2-List-coloring planar graphs without monochromatic triangles

AUTHOR: CARSTEN THOMASSEN

PRESENTER: PIOTR KALICIAK

List-coloring

Let *G* be a graph. Then for every *v in* G, let *L*(*v)* be a list of colors, that can be used to color *v*.

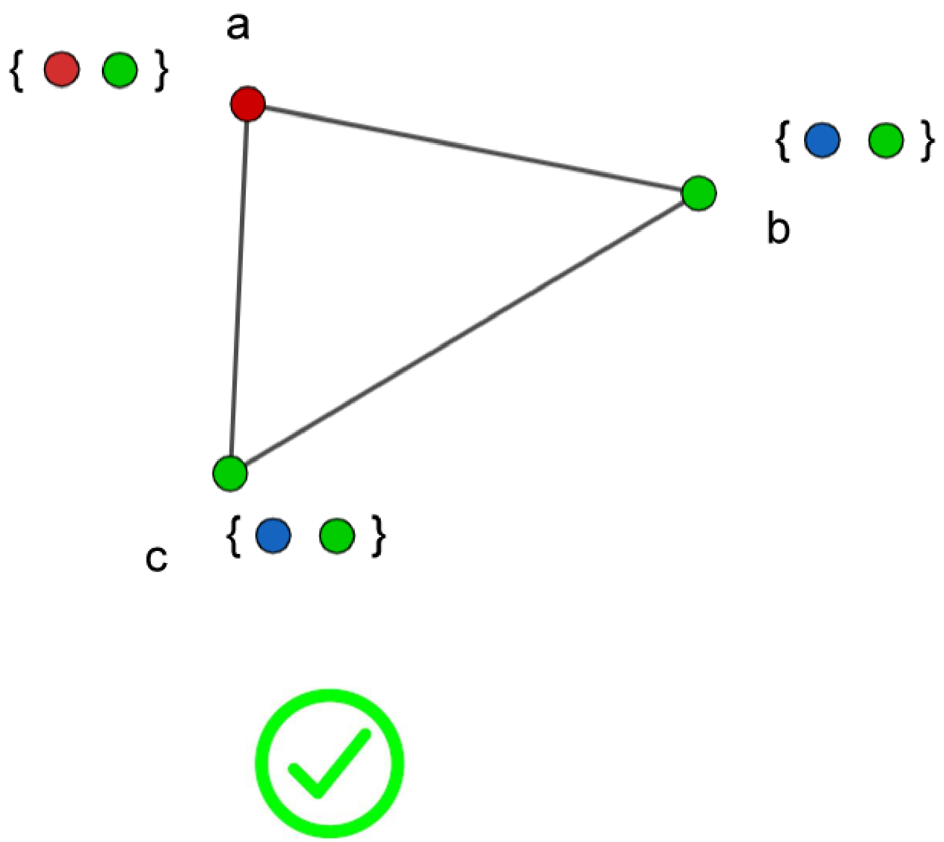
n-list-coloring means, that every list is of size *n*.

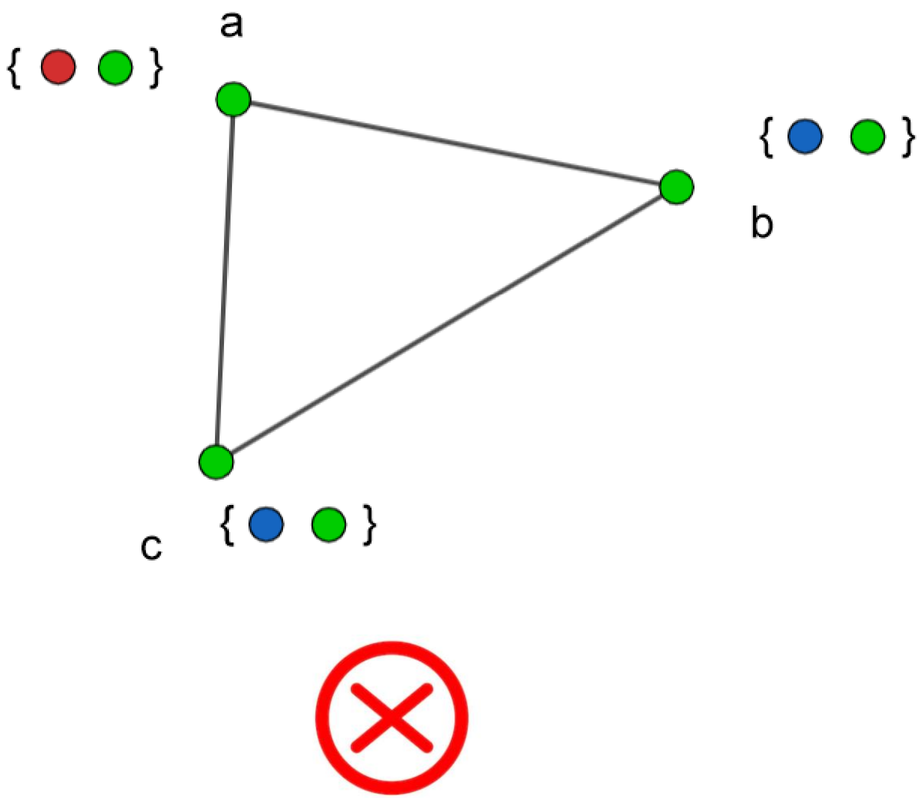
So in this problem, we consider a situation, where every vertex has list of 2 colors.

Unlike the most cases, we don’t mind having two vertices sharing a color.

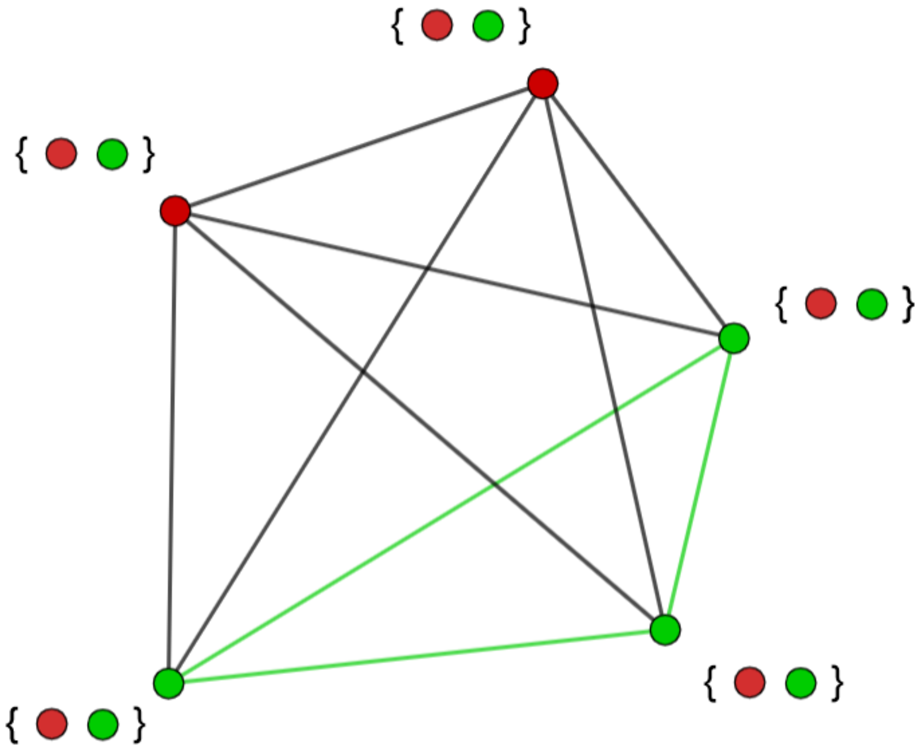
But coloring of a graph, where there are 3 vertices, adjacent to each other and having the same color is forbidden.

