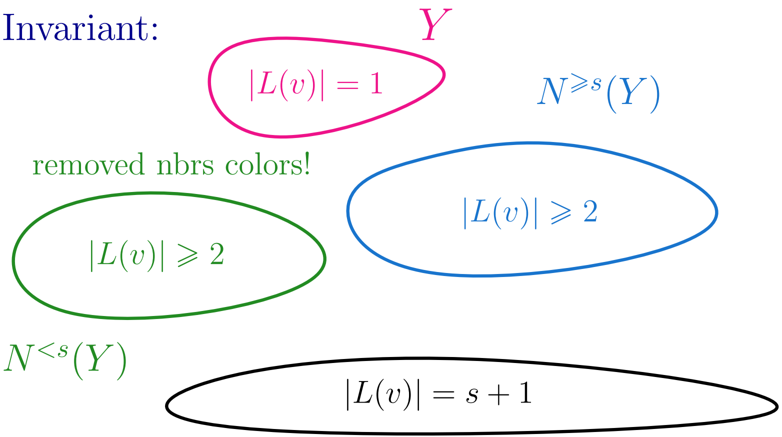
Invariant:



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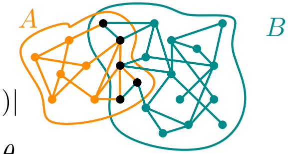

$\mathcal{T}$ - ...order $<\theta$

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(T3) $V(A)\neq V(G)$

$|N^{\geqslant s}(X)| \leqslant f(|X|,s,t)$

We start with $|L(v)| = s+1$

Iteratively we enlarge colored set Y (until not too big)

Invariant:

Y

$|L(v)| = 1$

$N^{\geqslant s}(Y)$

Start with $Y=\{v\}$ and any color

removed nbrs colors!

$|L(v)| \geqslant 2$

$|L(v)| \geqslant 2$

$N^{<s}(Y)$

$|L(v)| = s+1$

$\forall_{s,t,\omega} \exists_{\eta=\eta(s,t,\omega)} \forall_G$

- $\mathrm{tw}(G) \leqslant \omega$ (:= no tangle of order $\omega + 2$)
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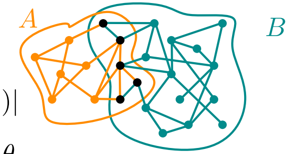

$\mathrm{ord}(A, B) := |V(A \cap B)|$

$\mathcal{T}$ - ...order $< \theta$

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$|N^{\geqslant s}(X)| \leqslant f(|X|, s, t)$

We start with $|L(v)| = s + 1$

Iteratively we enlarge colored set Y (until not too big)

Invariant:

Y

$|L(v)| = 1$

$N^{\geqslant s}(Y)$

$|L(v)| \geqslant 2$

removed nbrs colors!

$|L(v)| \geqslant 2$

$N^{<s}(Y)$

$|L(v)| = s + 1$

Start with $Y = \{v\}$ and any color

If $N^{\geqslant s}(Y) = \emptyset$ color some $z \in N^{<s}(Y)$

$\forall_{s,t,\omega} \exists_{\eta=\eta(s,t,\omega)} \forall_G$

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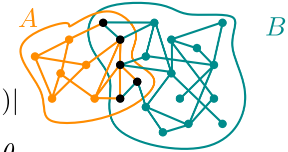

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Otherwise:

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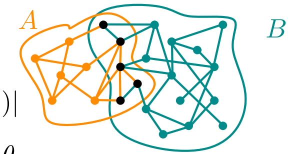

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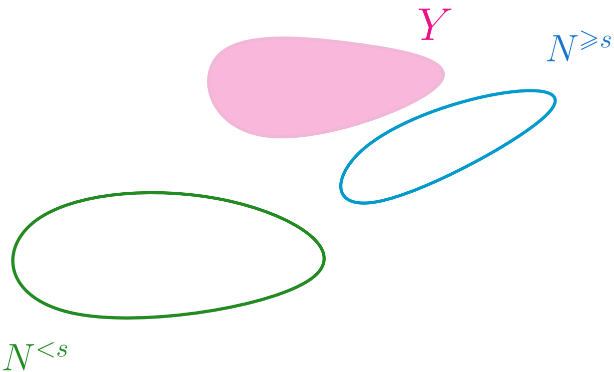

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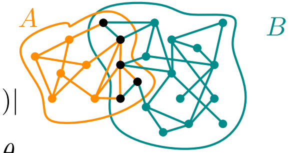

