Every graph contains a linearly sized induced subgraph with all degrees odd

Authors: Asaf Ferber, Michael Krivelevich

Rafał Pyzik

Gallai's Theorem

Let $G[V]$ be graph induced on vertices V.

Theorem 1. *(Gallai's Theorem)*

Let G be any undirected graph. Then:

1. there exists partition $V(G) = V_1 \cup V_2$ of vertices of G such that both graphs $G[V_1]$ and $G[V_2]$ have all their degrees even.
2. there exists partition $V(G) = V_o \cup V_e$ of vertices of G such that graph $G[V_o]$ has all degrees odd and $G[V_e]$ has all degrees even.

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From (1) it follows that G has induced subgraph of size at least $|V(G)|/2$ with all degrees even.

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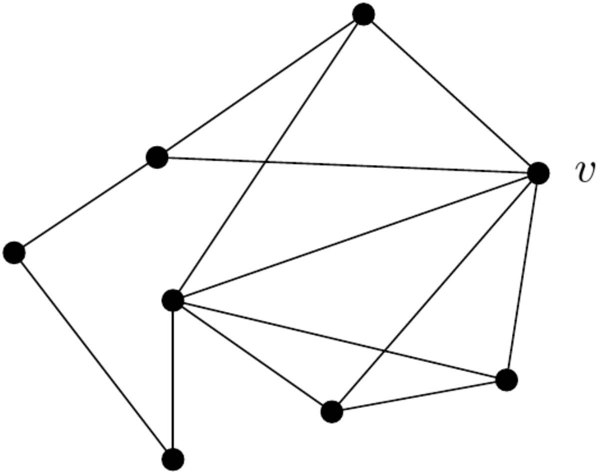
From induction on $|V(G)|$ we can partition $V(G_1) = W_1 \cup W_2$, W_1, W_2 span subgraphs of G_1 with even degrees. Now WLOG if $|N(v) \cap W_1|$ is even then

$$V_1 = W_1 \cup \{v\},\ V_2 = W_2$$

satisfies desired property.

Gallai's Theorem



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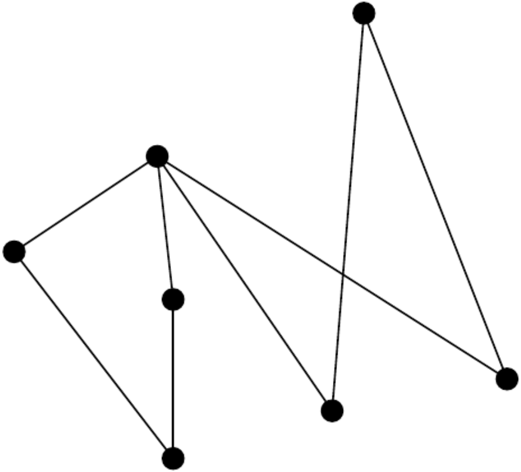

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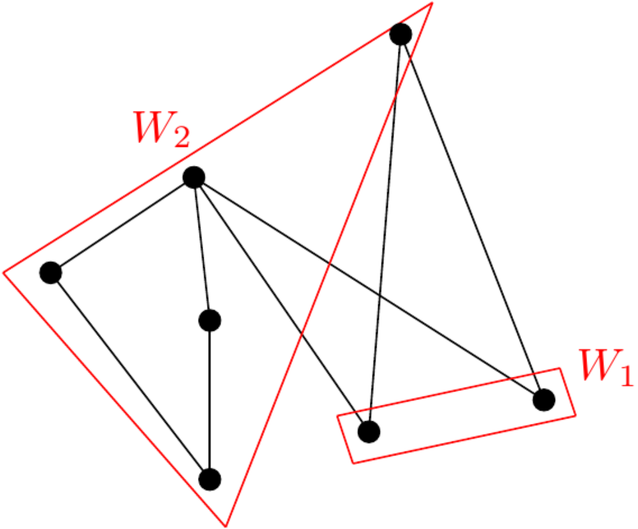