Forbidden cases





Not every graph can be colored this way, for example K_5:



But this graph isn't planar...

Motivations

1) The result implies the following conjecture of Mohar and Škrekovski (1999):
„If *G* is a planar graph such that every edge is in a triangle, and *L* is a list assignment such that each list has two colors, then there is a list-coloring such that no maximal complete subgraph is monochromatic”

2) It also implies the following conjecture of Kündgen and Ramamurthi (2002):
„If *G* is a planar, connected graph with at least one edge and *L* is a list assignment such that each list has two colors, then there is a list-coloring such that no face boundary is monochromatic”
(it sufficies to verify for triangulations)

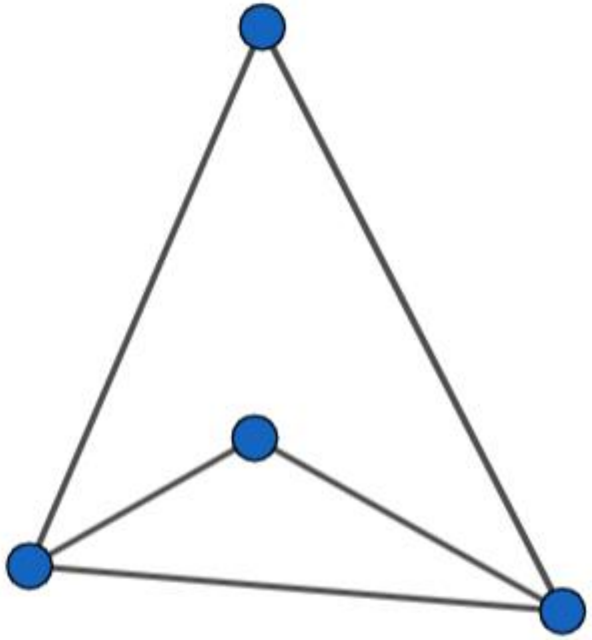
Planar graph vs plane graph

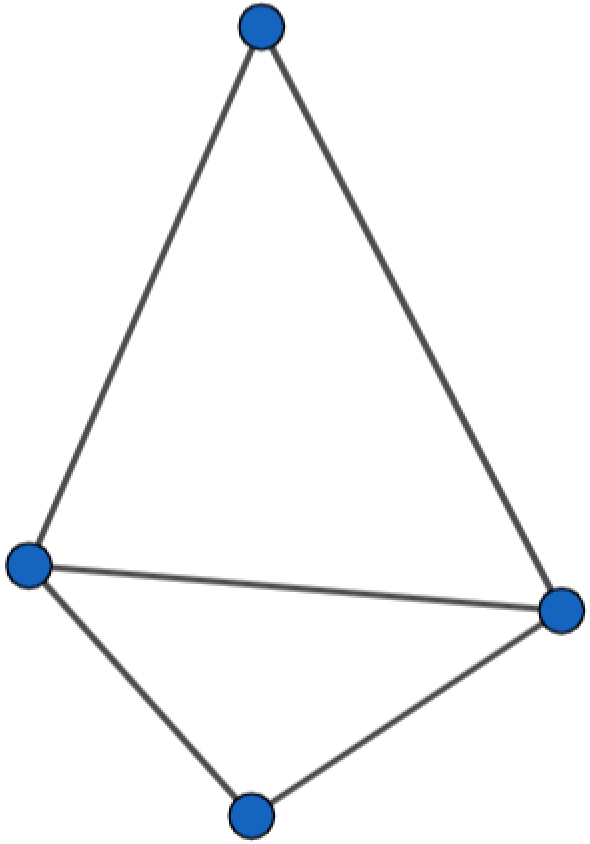
Graph is **planar** when it can be drawn on a plane in such a way that its edges intersect only at their endpoints.

A drawing of a planar graph is called a **plane** graph.

Planar graph vs plane graph

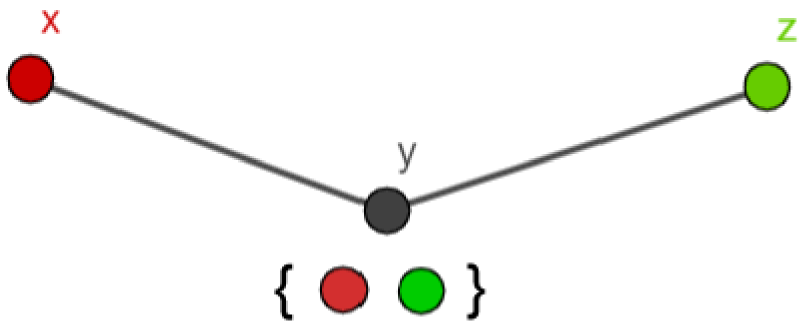
A single planar graph can be represented by different plane graphs:





