On List-Coloring Outerplanar Graphs

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Agenda

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2. LIST-COLORING BIPARTITE OUTERPLANAR GRAPHS
3. LIST-COLORING GENERAL OUTERPLANAR GRAPHS
4. SUMMARY

INTRODUCTION

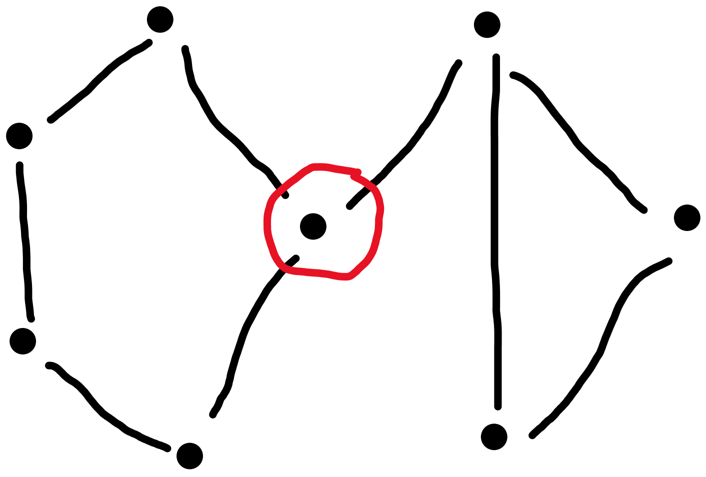
List-coloring definitions

Suppose each vertex v of a graph G has list of colors L(v). Then G is said to be **L-list-colorable** when it there exists a coloring in which each vertex v has assigned to a color from its list L(v). Adjacent vertices cannot get the same color (as in usual coloring).

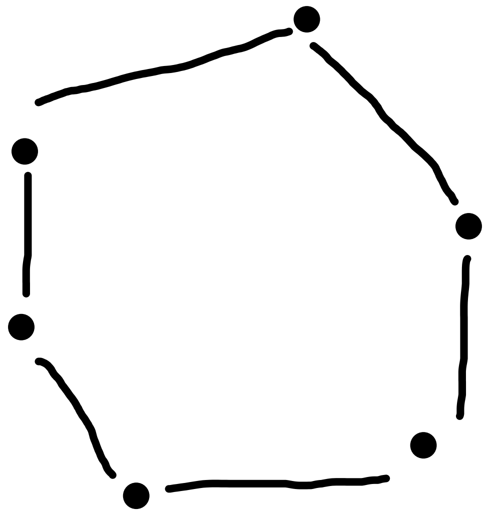
If each list of L has at least k colors and G is L-list-colorable, then G is said to be **k-list-colorable** or **k-choosable**. The minimum k for which G is k-choosable is called its **choice number** or **list-chromatic number.**

Connectivity definitions

Graph is called **2-connected** if it contains no vertex whose removal leaves a graph disconnected.



1-connected graph

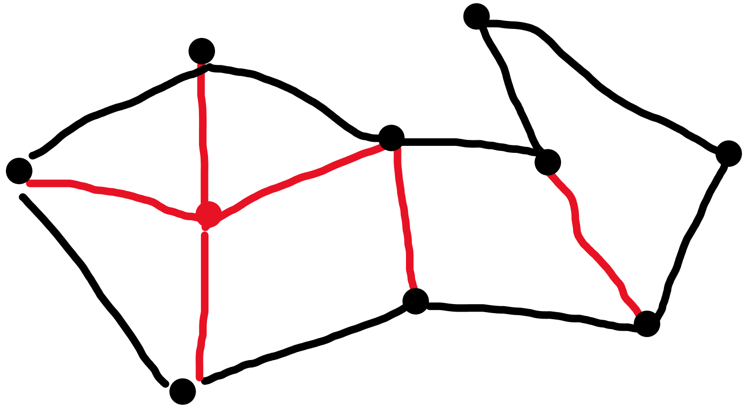


2-connected graph

Planarity definitions

In every plane graph there is exactly one unbounded face called **infinite** or **external face**. All others are **finite** or **internal faces**.

An edge/a vertex is called **external** (respectively, **internal**) when it lies (resp., does not lie) on the boundary of the external face.



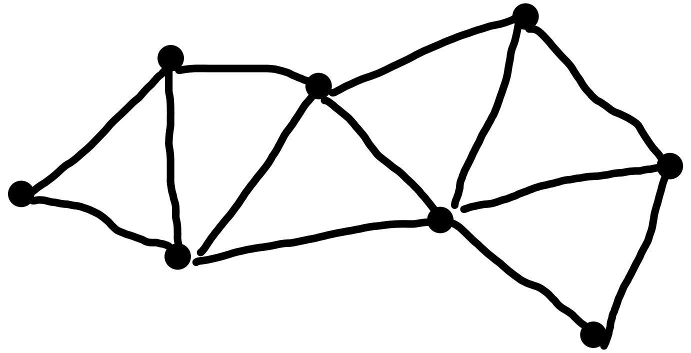
Example of planar graph

- Red edges/vertices are internal
- Black edges/vertices are external

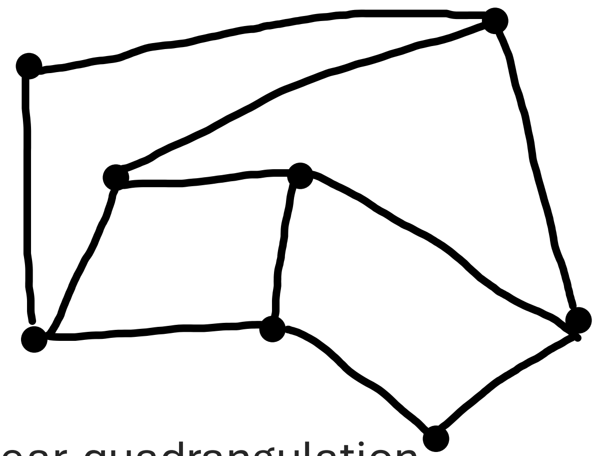
External face is bouded by black edges.
Other faces are internal.

Planarity definitions

A **near-triangulation** (respectively, **near-quadrangulation**) is a plane graph in which all finite faces are triangles (resp., quadrilaterals).



