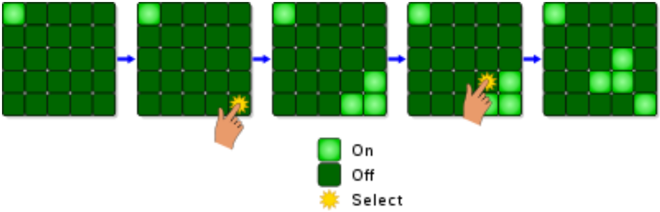
A counterexample to the lights out problem
by János Nagy, Péter Pál Pach

Piotr Kaliciak

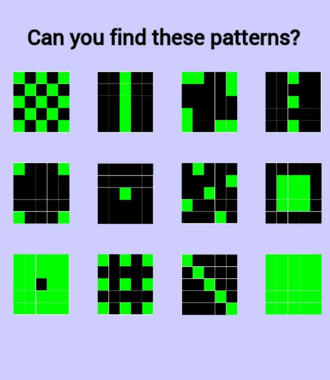
Jagiellonian University

November 17, 2022

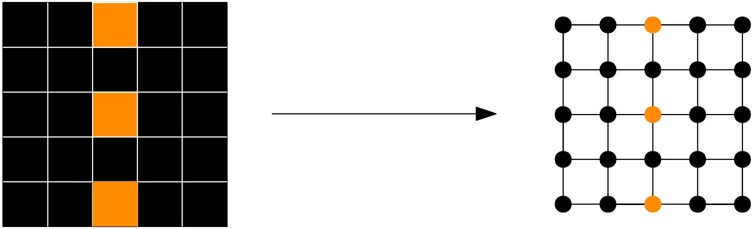
Genesis


On
Off
Select

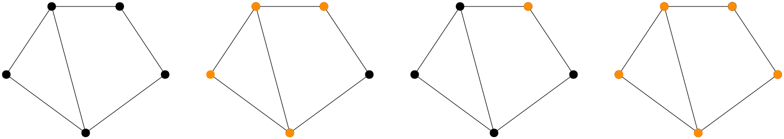

LIGHTS OUT
ON/OFF
START
SOUND
HELP
SELECT
TIGER


Can you find these patterns?

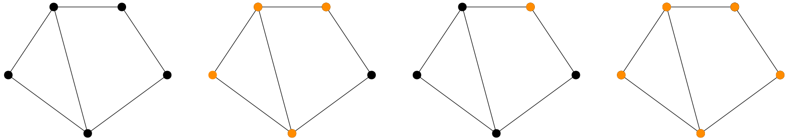
Translation



Generalization



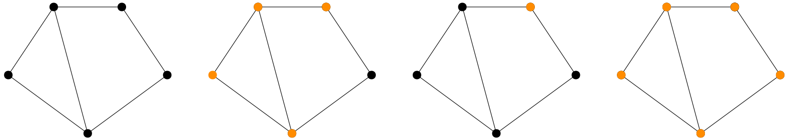
Generalization



Observations:

- Order of light switches doesn't matter.
- Only parity of number of light switches in node matters.

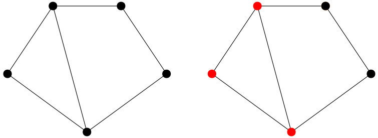
Generalization



Observations:

- Order of light switches doesn't matter.
- Only parity of number of light switches in node matters.
- It appears that when all lights are off, it is always possible to light on the whole graph.

Graph theory notation

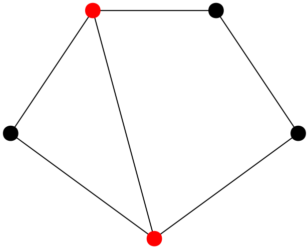


In graph theory notation, equivalent question is, if there exist a subset of vertices X, such that $|N[v] \cap X| \not\equiv 0 \pmod 2$ for every vertex v in G.

$N[v] = \{v\} \cup N(v)$ is the closed neighbourhood of v.

Dominating set

A **nowhere 0 mod p dominating set** for graph G is a subset of vertices, such that $|N[v] \cap X| \not\equiv 0 \pmod{p}$ for every v in G.



Nowhere 0 mod 3 dominating set

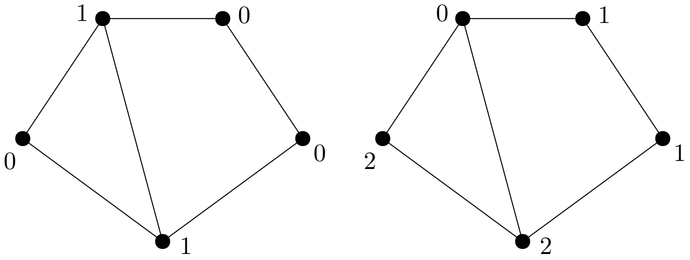
Dominating function

A **nowhere 0 mod p dominating set** for graph G is a subset of vertices, such that $|N[v] \cap X| \not\equiv 0 \pmod{p}$ for every v in G.