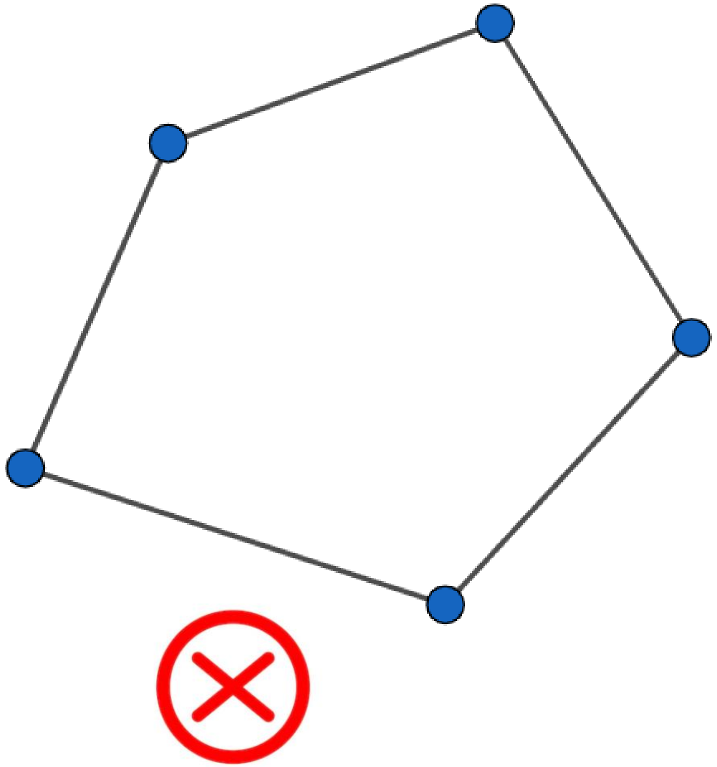
Bad 2-path

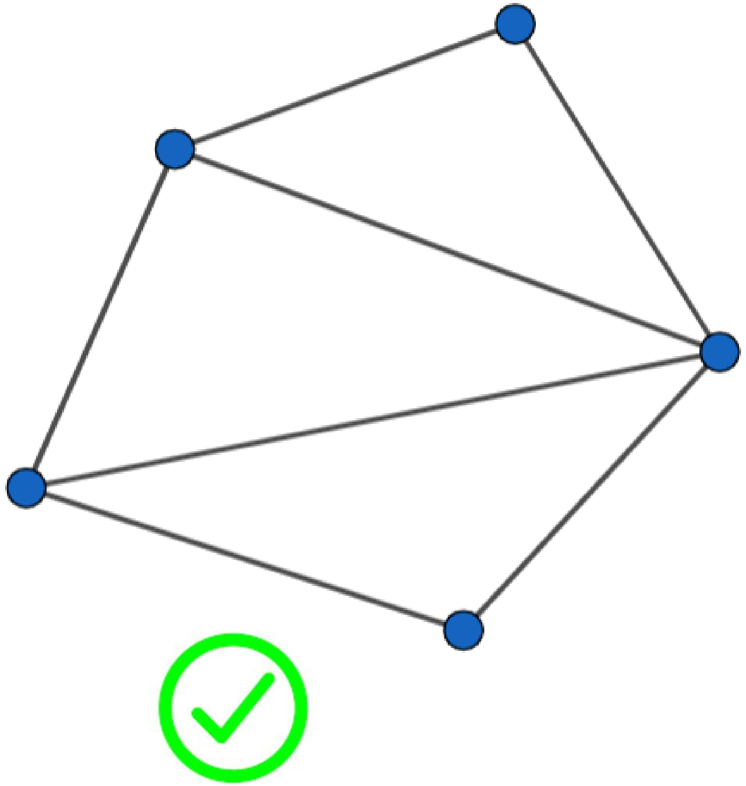
Let x,y,z be vertices of graph G. We call a path xyz a **bad 2-path** if x, z are precolored with distinct colors α, β and $L(y)=\{\alpha, \beta\}$.



Near-triangulation plane

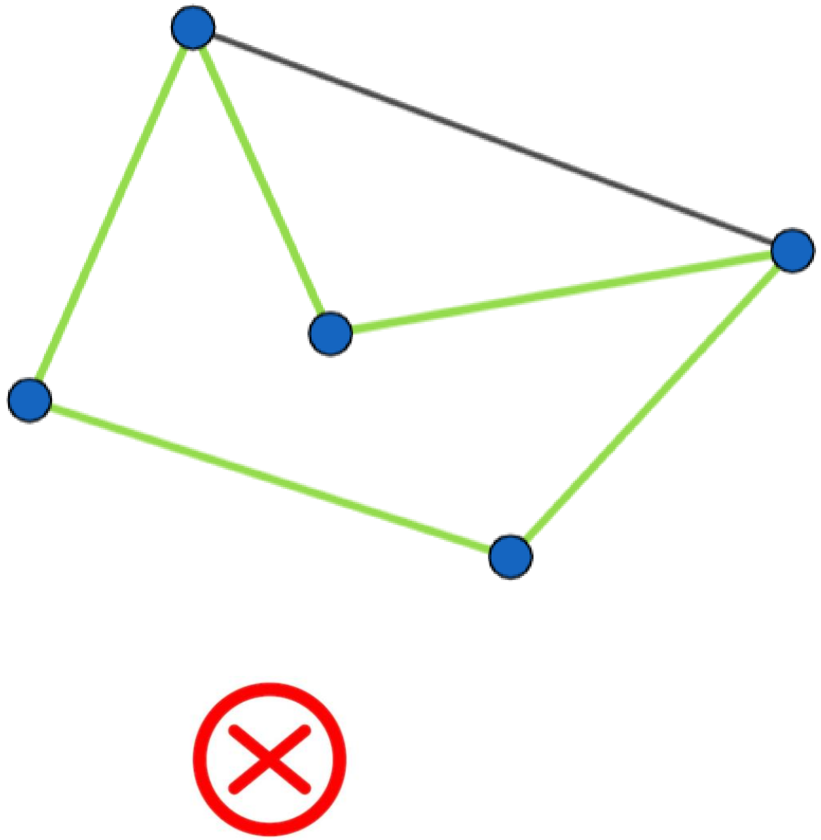
A plane graph is said to be a plane near-triangulation if all its faces, except possibly the exterior face, are triangles.

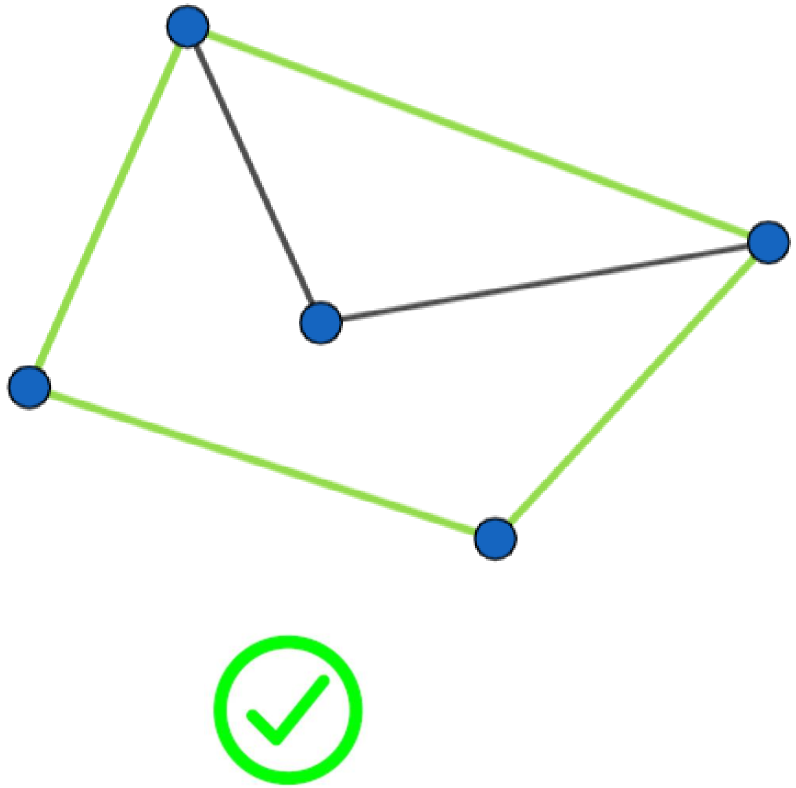




Outer cycle

A cycle is outer if all of its vertices belong to the outer face of the drawing.