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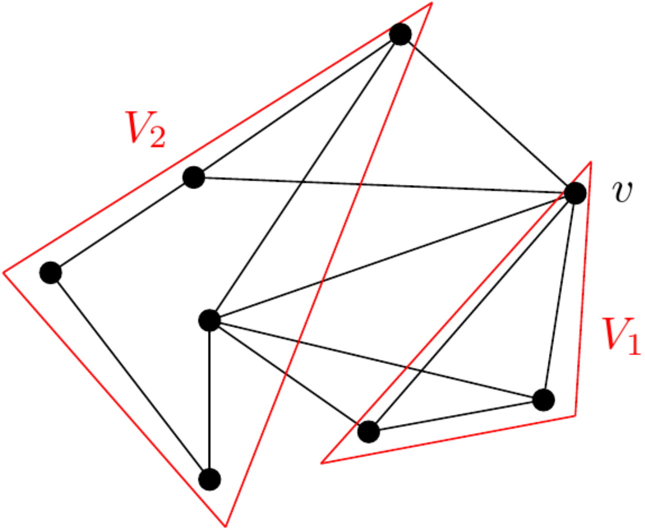

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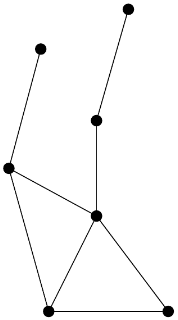

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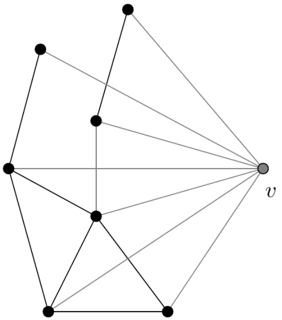

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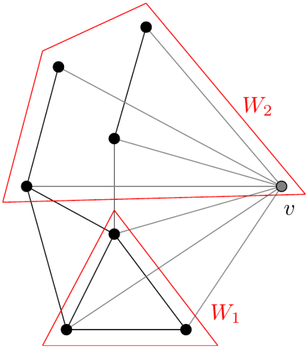

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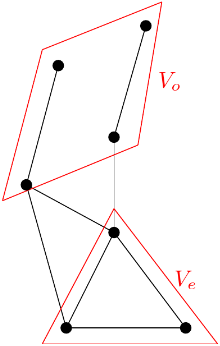
   WLOG if $v \in W_2$ then $V_e = W_1$, $V_o = W_2 \setminus \{v\}$ is desired partition.



Definitions

Given graph $G = (V, E)$ lets define

$$f_o(G) = \max\{|V_0| : V_0 \subseteq V, G[V_0] \text{ has all degrees odd}\}$$

and

$$f_o(n) = \min\{f_o(G) : |V(G)| = n \text{ and } \delta(G) \geq 1\}.$$

Main conjecture

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Partial results:

- $f_o(n) = \Omega(\sqrt{n})$ – Y. Caro (1991)
- $f_o(n) = \Omega(n/\log n)$ – A. D. Scott (1992)
- trees
- maximum degree ≤ 3
- $c \leq \frac{2}{7}$

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Every graph G without isolated vertices satisfies $f_o(G) \geq cn$ for $c = \frac{1}{10000}$.

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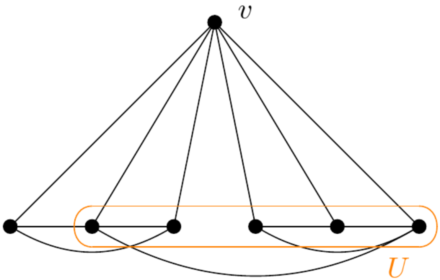

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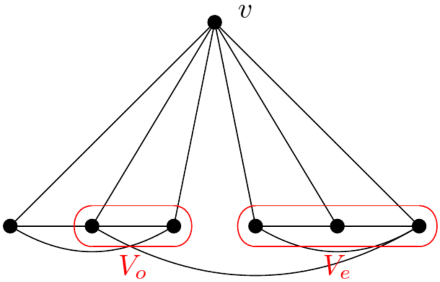