A **nowhere 0 mod p dominating function** for graph G is a function $f : V \to \mathbb{Z}_p$, such that $\sum_{u \in N[v]} f(u) \not\equiv 0 \pmod{p}$ for every v in G.

Dominating function

A **nowhere 0 mod p dominating function** for graph G is a function $f : V \to \mathbb{Z}_p$, such that $\sum_{u \in N[v]} f(u) \not\equiv 0 \pmod{p}$ for every v in G.



Nowhere 0 mod 3 dominating functions

Main questions

Question 1. Let G be a finite, simple graph and $p \geq 3$. Does there always exist nowhere 0 mod p dominating set?

Question 2. Let G be a finite, simple graph and $p \geq 3$. Does there always exist nowhere 0 mod p dominating function?

Main theorem

Theorem 1. Let $p \geq 3$ be an arbitrary prime and let $M \subseteq \mathbb{Z}_p \backslash \{0\}$. Then there exists a simple connected graph G_M, with vertex set V_M such that there does not exist function $f : V_M \to \mathbb{Z}_p$ such that $\sum_{u \in N[v]} f(u) \in M$ for each $v \in V_M$.

Main theorem

Theorem 1. Let $p \geq 3$ be an arbitrary prime and let $M \subseteq \mathbb{Z}_p \backslash \{0\}$. Then there exists a simple connected graph G_M, with vertex set V_M such that there does not exist function $f : V_M \to \mathbb{Z}_p$ such that $\sum_{u \in N[v]} f(u) \in M$ for each $v \in V_M$.

When $M = \mathbb{Z}_p \backslash \{0\}$ then $G_{\mathbb{Z}_p \backslash \{0\}}$ is counterexample to Questions 1 and 2.

Proof

The statement will be proven by induction on the cardinality $|M|$.

Proof

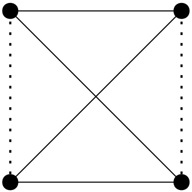
The statement will be proven by induction on the cardinality $|M|$.

Case 1. $M = \emptyset$.
Any nonempty graph is going to hold conditions. We assume that $G_\emptyset$ has only one vertex:

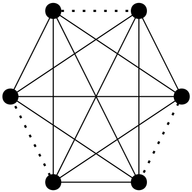
$G_\emptyset$

$|M| = 1$

Case 2. $|M| = 1$.
Let $M = \{a\}$ where $a \neq 0$.
Let G_M be obtained from K_{p+1} by deleting exactly one edge adjacent to every vertex.



G_M for $p = 3$



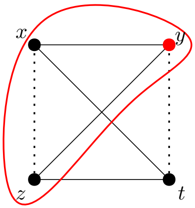
G_M for $p = 5$

$|M| = 1$

Case 2. $|M| = 1$.
Let $M = \{a\}$ where $a \neq 0$.
Let G_M be obtained from K_{p+1} by deleting exactly one edge adjacent to every vertex.
For the sake of contradiction lets assume that there exist function $f : V \to \mathbb{Z}_p$ such that $\sum_{u \in N[v]} f(u) = a$ for every $v \in V_M$.
Then we get:

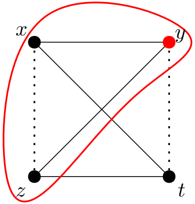
$|M| = 1$

$$\sum_{v \in V_M} \sum_{u \in N[v]} f(u) = \sum_{v \in V_M} a = (p+1)a = a$$

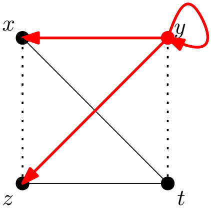


$|M| = 1$

$$\sum_{v \in V_M} \sum_{u \in N[v]} f(u) = \sum_{v \in V_M} a = (p+1)a = a$$



$$a = \sum_{v \in V_M} \sum_{u \in N[v]} f(u) = \sum_{u \in V_M} f(u) \sum_{v \in N[u]} 1 = \sum_{u \in V_M} f(u)p = 0$$



$|M| = k$

Case 3. $|M| = k \geq 2$.