Near-triangulation



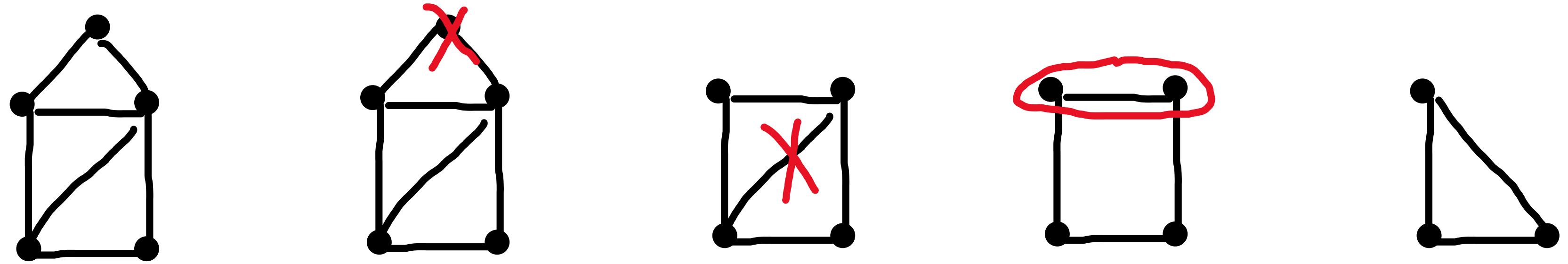
Near-quadrangulation

Complete graphs notation

K_n - complete graph on n vertices

$K_{m,n}$ - complete bipartite graph on m + n vertices

A graph is said to be **K_n-minor-free** if it cannot be transformed into K_n by deleting vertices and edges and by contracting pairs of adjacent vertices.

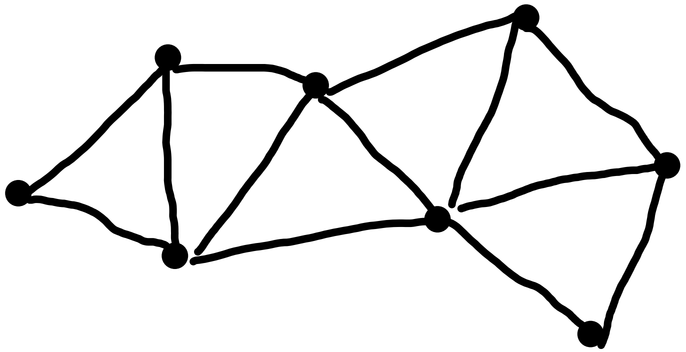


Vertex deletion | Edge deletion | Contraction | K_3

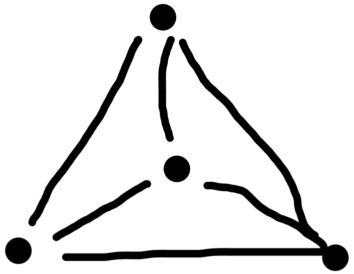
Outerplanarity definitions

A graph is called **outerplanar** if all vertices are external.

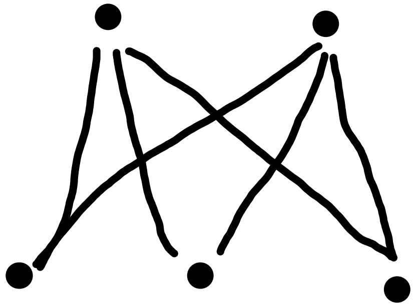
Known fact: Graphs is outerplanar iff is K_4-minor-free and $K_{2,3}$–minor-free.



Outerplanar graph



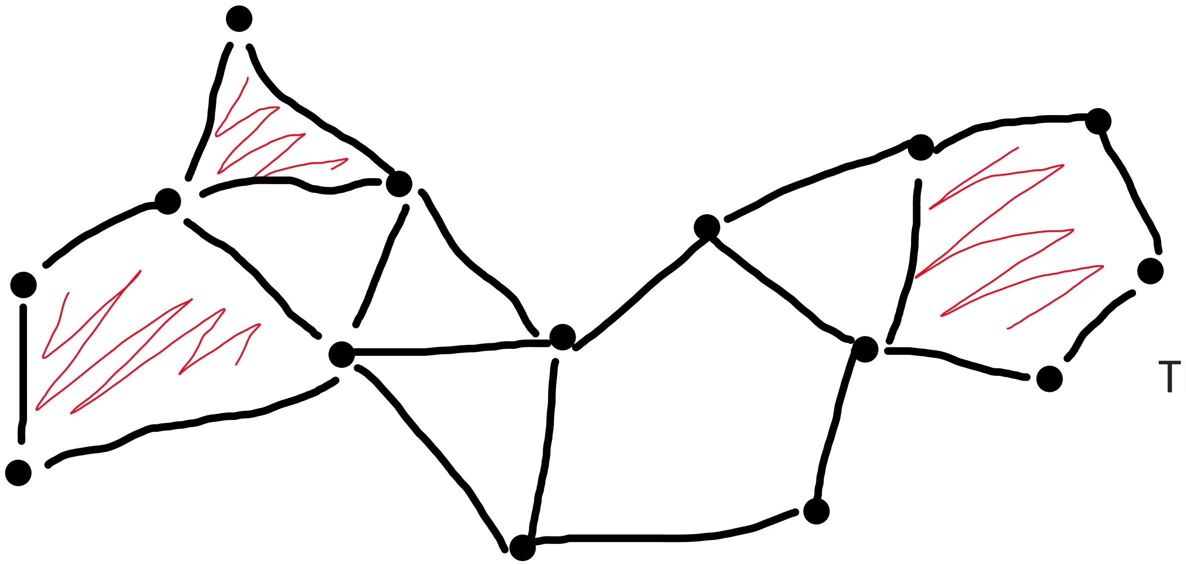
K_4 - the smallest planar graph that is not outerplanar



$K_{2,3}$ – not outerplanar graph

Outerplanarity definitions

An **ear** is a finite face incident along an edge with just one other finite face of the graph.



The red faces are ears

LIST-COLORING BIPARTITE OUTERPLANAR GRAPHS

Main Theorem for bipartite outerplanar

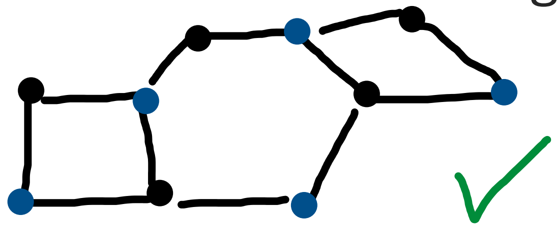
Theorem 1.

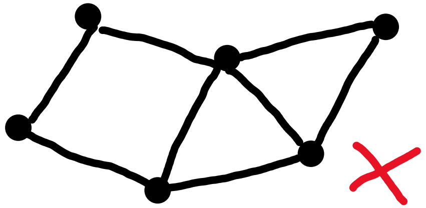
Let G be a 2-connected, outerplanar bipartite graph. If $|L(v)| \geq \min\{deg(v), 4\}$ for each vertex v, then G is L-list-colorable.