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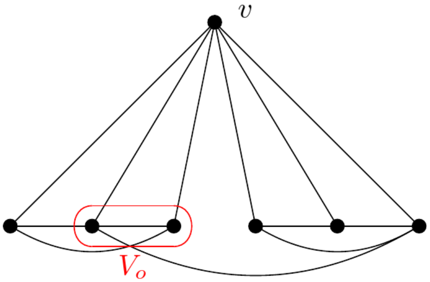

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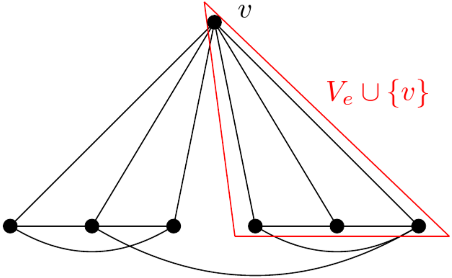

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Idea of proof similar to Lemma 3.

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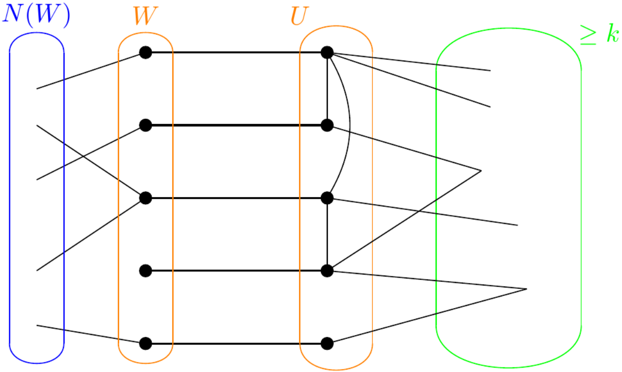

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Take $G[X_1 \cup U_0]$. In it all vertices in X_1 have odd degree. For every vertex in U_0 with even degree we can add it's matched edge from M.

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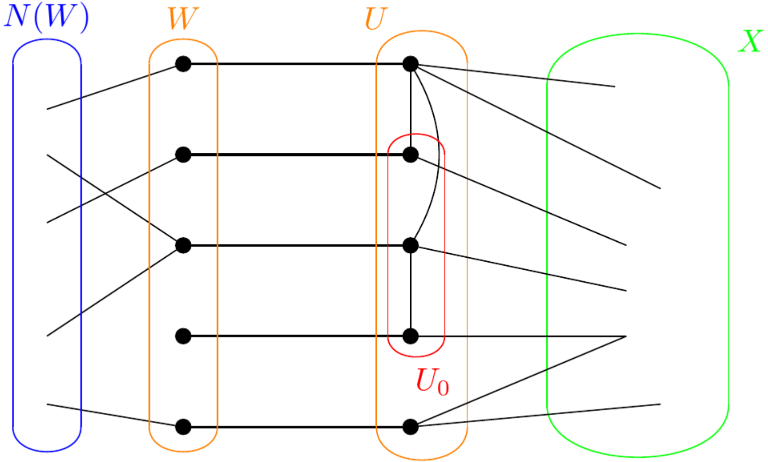

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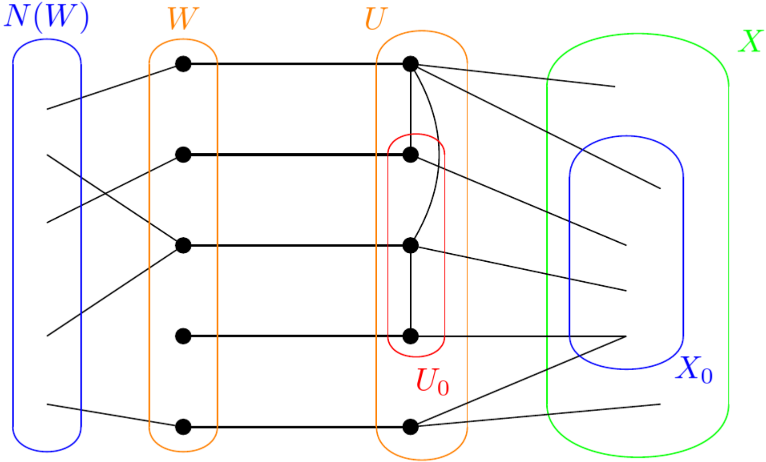