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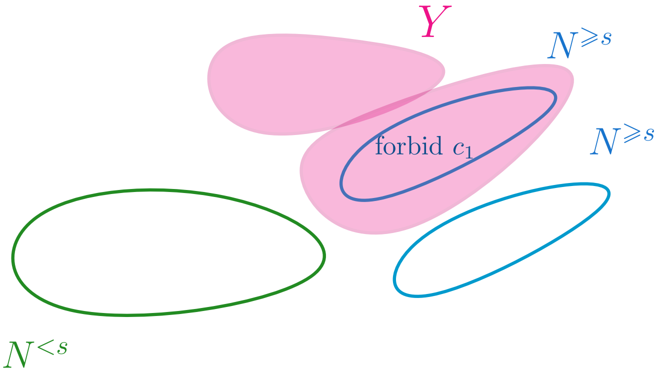

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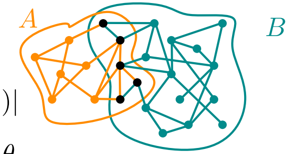

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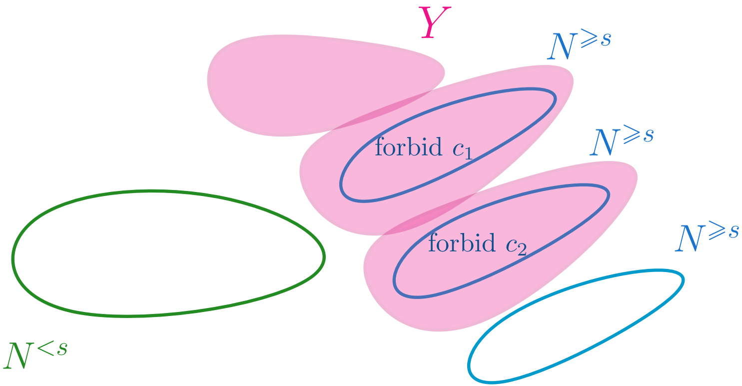

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Otherwise:

$\forall_{s,t,\omega} \exists_{\eta=\eta(s,t,\omega)} \forall_G$

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- G has no $K_{s,t}$-subgraph

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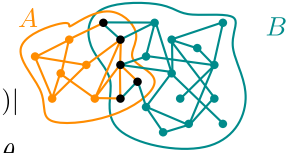
$\mathrm{ord}(A, B) := |V(A \cap B)|$



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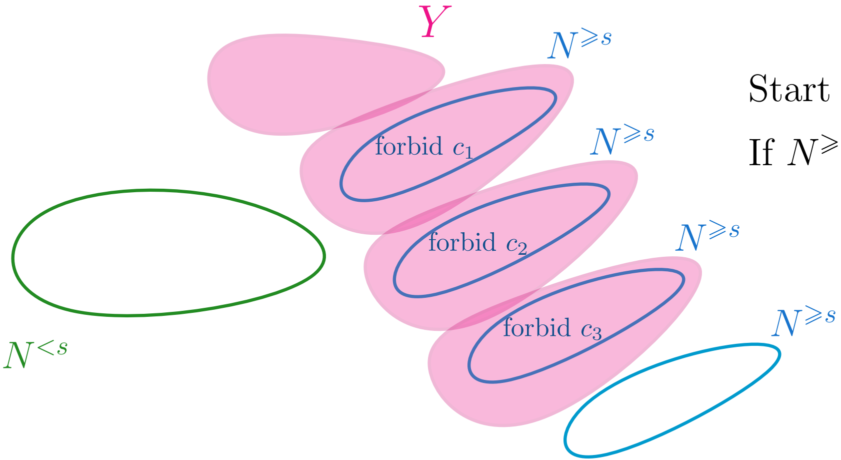

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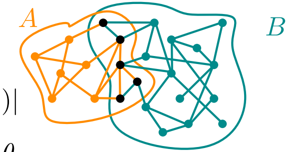
ord(A, B) $:= |V(A \cap B)|$