Sketch of the proof theorem 1

Observations:

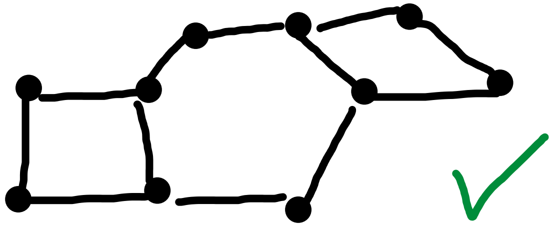
- Each internal face of the graph is limited by a cycle of even length

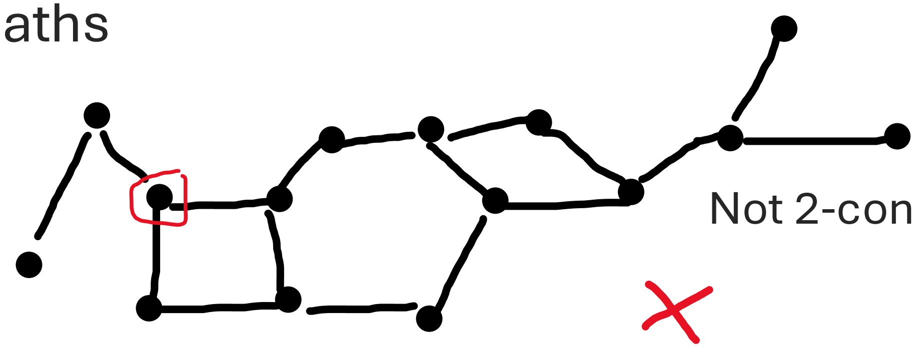




Not bipartite

- Graph consists only of cycles, no paths



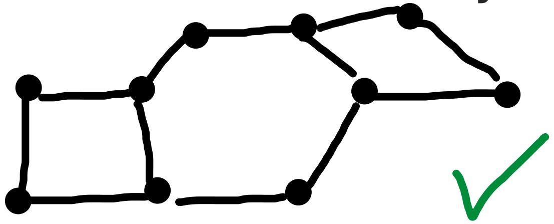


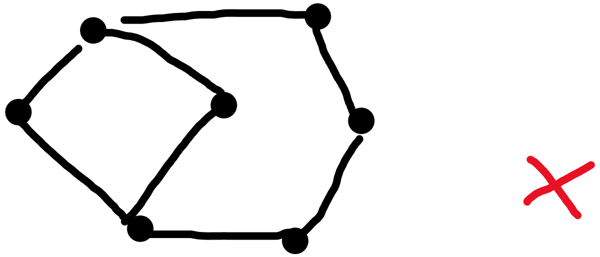
Not 2-connected

Sketch of the proof theorem 1

Observations:

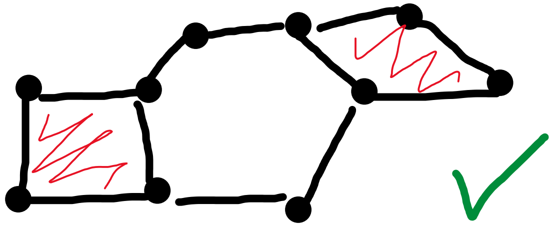
- Adjacent faces has only one edge in common

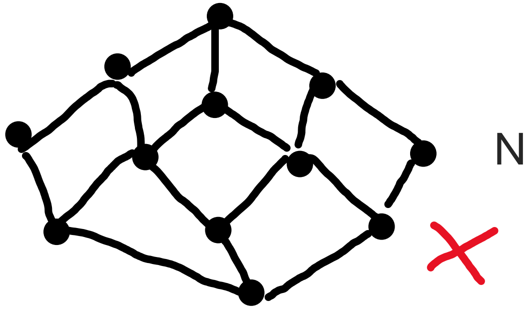




Not outerplanar

- Graph contains an ear

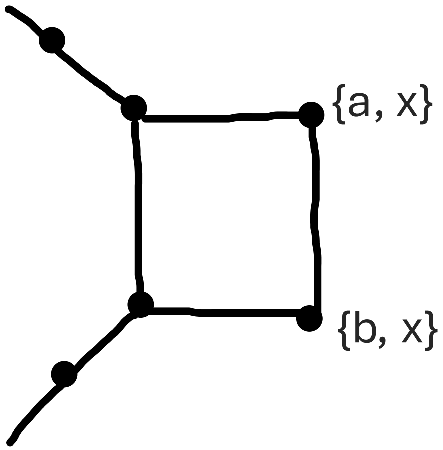


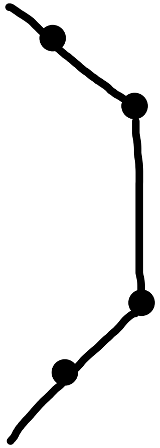


Not outerplanar

Sketch of the proof theorem 1

The ear of the graph can be reduced by adding **forbidden coloring** to the internal edge. **Forbidden coloring** - f(u, v) = (c, d) means that c (respectively, d) is a color forbidden at u (resp., v).





Forbidden coloring for edge:
f(e) = (a, b)

Sketch of the proof theorem 1

Proof by induction on number of internal faces |f|

- If |f| = 1

Even-length cycle

- If |f| > 2

G contains an ear, which is a 2i-sided face, $i \geq 2$, $(v_1, v_2, \ldots, v_{2i})$. Remove the path of vertices $P = v_2, v_3, \ldots, v_{2i-1}$ (of at least two vertices) to form G' with forbidden coloring and invoke the inductive assumption.

Theorem 1 – all conditions are needed

Theorem 1.

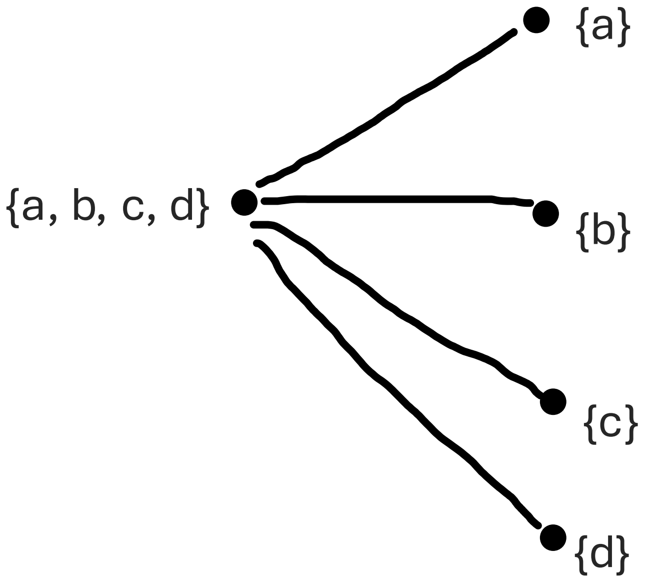
Let G be a

- 2-connected,
- outerplanar,
- bipartite graph,
- $|L(v)| \geq \min\{\deg(v), 4\}$ for each vertex v

then G is L-list-colorable.