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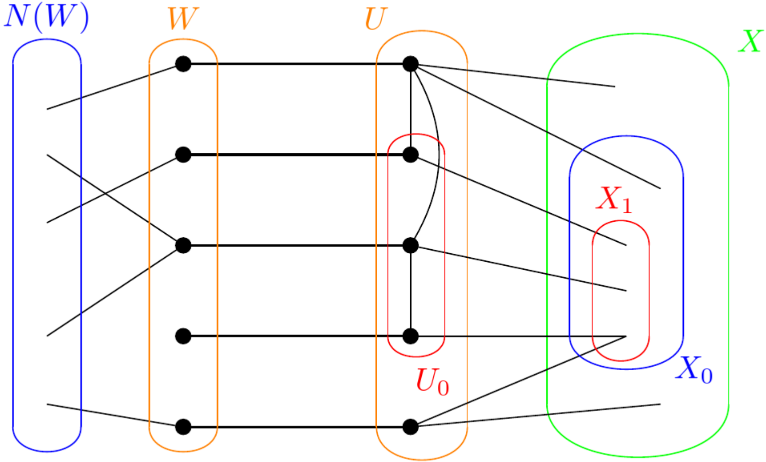

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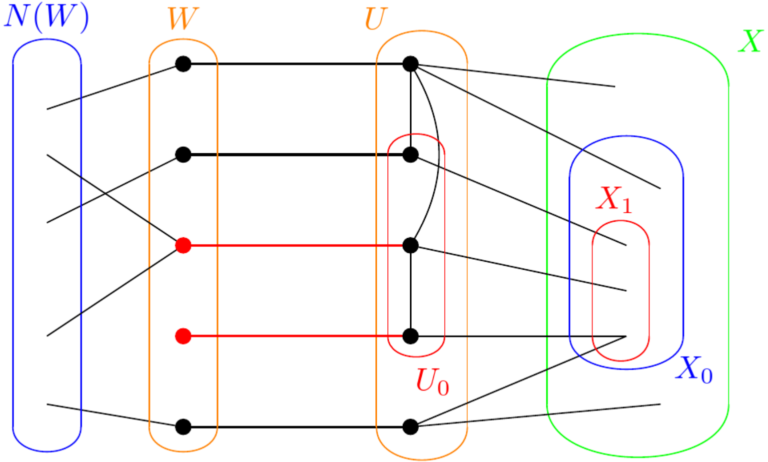

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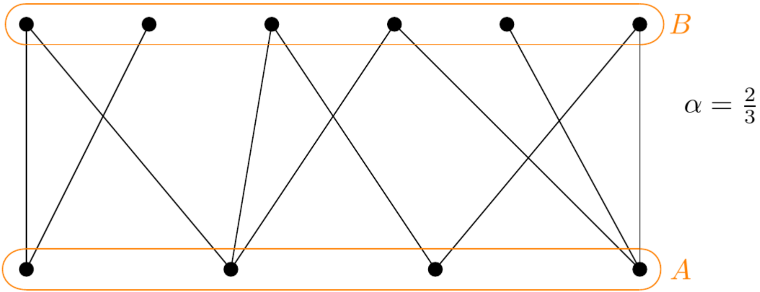