Main theorem

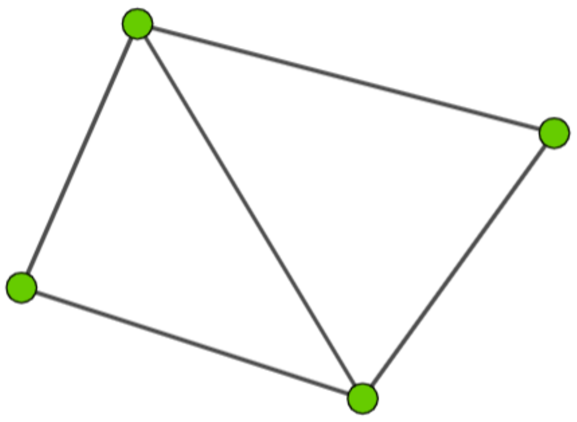
It sufficies to prove this result for near-triangulation planes.
For technical reasons the stronger result is proved.

Theorem 1

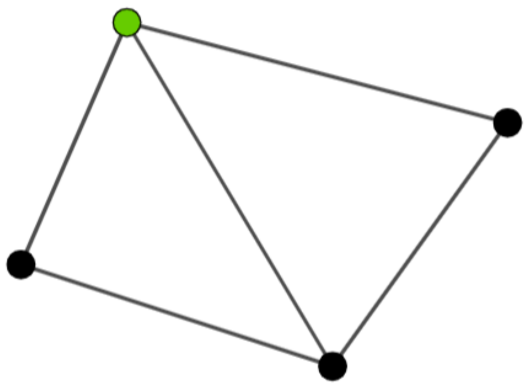
Let G be a plane near-triangulation with outer cycle $C : v_1v_2 \ldots v_kv_1$.

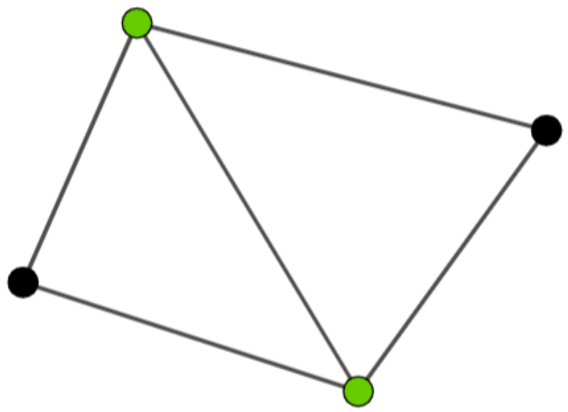
Theorem 1

Let *G* be a plane near-triangulation with outer cycle *C* : $v_1v_2 \ldots v_kv_1$.
Let *c* be a coloring of a nonempty vertex set *A* on *C* such that the subgraph *G(A)* induced by *A* either equals the outer cycle *C* or else the edges in *G(A)* (if any) induce a path which is denoted by *P*.



A=C





Theorem 1

Let G be a plane near-triangulation with outer cycle $C : v_1v_2 \dots v_kv_1$.
Let c be a coloring of a nonempty vertex set A on C such that the subgraph $G(A)$ induced by A either equals the outer cycle C or else the edges in $G(A)$ (if any) induce a path which is denoted by P.
Let A_i denote the set of vertices of degree i in the subgraph $G(A)$ for $i = 0, 1, 2$.

Theorem 1

Let G be a plane near-triangulation with outer cycle $C : v_1v_2 \ldots v_kv_1$.
Let c be a coloring of a nonempty vertex set A on C such that the subgraph $G(A)$ induced by A either equals the outer cycle C or else the edges in $G(A)$ (if any) induce a path which is denoted by P.
Let A_i denote the set of vertices of degree i in the subgraph $G(A)$ for $i = 0, 1, 2$.
For each vertex v in G, let $L(v)$ be a list of colors. If v is in A, then $L(v)$ consists of $c(v)$ only. Otherwise, $L(v)$ has precisely two colors.

Theorem 1

Let *G* be a plane near-triangulation with outer cycle *C* : $v_1 v_2 \ldots v_k v_1$.
Let *c* be a coloring of a nonempty vertex set *A* on *C* such that the subgraph *G(A)* induced by *A* either equals the outer cycle *C* or else the edges in *G(A)* (if any) induce a path which is denoted by *P*.
Let A_i denote the set of vertices of degree *i* in the subgraph *G(A)* for *i* = 0, 1, 2.
For each vertex *v* in *G*, let *L(v)* be a list of colors. If *v* is in *A*, then *L(v)* consists of *c(v)* only. Otherwise, *L(v)* has precisely two colors.
Assume that *G* has no bad 2-path with both ends in $A_0 \cup A_1$. (Such a path will be called a forbidden bad 2-path.)

Theorem 1

Let G be a plane near-triangulation with outer cycle $C : v_1v_2 \ldots v_kv_1$.
Let c be a coloring of a nonempty vertex set A on C such that the subgraph $G(A)$ induced by A either equals the outer cycle C or else the edges in $G(A)$ (if any) induce a path which is denoted by P.
Let A_i denote the set of vertices of degree i in the subgraph $G(A)$ for $i = 0, 1, 2$.
For each vertex v in G, let $L(v)$ be a list of colors. If v is in A, then $L(v)$ consists of $c(v)$ only. Otherwise, $L(v)$ has precisely two colors.
Assume that G has no bad 2-path with both ends in $A_0 \cup A_1$. (Such a path will be called a forbidden bad 2-path.)
Let q be the number of vertices of P, and let m be the number of monochromatic edges of P.
Assume that $q + m \leq 6$ if P is a path, and that $q + m \leq 5$ if $P = C$.

Theorem 1

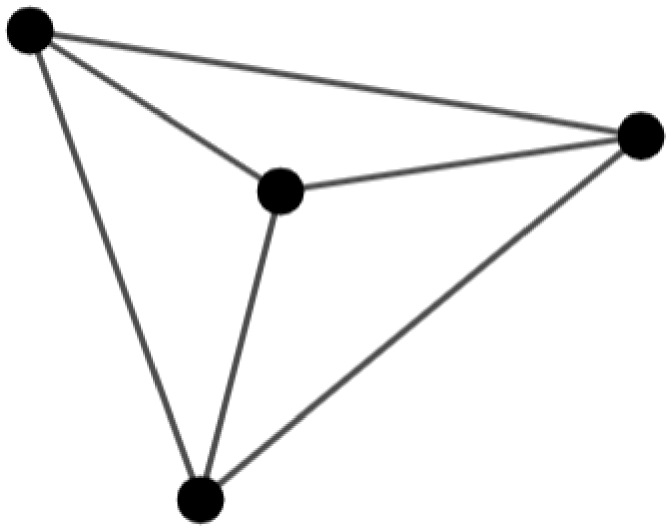
Under all these conditions *c* can be extended to an *L*-coloring of *G* such that no triangle in *G* is monochromatic.
To obtain the main result, we can precolor one of the vertices on the outer cycle and apply this theorem.