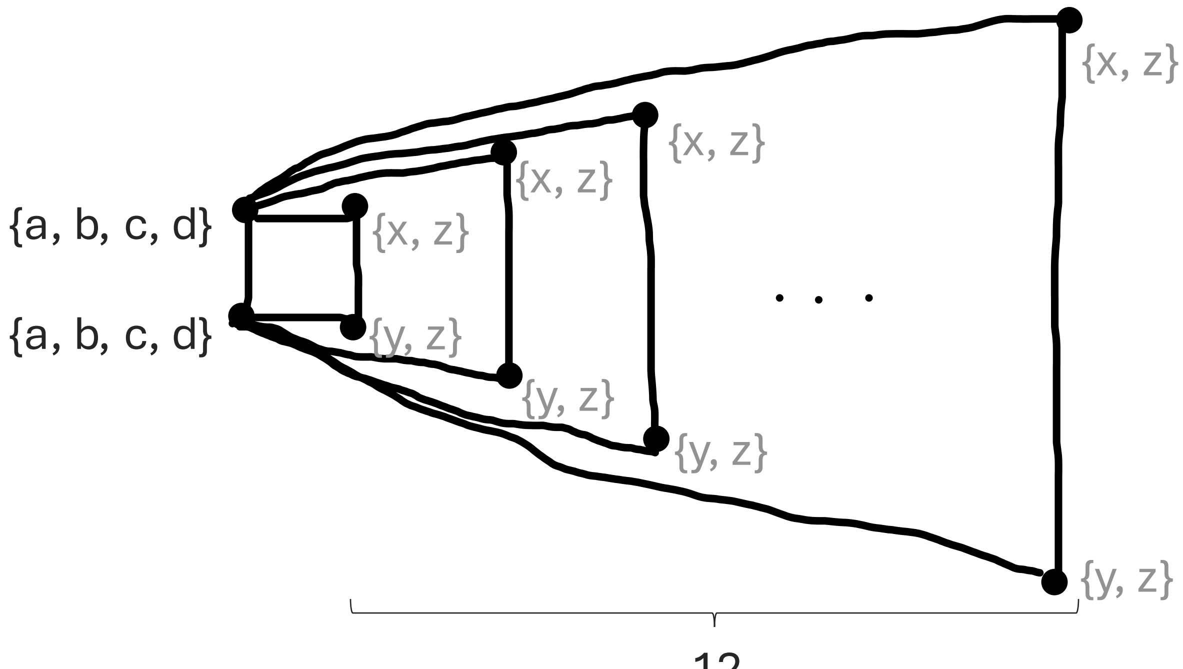
Theorem 1
"2-connected" condition

Example of 1-connected, outerplanar bipartite graph with $|L(v)| \geq \min\{deg(v), 4\}$.



Theorem 1
"outerplanar" condition

Example of 2-connected, K_4-minor-free, bipartite graph with $|L(v)| \geq \min\{deg(v), 4\}$.

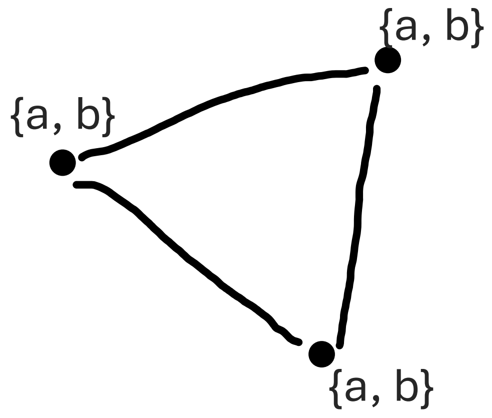


Twelve 4-cycles C_4 with common edge

x and y ($x \neq y$) varies over all 12 ordered pairs taken from {a, b, c, d}.

Theorem 1
"bipartite" condition

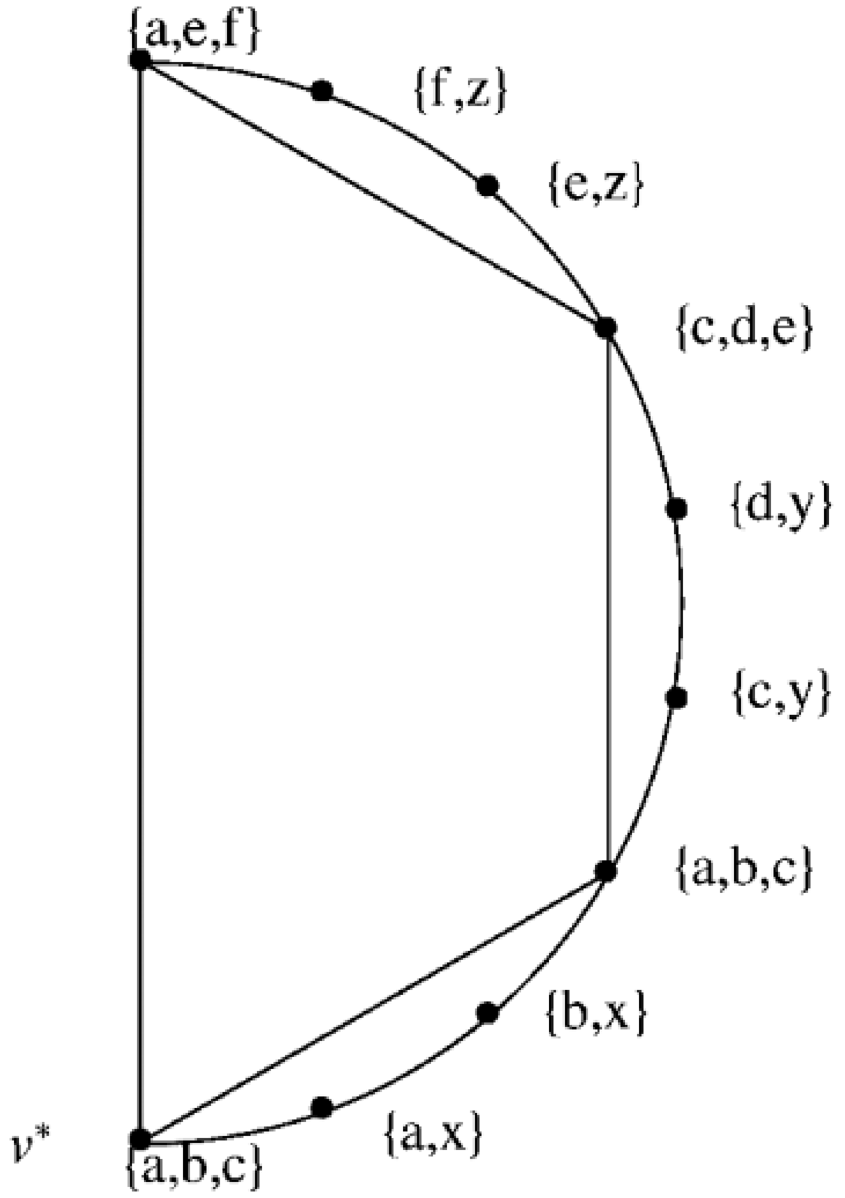
Example of 2-connected, outerplanar graph with $|L(v)| \geq \min\{deg(v), 4\}$.