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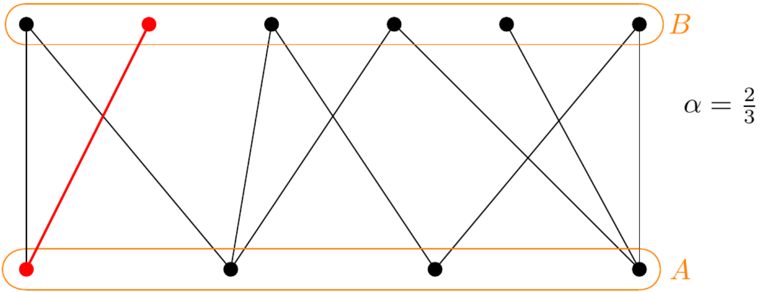

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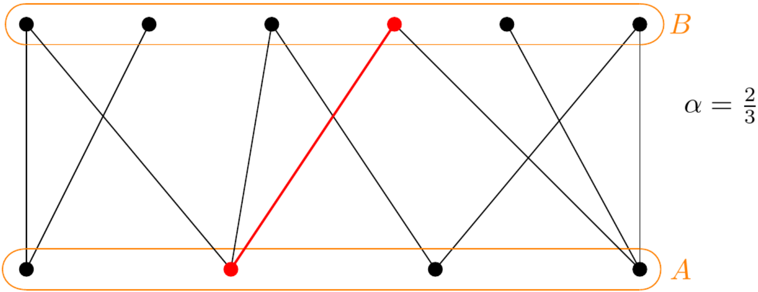

Proof.

$$\sum_{ab \in E(G)} \left(\frac{1}{d(b)} - \frac{1}{d(a)} \right) = \sum_{b \in B} d(b) \cdot \frac{1}{d(b)} - \sum_{a \in A, d(a) > 0} d(a) \cdot \frac{1}{d(a)} \geq$$

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Hence there is an edge ab for which $\frac{1}{d(b)}-\frac{1}{d(a)}\geq(1-\alpha)\frac{1}{d(b)}$, implying $d(a)\geq\frac{d(b)}{\alpha}$.

Lemma 5.

For graph G and $\beta > 0$, define

$$L = L(G;\beta) = \{v \in V : \exists u \in V, uv \in E(G), \\ |N(u) \setminus N(v)| \geq \beta |N(u) \cup N(v)| \ \}.$$

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Set $\beta = \frac{1}{20}, \gamma = \frac{1}{14}$.

Lemma 5.

Let G be a graph on n vertices with $\delta(G) \geq 1$ and $|L(G;\beta)| \leq \gamma n$. Then $f_o(G) \geq n/61$.

Proof of Theorem 2.

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Start with $M_0 = \varnothing$. Given matching $M_i = U_i \sqcup W_i$, W_i has only one neighbour in M_i, we define

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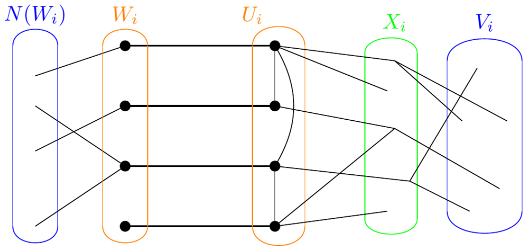

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After the process we have $|X_i| \geq n/80$ – apply Lemma 3.

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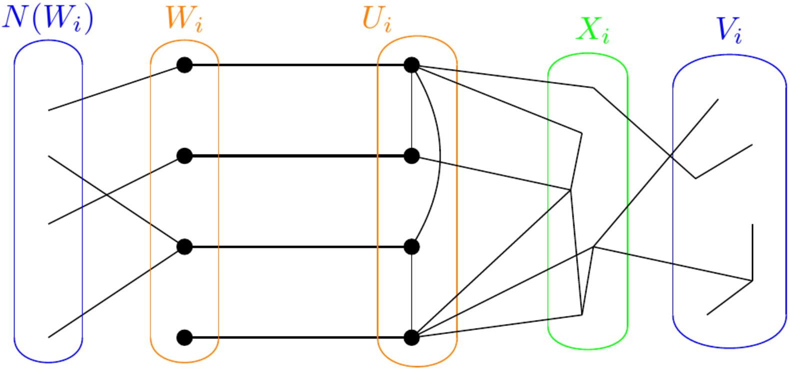
Now we will try to add an edge to M_i while maintaining $|X_{i+1}| \geq \frac{|V \setminus V_{i+1}|}{40}$.