Sketch of the proof

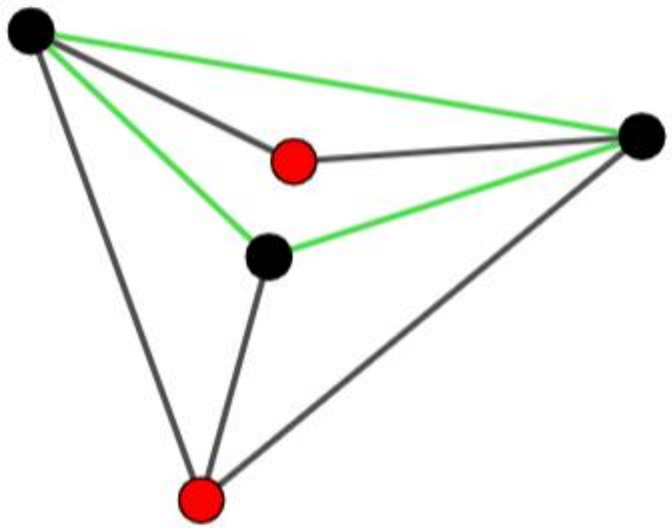
The theorem is proven by induction on the numer of edges in *G*. Lets assume by contradoction that *G* is a counterexample with the fewest egdes among all counterexamples.
To proof the main theorem, the author first shows 11 claims, most of them proven by contradiction and application of the inductive thesis, while considering many cases, and then combines them to finally prove the result.

Separating triangle

A triangle of a plane graph is a separating triangle if its removal separates the graph.