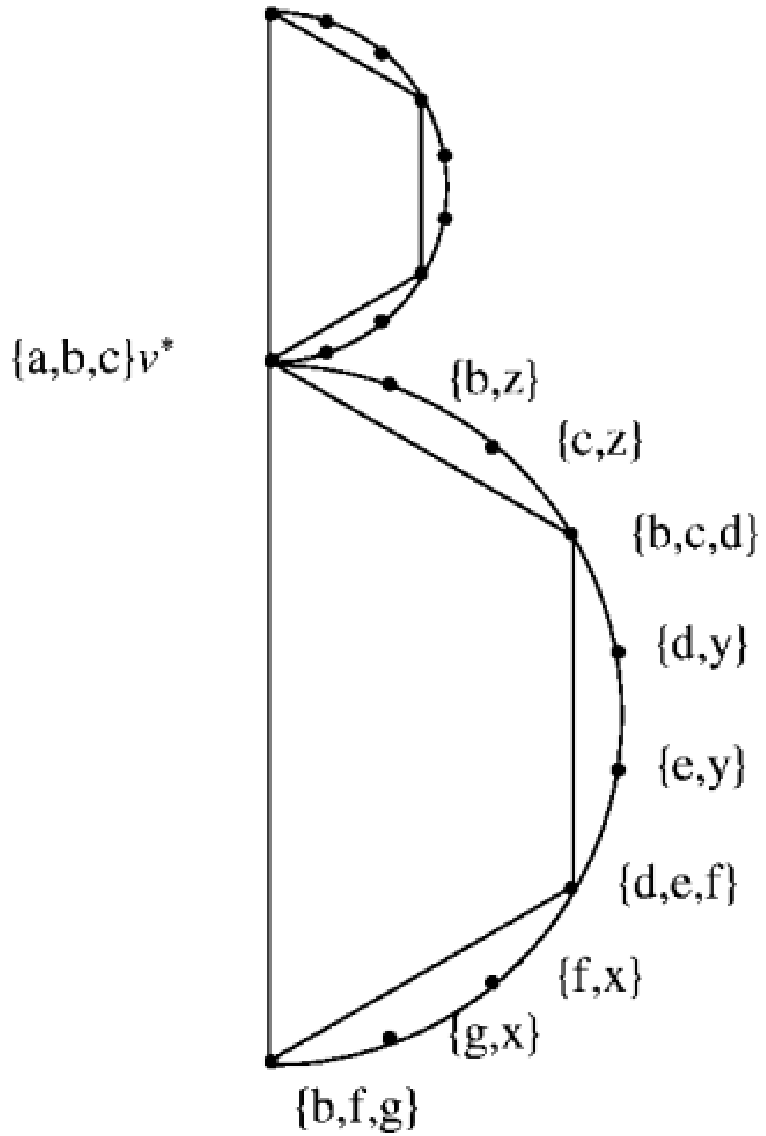
Theorem 1
"$|L(v)| \geq \min\{deg(v), 4\}$" condition

Example of 2-connected, outerplanar, bipartite graph with $|L(v)| = \min\{deg(v), 3\}$.

Pictures by Joan P. Hutchinson

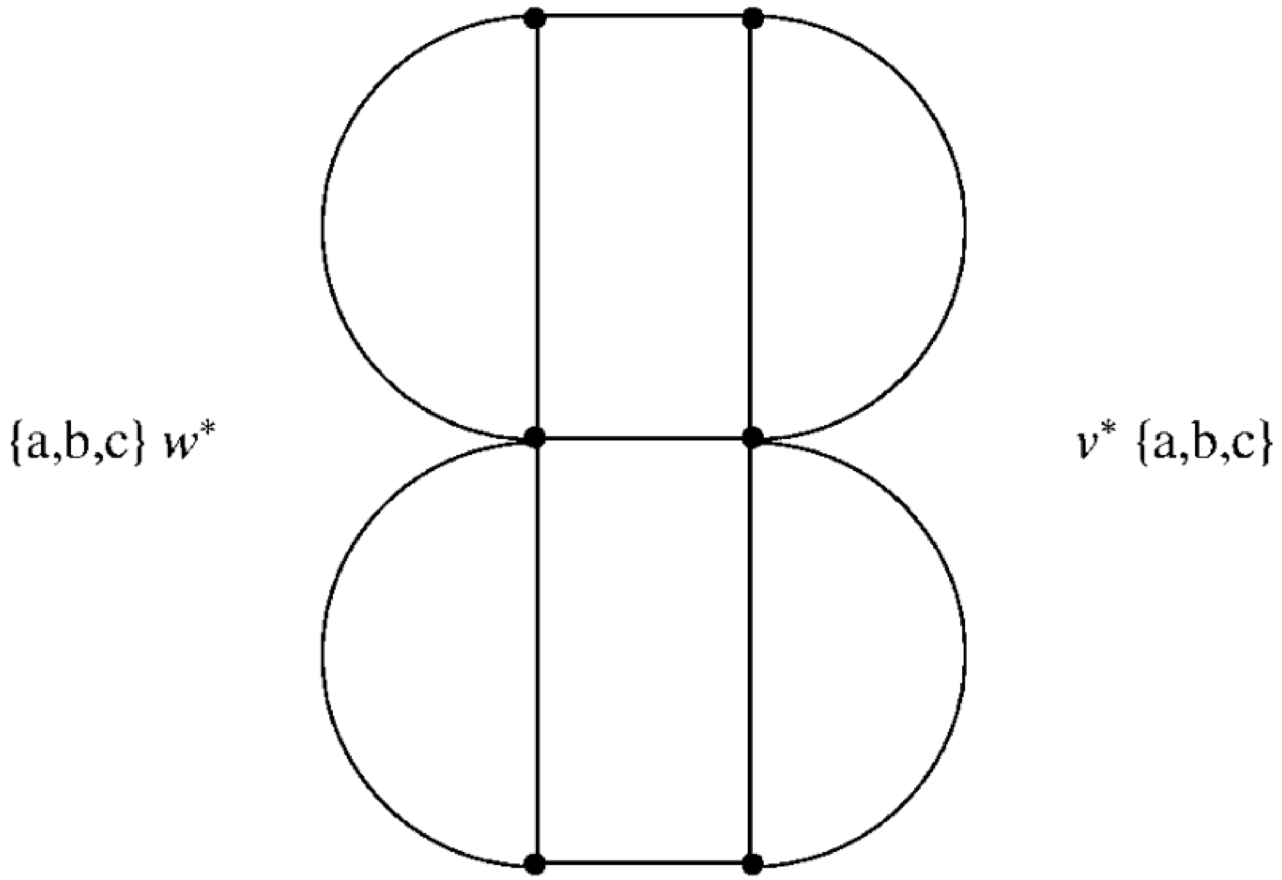


v* cannot be colored by "a"



v* cannot be colored also by "b"