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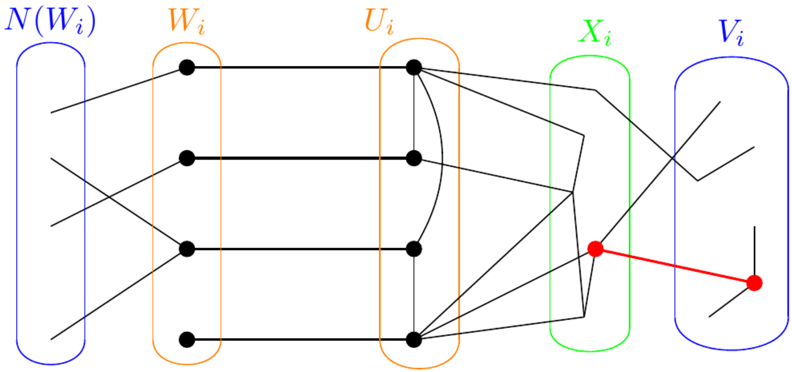

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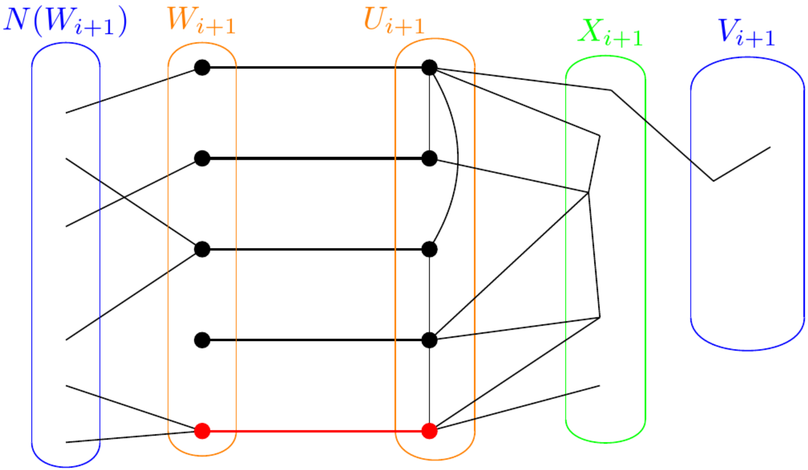

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$|X_i| \leq \frac{n}{2500} \leq \frac{|L|}{44}$, so by Lemma 4. for bipartite graph between X_i and L there exists $x \in X_i$ and $v \in L$ satisfying

$$d(x, L) \geq 44d(v, X_i) \geq 1.1d(v, V_i) > 0.$$

Lets add edge xv to M_i, $U_{i+1} = U_i \cup \{x\}, W_{i+1} = W_i \cup \{v\}$.
One can show

$$|X_{i+1}| > |X_i| + \frac{3d(x, V_i)}{44},$$
$$|V_{i+1}| \geq |V_i| - \frac{21d(x, V_i)}{11},$$

so at least $1/40$ of vertices deleted from V_i go to X_{i+1}.

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Lets add edge uv to M_i, $U_{i+1} = U_i \cup \{u\}, W_{i+1} = W_i \cup \{v\}$.
One can show

$$|X_{i+1}| > |X_i| + \frac{1}{40}|N(u, V_i) \cup N(v, V_i)|,$$
$$|V_{i+1}| \geq |V_i| - |N(u, V_i) \cup N(v, V_i)|,$$

so at least $1/40$ of vertices deleted from V_i go to X_{i+1}.