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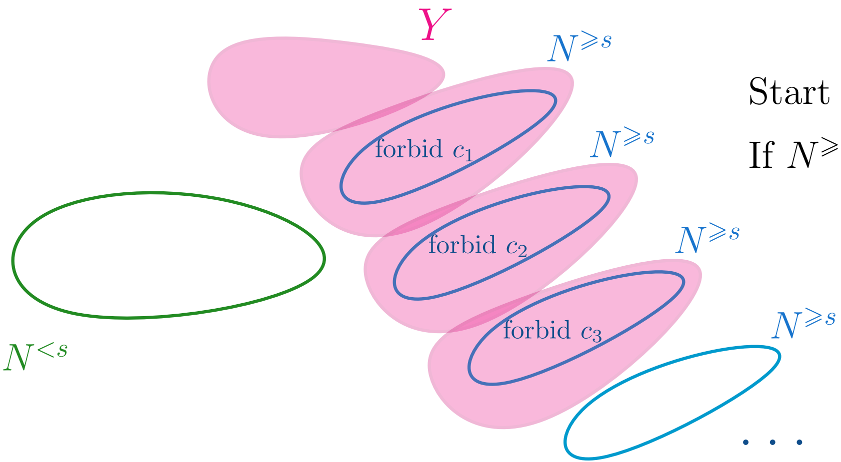

Start with $Y = \{v\}$ and any color

If $N^{\geqslant s}(Y) = \emptyset$ color some $z \in N^{<s}(Y)$

Otherwise:

Invariant preserved

Size of new Y controlled

Monochromatic components in Y controlled

$\forall_{s,t,\omega} \exists_{\eta=\eta(s,t,\omega)} \forall_G$

- $\mathrm{tw}(G) \leqslant \omega$ (:= no tangle of order $\omega + 2$)
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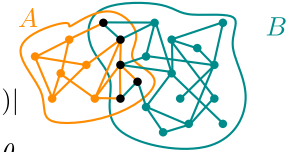

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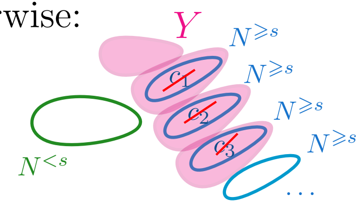

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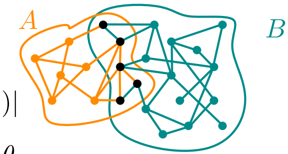

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We start with $|L(v)| = s + 1$

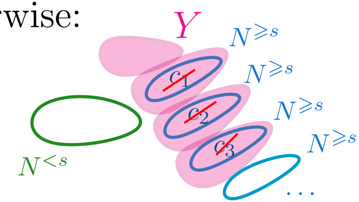
Iteratively we enlarge colored set Y $\boxed{\text{(until not too big)}}$

$\mathcal{T}_\theta := \{(A, B)_\theta : |V(A) \cap Y| \leqslant 3\theta\}$ $|Y| > 9\theta$ $\theta := \omega + 2$

Start with $Y = \{v\}$ and any color

If $N^{\geqslant s}(Y) = \emptyset$ color some $z \in N^{<s}(Y)$

Otherwise:



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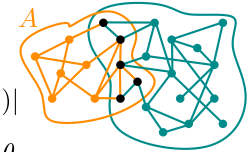

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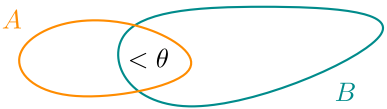
Iteratively we enlarge colored set Y (until not too big)

$\mathcal{T}_\theta := \{(A, B)_\theta : |V(A) \cap Y| \leqslant 3\theta\}$ $|Y| > 9\theta$ $\theta := \omega + 2$

is not a tangle!

But:



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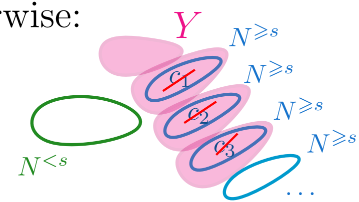

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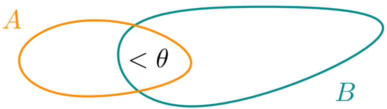

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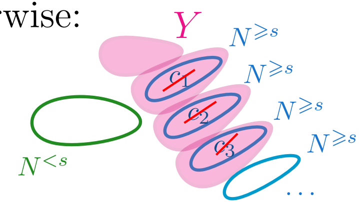

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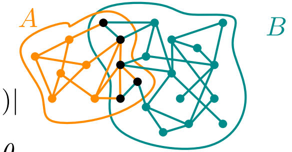

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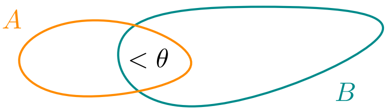

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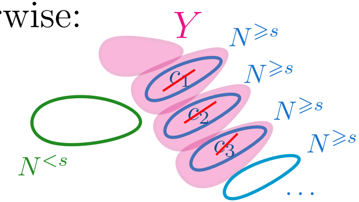