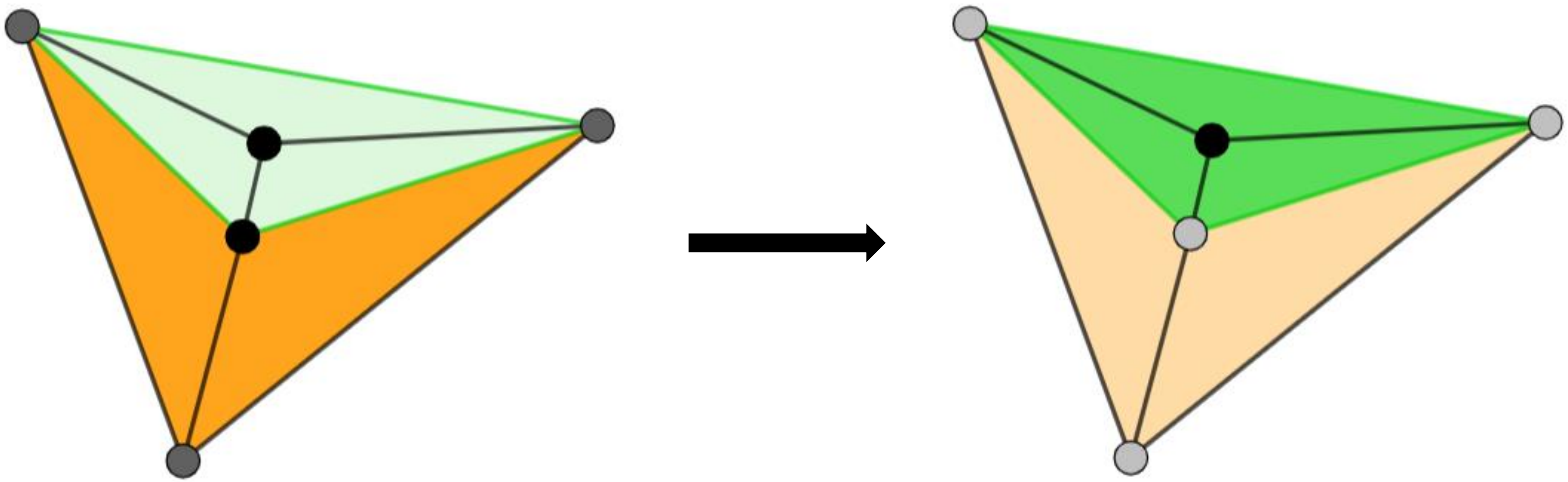
No separating triangle



Separating triangle marked green

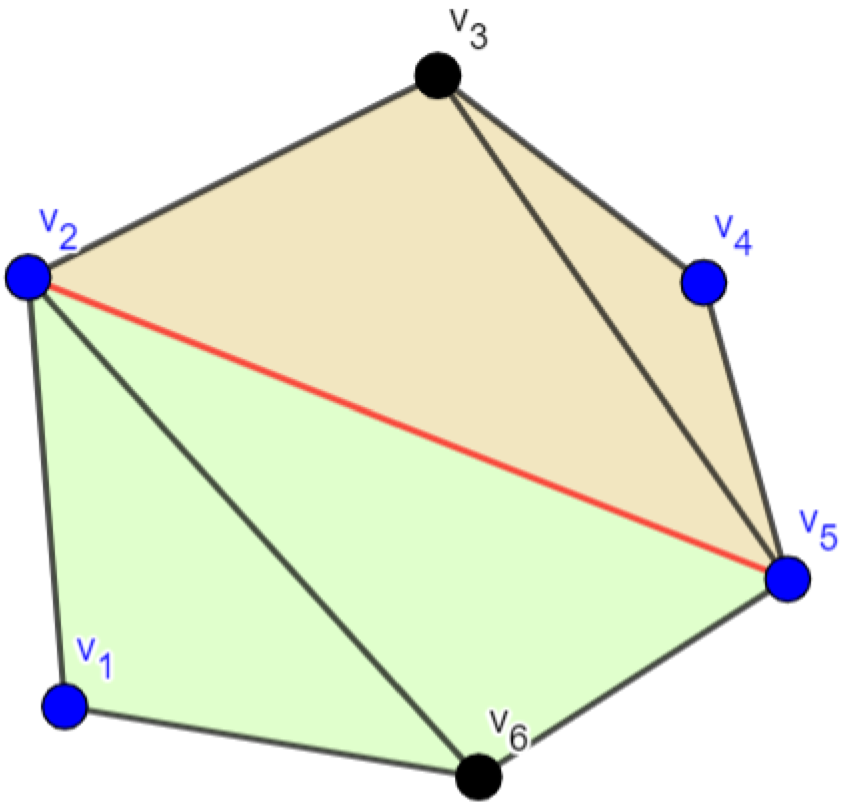
Claim 1

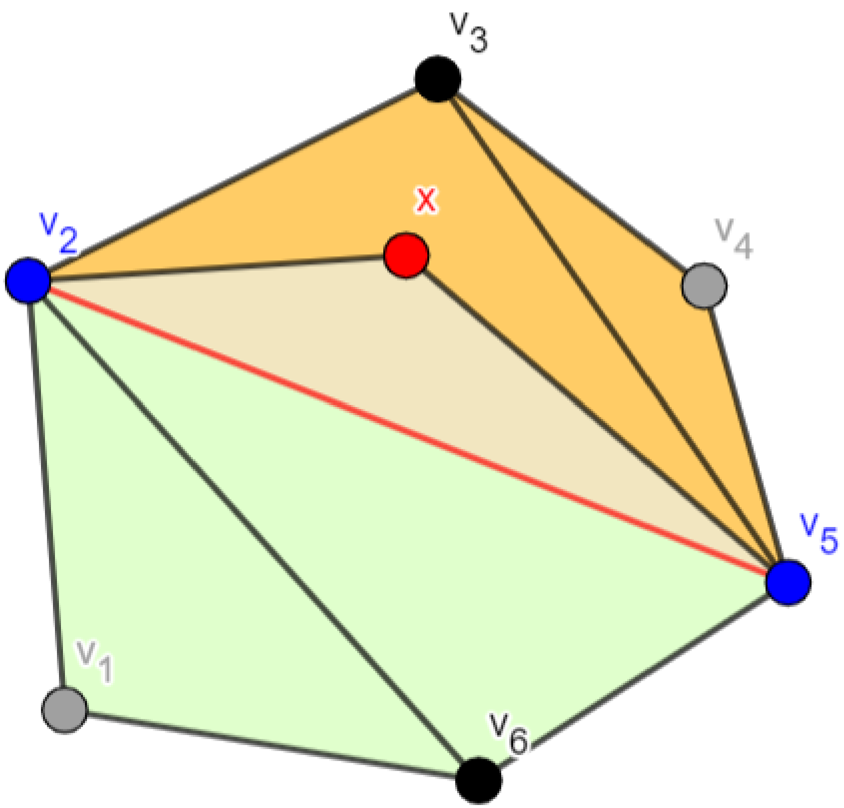
<u>*G* has no separating triangle</u>



Claim 2.a

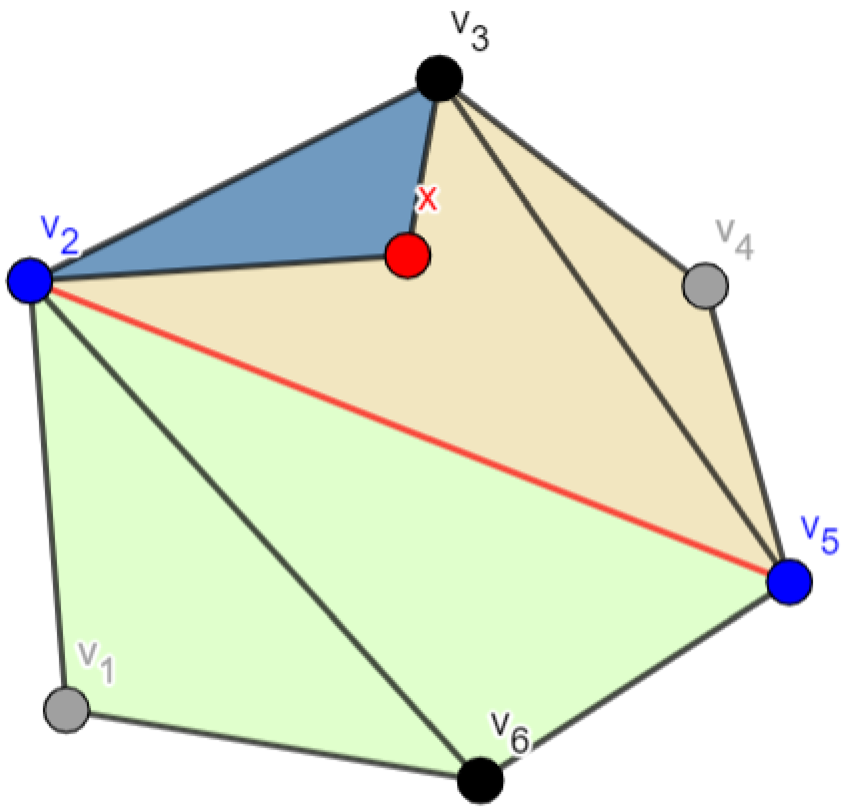
<u>*P* has no edge which is a chord of *C*</u>





Claim 2.a

We choose such *x* that blue graph has as few edges as possible.



Claim 2.a

Now we can choose the notation such that either $P = C$,
or P is the path $v_1v_2 \dots v_q$.

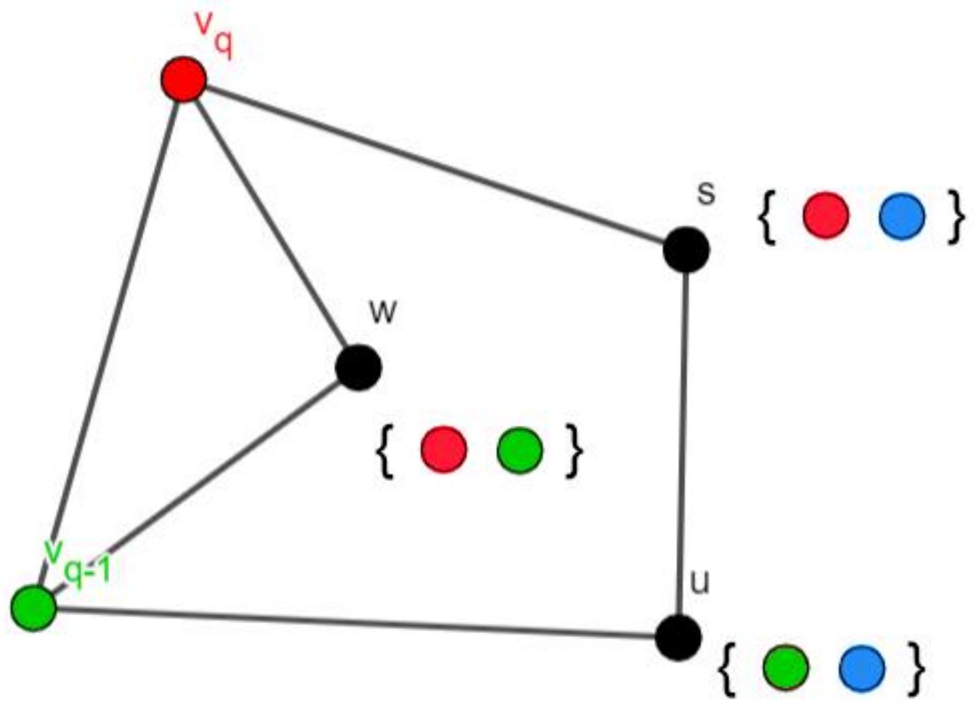
Claims 2-5

2. G cannot be separated by a path of length 1 or 2 except in a very special way.
3. P has no monochromatic edge $v_i v_{i+1}$.
4. $q \geq 4$
5. If P is a path, and if Q is a path of length 2 or 3 joining two vertices in $A_0 \cup A_1$, then Q is of length 3 and is contained in the outer cycle C.

We now focus on the edge $v_{q-1} v_q$. By *Claim 3*, v_q , v_{q-1} have distinct colors.
$c(v_q) = 1$, $c(v_{q-1}) = 2$.
Let $wv_q v_{q-1} w$ be the unique triangle containing the edge $v_q v_{q-1}$.