Picture by Joan P. Hutchinson



Both v* and w* should be in "c" color

LIST-COLORING GENERAL OUTERPLANAR GRAPHS

Main Theorem for general outerplanar

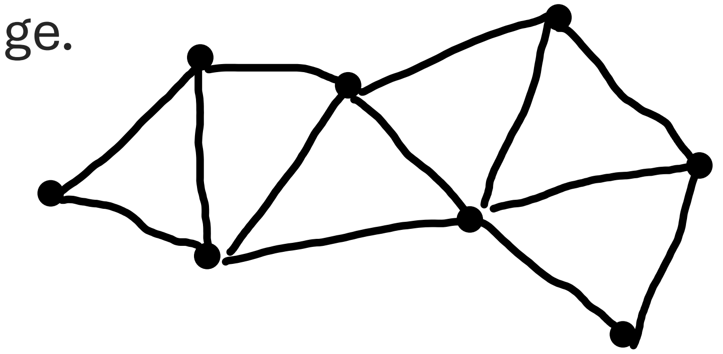
Theorem 3

If the graph is a 2-connected, outerplanar near-triangulation (or equivalently, edge-maximal outerplanar) and satisfies $|L(v)| \geq \min\{deg(v), 5\}$ for every vertex v, then the graph is L-list-colorable except for K_3 with three identical 2-lists.

Sketch of the proof theorem 3

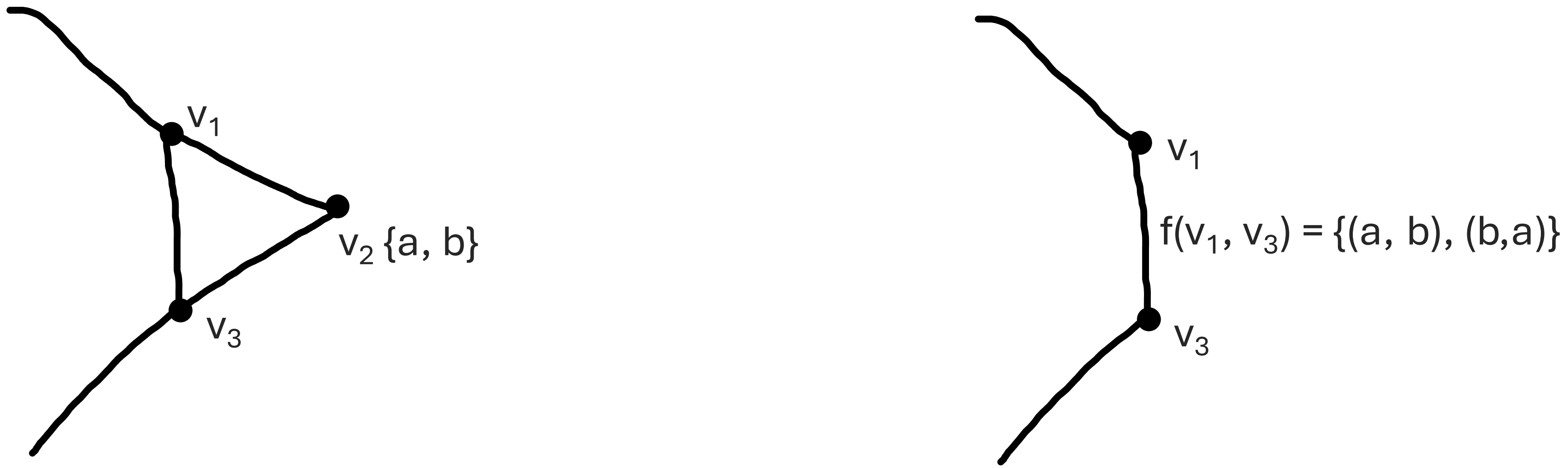
Observation:

- Each internal face of the graph is limited by a triangle.
- Adjacent triangles has only one common edge.
- Graph contains traingle ear.



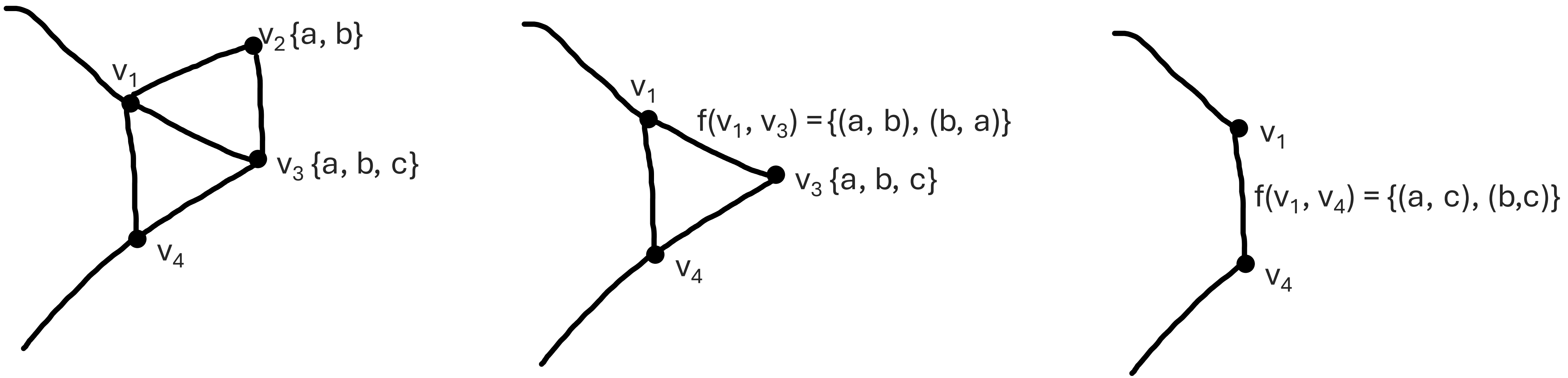
Sample graph which meets the assumptions of the Theorem 3