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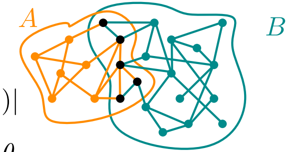

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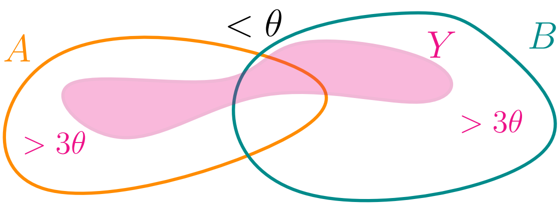

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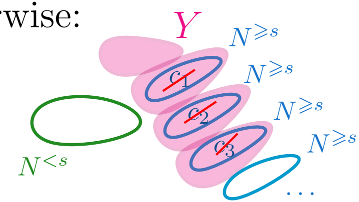

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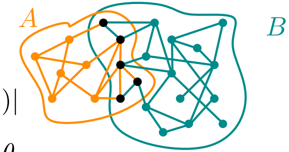

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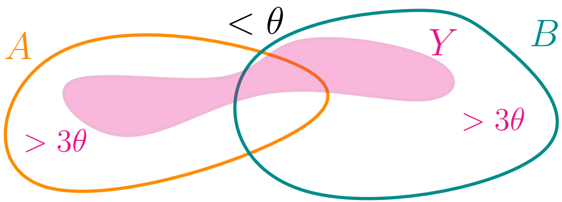

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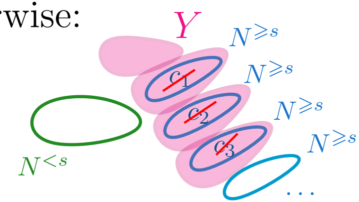

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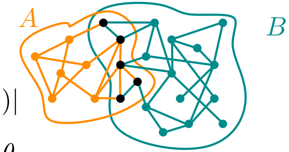

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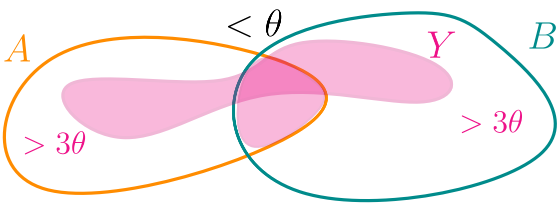

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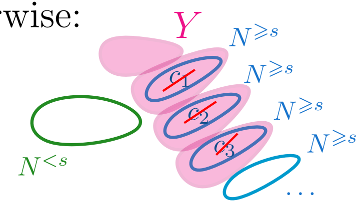

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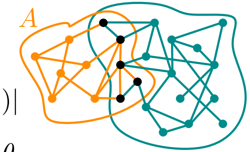

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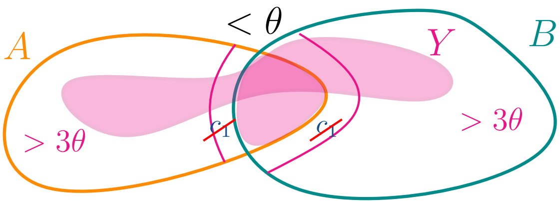

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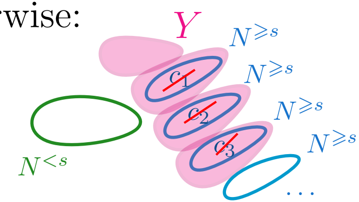

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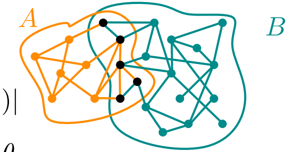

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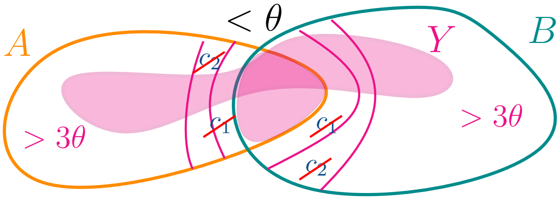

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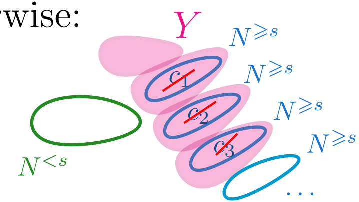

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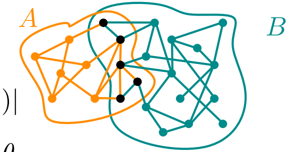