Claim 6

G has a 4-cycle $v_q v_{q-1} u s v_q$ whose interior has precisely one vertex *w*. Moreover, $L(s) = \{1, 3\}$, $L(u) = \{2, 3\}$, and $L(w) = \{1, 2\}$

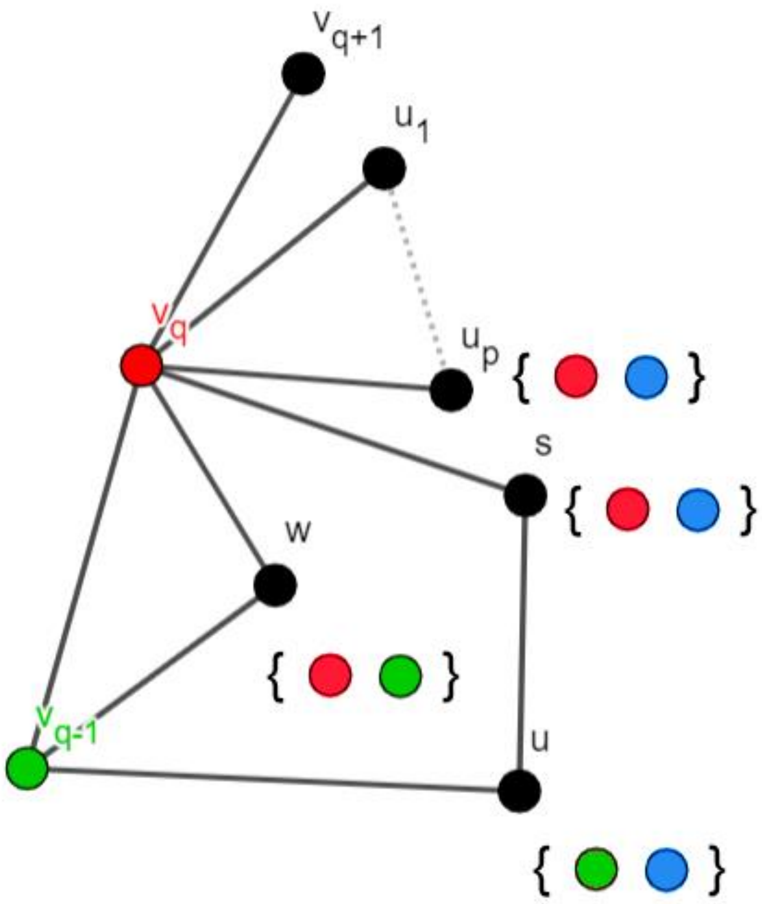


Claim 7

Let v_{q+1}, u_1, u_2, ...,u_p , s, w, v_{q-1} be the neighbors of v_q in clockwise order.

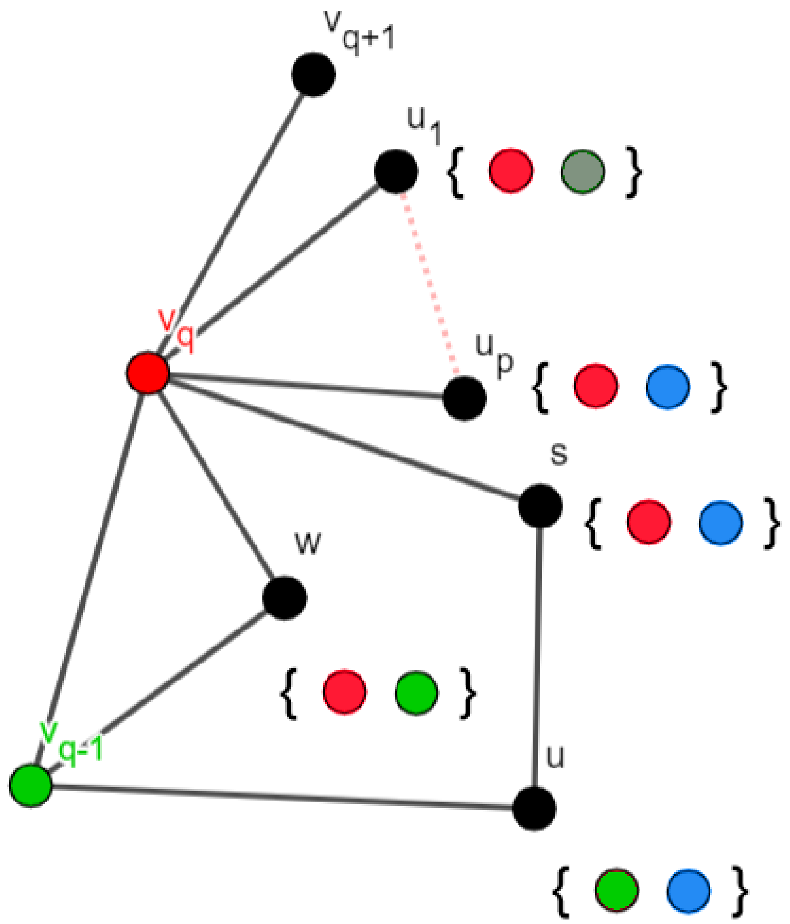
$u_0 = v_{q+1}$, $u_{p+1} = s$.

$p \geq 1$, and $L(u_p) = \{1, 3\}$.



Claim 8

Each list $L(u_i)$, $1 \leq i \leq p$, contains the color 1.

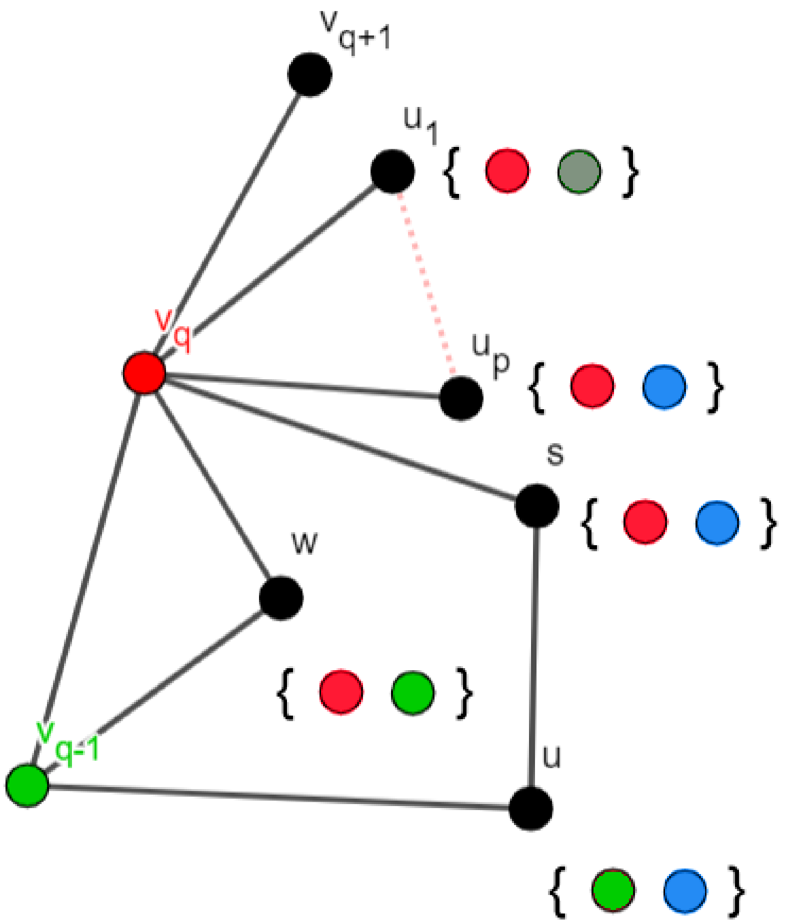


Claim 9

None of u_0, u_1, ..., u_p , s is joined to a precolored vertex distinct from v_q.