Sketch of the proof theorem 3



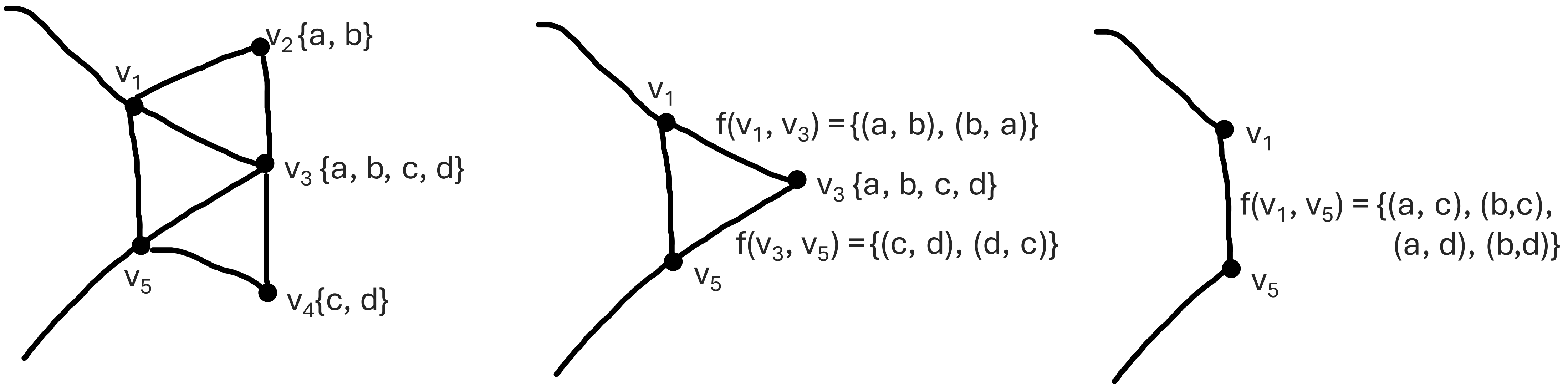
Triangle ear elliminaton

Sketch of the proof theorem 3



Two triangle ears elliminaton

Sketch of the proof theorem 3



Three triangle ears elliminaton

Sketch of the proof theorem 3

Proof by induction on number of vertices

- If |V| = 3

Triangle with possibly multiple forbidden colorings on edges.

- If |V| > 2

G contains an triangle ear. Elliminate ear, add fordidden colorings to new external edge and invoke inductive assumption.

Theorem 3 – all conditions are needed

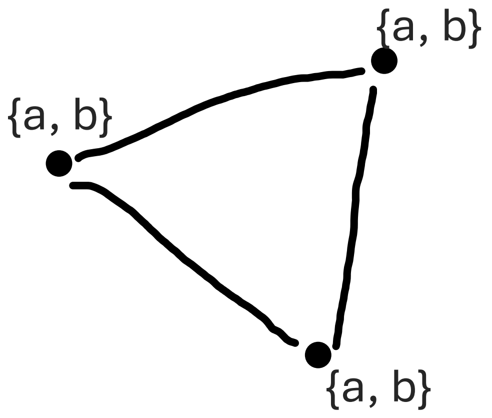
Theorem 3

If the graph is

- 2-connected,
- outerplanar,
- near-triangulation
- satisfies $|L(v)| \geq \min\{deg(v), 5\}$ for every vertex v,

then the graph is L-list-colorable.

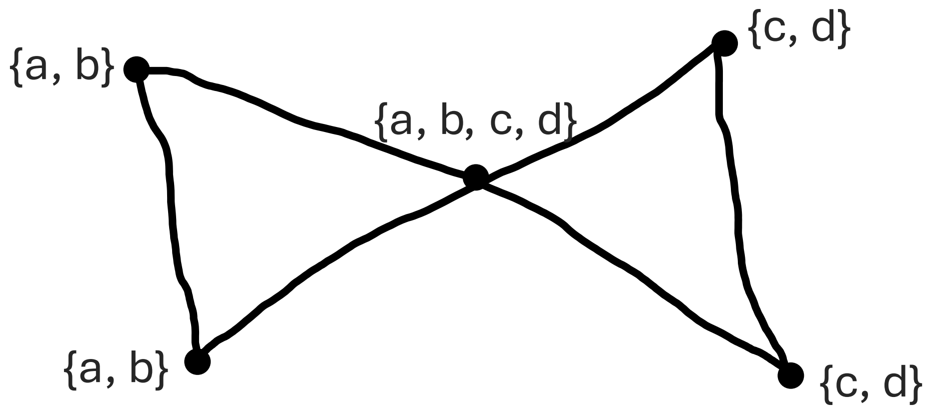
Exception - K_3 with three identical 2-lists.



Exception - K_3

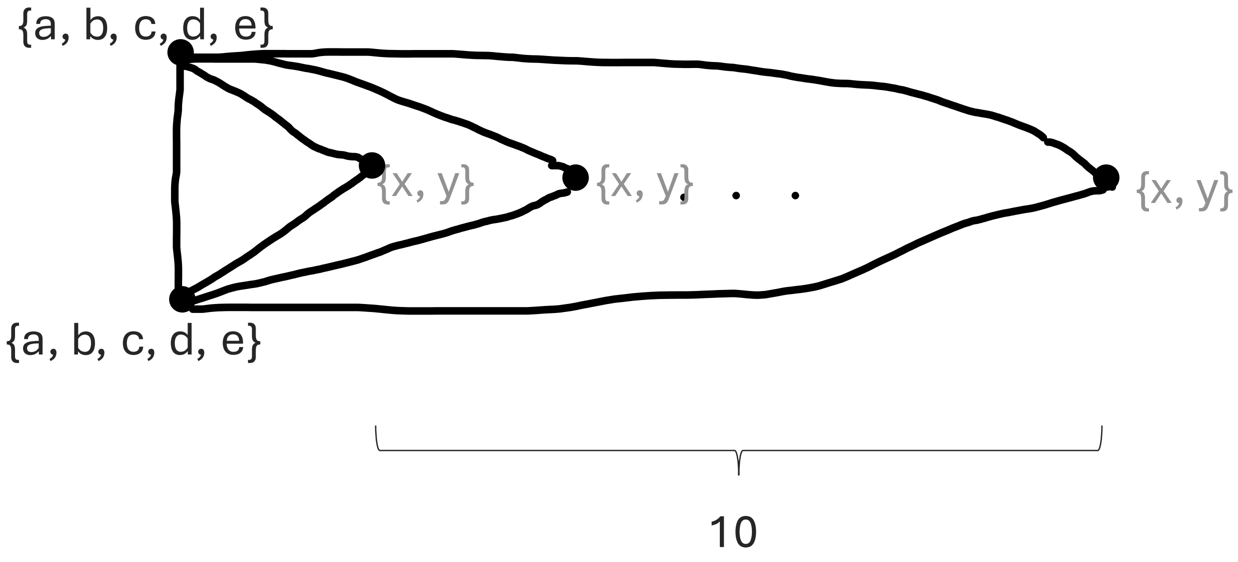
Theorem 3
"2-connected" condition

Example of 1-connected, outerplanar, near-triangulation graph with $|L(v)| \geq \min\{deg(v), 5\}$.