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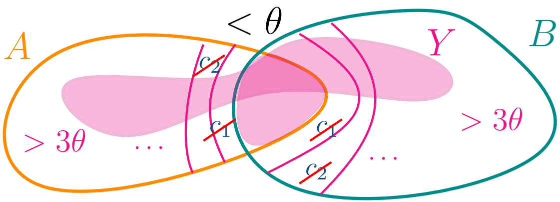

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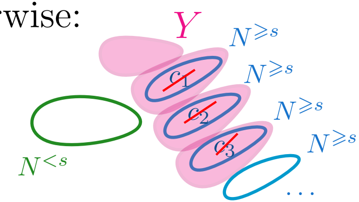

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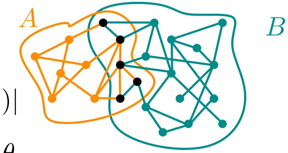

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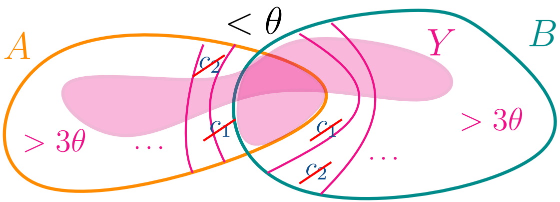

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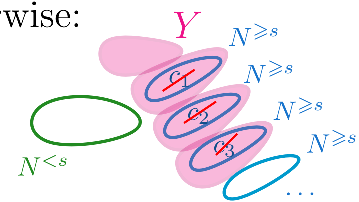

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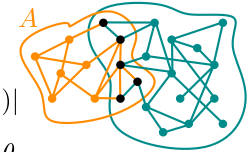

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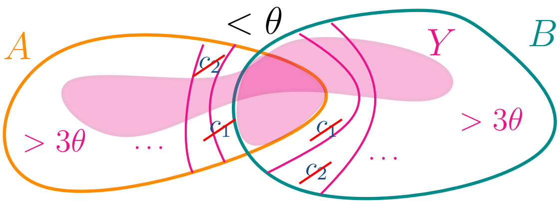

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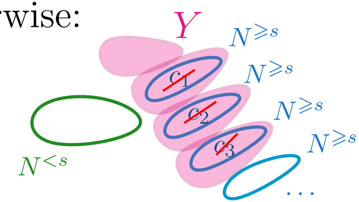

Monochromatic components can't glue!

Hadwiger's clustered conjecture:

$\mathcal{G}_s = K_{s+1}$ minor free graphs

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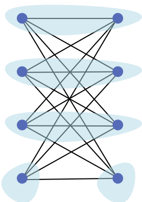

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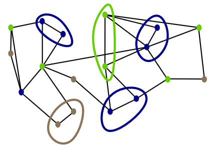

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Technical statement: (optimal)

$$\forall_{s,t,\overset{\omega}{\cancel{H}}} \exists_{\eta=\eta(s,t,\overset{\omega}{\cancel{H}})} \forall_G$$

~~planar~~

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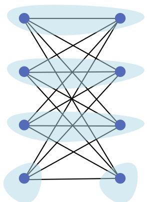

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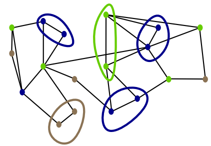

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- All tools we already used

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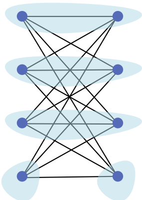
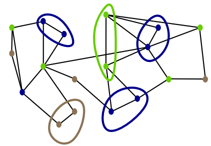


$K_{s,s}$, has K_{s+1} minor



We proved

Technical statement: (optimal)

$$\forall_{s,t,\overset{\omega}{\cancel{H}}}\ \exists_{\eta=\eta(s,t,\overset{\omega}{\cancel{H}})}\ \forall_G$$

~~planar~~

$\mathrm{tw}(G) \leqslant \omega$

~~G has no H-minor~~

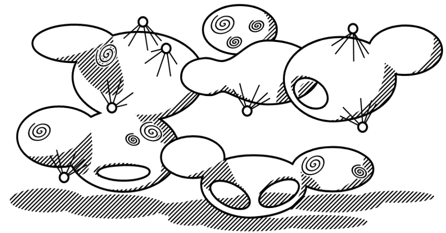
G has no $K_{s,t}$-subgraph

$(s+1)$ choosable

$\Rightarrow G$ is $\cancel{(s+2)}$-~~colorable~~ with clustering η

How to do the general case?