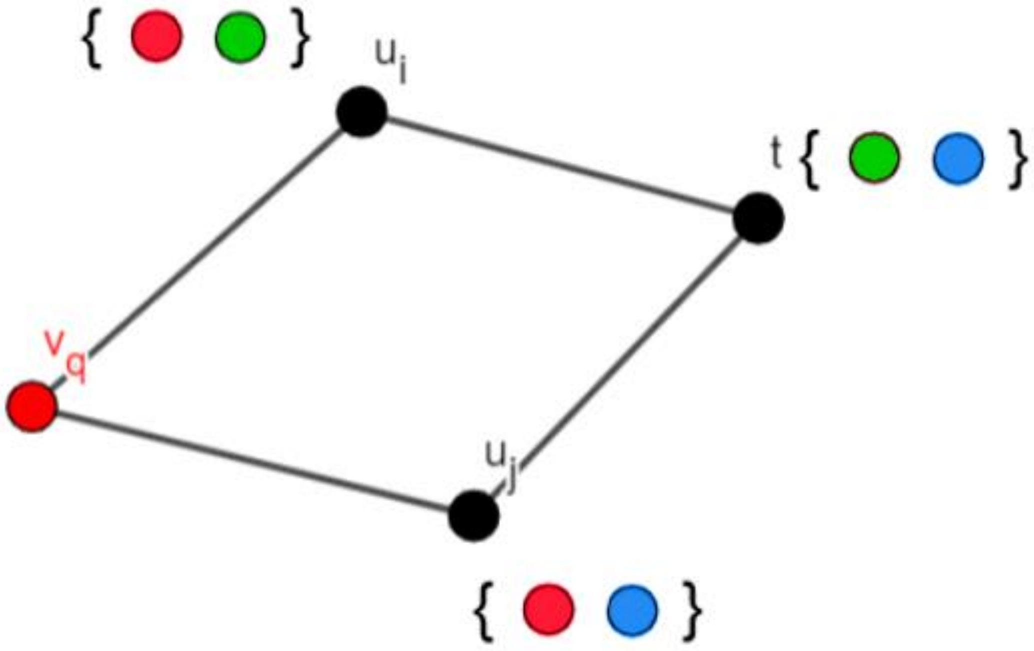
Claim 10

There is no edge $u_i u$ with $0 \leq i \leq p$.



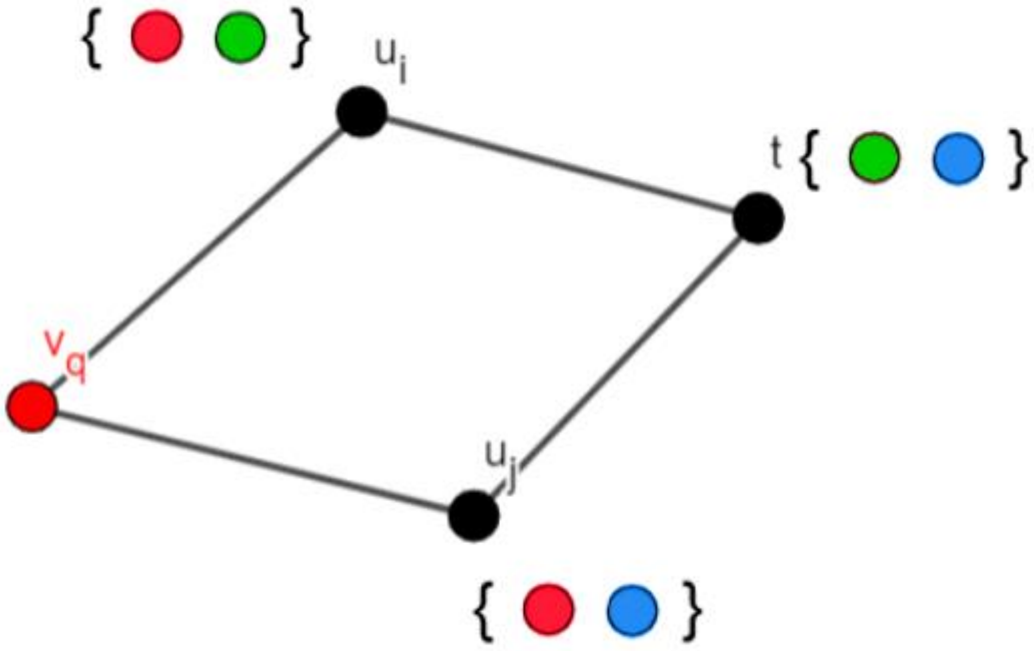
Problematic 4-cycle

The idea is to give some of the u_k a color distinct from 1, but this way we may create a bad 2-path.



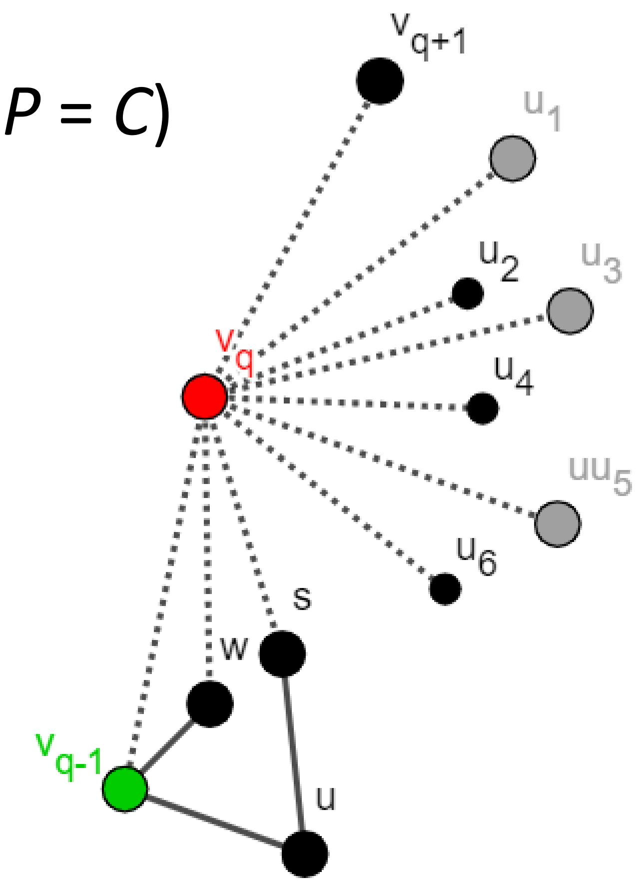
Problematic 4-cycle

If we give u_i the color distinct from 1, and then the coloring can be extended to the interior of the cycle, then we say that this cycle is of **type 1**.
Otherwise it is of **type 2**.
But as proven in the **claim 11**, coloring of cycles of type 2 can be extended if we give a color to u_j.



Final concept

Now the idea is to give of u_1, u_3, ... (or each of u_2, u_4, ... if $P = C$) a color distinct from 1 and then delete v_q.
Then the author separates G into smaller graphs depending on the location of problematic 4-cycles.
After consideration of all cases and making use of the previously proven claims, he comes to the contradicion, that proves the theorem.



Summary

References

Carsten Thomassen (2008), 2-List-coloring planar graphs without monochromatic triangles