Theorem 3
"outerplanar" condition

Example of 2-connected, K_4-minor-free, near-triangulation graph with $|L(v)| \geq \min\{deg(v), 5\}$.

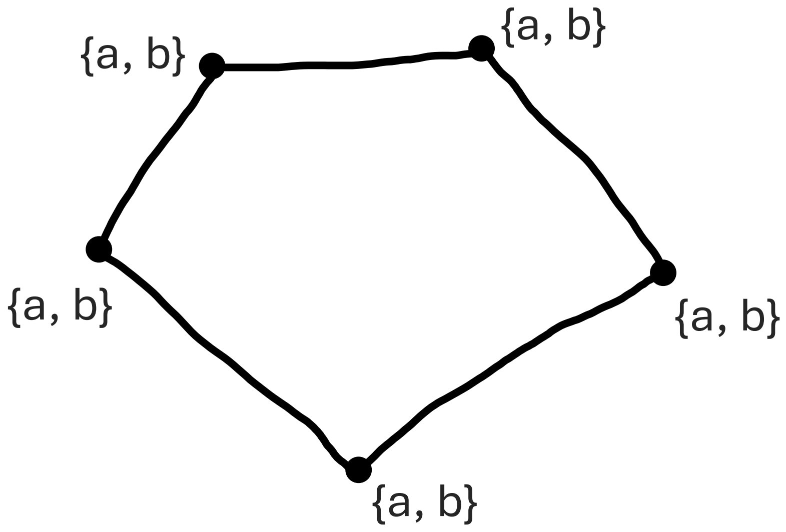


Ten 3-cycles C_3 with common edge

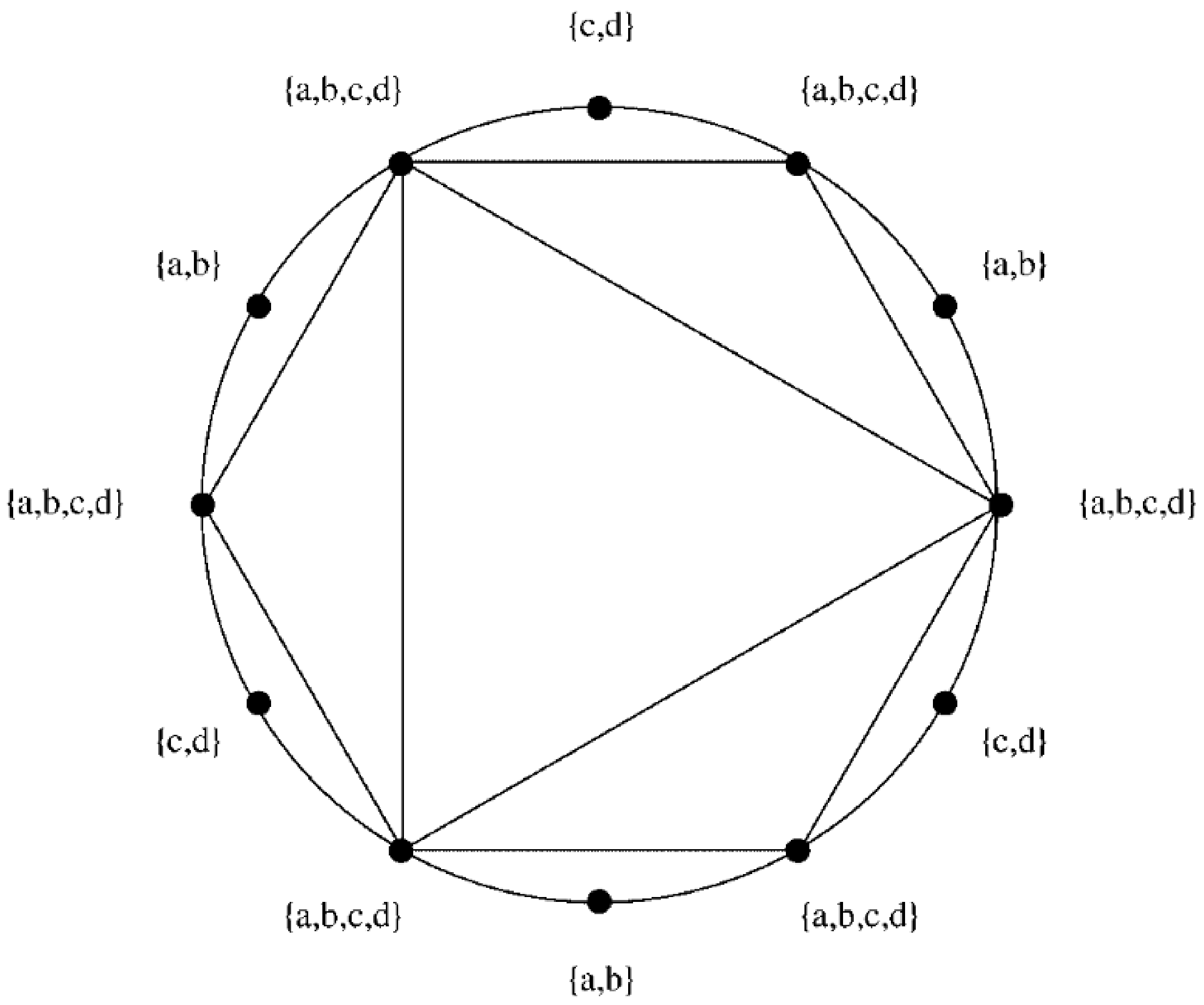
x and y ($x \neq y$) varies over all 10 unordered pairs taken from {a, b, c, d, e}.

Theorem 3
"near-triangulation" condition

Example of 2-connected, outerplanar graph with |L(v)| ≥ min{deg(v), 5}.



Picture by Joan P. Hutchinson



Theorem 3 "|L(v)| ≥ min{deg(v),5}" condition

Example of 2-connected, outerplanar, near-triangulation graph with $|L(v)| = \min\{\deg(v), 4\}$.

SUMMARY

References

Hutchinson, J.P. (2008), On list-coloring outerplanar graphs. J. Graph Theory, 59: 59-74. https://doi.org/10.1002/jgt.20326