- All tools we already used
- Graph structure theorem



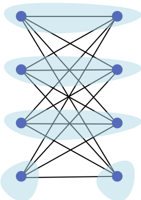
source: Felix Reidl's website

Hadwiger's clustered conjecture:

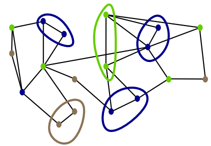
$\mathcal{G}_s = K_{s+1}$ minor free graphs

$\chi_c(\mathcal{G}_s) \leqslant s$

$$\boxed{\chi_c(\mathcal{G}_s) \leqslant s+2}$$



$K_{s,s,}$ has K_{s+1} minor



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G has no $K_{s,t}$-subgraph

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How to do the general case?

- All tools we already used
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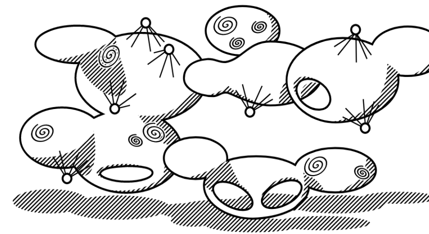

source: Felix Reidl's website

- Results from $\sim$100 pages long companion paper

Claim 21.58.

$$(Y^{(i,-1,k+1)} - Y^{(i,-1,k)}) \cap X_{V(T_t)} \subseteq N_G[W_0^{(i,-1,k)}] \cap X_{V(T_t)} \subseteq N_G[\bigcup_{j'=1}^{|\mathcal{V}|-1} \bigcup_{S \in S^\circ_{j'}} S] \subseteq \bigcup_{j'=1}^{|\mathcal{V}|-1} \overline{I^\circ_{j'}} \subseteq \bigcup_{j'=1}^{|\mathcal{V}|-1} I_{j'}.$$

So for every $k \in [0, w_0 - 1]$ and $q \in [0, s+1]$,

$$\begin{aligned}(Y^{(i,-1,k,q+1)} - Y^{(i,-1,k,q)}) \cap I_j \cap X_{V(T_t)} - X_t &\subseteq A_{L^{(i,-1,k,1)}}(Y_1^{(i,-1,k,q)} \cap \overline{I^\circ_j}) \cap I_j \cap X_{V(T_t)} - X_t \\ &\subseteq N_G^{\geqslant s}(Y^{(i,-1,k,q)} \cap \overline{I^\circ_j}) \cap I_j \cap X_{V(T_t)} - X_t \\ &\subseteq N_G^{\geqslant s}(Y^{(i,-1,k,q)} \cap \overline{I^\circ_j} \cap X_{V(T_t)}).\end{aligned}$$

Hence for every $k \in [0, w_0 - 1]$ and $q \in [0, s+1]$, $|(Y^{(i,-1,k,q+1)} - Y^{(i,-1,k,q)}) \cap I_j \cap X_{V(T_t)} - X_t| \leqslant |N_G^{\geqslant s}(Y^{(i,-1,k,q)} \cap \overline{I^\circ_j} \cap X_{V(T_t)})| \leqslant f(|Y^{(i,-1,k,q)} \cap I_j \cap X_{V(T_t)}|)$, so

$$\begin{aligned}|(Y^{(i,-1,k,q+1)} - Y^{(i,-1,k,q)}) \cap I_j \cap X_{V(T_t)}| &\leqslant |(Y^{(i,-1,k,q+1)} - Y^{(i,-1,k,q)}) \cap I_j \cap X_{V(T_t)} - X_t| + |X_t \cap I_j| \\ &\leqslant f(|Y^{(i,-1,k,q)} \cap I_j \cap X_{V(T_t)}|) + w_0.\end{aligned}$$

So

$$\begin{aligned}|Y^{(i,-1,k,q+1)} \cap I_j \cap X_{V(T_t)}| &\leqslant |(Y^{(i,-1,k,q+1)} - Y^{(i,-1,k,q)}) \cap I_j \cap X_{V(T_t)}| + |Y^{(i,-1,k,q)} \cap I_j \cap X_{V(T_t)}| \\ &\leqslant f(|Y^{(i,-1,k,q)} \cap I_j \cap X_{V(T_t)}|) + w_0 + |Y^{(i,-1,k,q)} \cap I_j \cap X_{V(T_t)}| \\ &= f_1(|Y^{(i,-1,k,q)} \cap I_j \cap X_{V(T_t)}|) + w_0.\end{aligned}$$