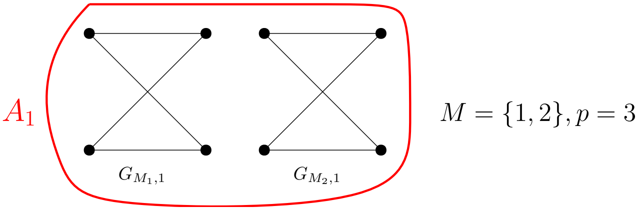
$|M| = k$

Case 3. $|M| = k \geq 2$.
Let $M_c = M \cap (M - c)$ for $c \in \mathbb{Z}_p \backslash \{0\}$.
$|M_c| < |M| = k$, since c generates $\mathbb{Z}_p$ (as an additive group).

$|M| = k$

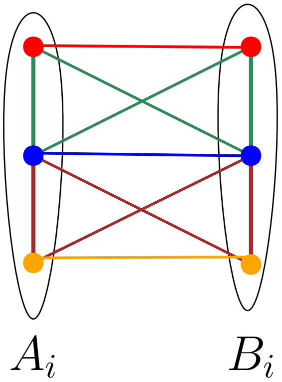
Case 3. $|M| = k \geq 2$.
Let $M_c = M \cap (M - c)$ for $c \in \mathbb{Z}_p \backslash \{0\}$.
$|M_c| < |M| = k$, since c generates $\mathbb{Z}_p$ (as an additive group).
Let $G_{M_c,i}$, be independent copy of G_{M_c}, for $1 \leq i \leq p+1$.
Let

$$A_i = \cup_{c \in \mathbb{Z}_p \backslash \{0\}} G_{M_c,i}$$



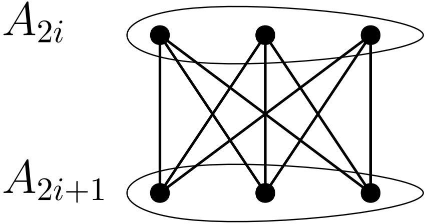
$|M| = k$

Let B_i be a copy of A_i, such that if for $v, u \in A_i$, (v, u) is an edge, than if $v', u' \in B_i$ are copies of v and u, then (v, u') is an edge and (v', u) is an edge. Also we add edge (v, v')
Let h_i be the bijection $A_i \to B_i$, such that $h_i(v) = v'$, when v' is copy of v.



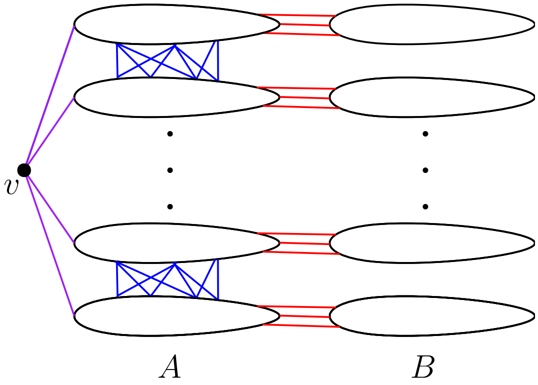
$|M| = k$

Next for every $1 \leq i \leq \frac{p+1}{2}$ we add edge between every pair of vertices from $(A_{2i} \times A_{2i+1})$.



$|M| = k$

At the end we add new vertex v such that it has edge to every vertex in every A_i.
Now graph G_M looks like this:



$|M| = k$

For the sake of contradiction lets assume that there exists function $f : V_M \to \mathbb{Z}_p$ such that $\sum_{u \in N[w]} f(u) \in M$ for each $w \in V_M$.

$|M| = k$

For the sake of contradiction lets assume that there exists function $f : V_M \to \mathbb{Z}_p$ such that $\sum_{u \in N[w]} f(u) \in M$ for each $w \in V_M$.

Now we want to prove that for every $1 \leq i \leq p + 1$, we have $f(v) + \sum_{u \in A_i} f(u) = 0$.

$|M| = k$

For the sake of contradiction lets assume that there exists function $f : V_M \to \mathbb{Z}_p$ such that $\sum_{u \in N[w]} f(u) \in M$ for each $w \in V_M$.

Now we want to prove that for every $1 \leq i \leq p+1$, we have $f(v) + \sum_{u \in A_i} f(u) = 0$. When this is true, than we have:

$$0 = \sum_{1 \leq i \leq p+1} \left(f(v) + \sum_{u \in A_i} f(u) \right) = \sum_{u \in N[v]} f(u) \in M$$

which leads to contradiction.

$|M| = k$

For the sake of contradiction lets assume that for $i = 1$ we have

$$f(v) + \sum_{u \in A_1} f(u) = c \neq 0.$$

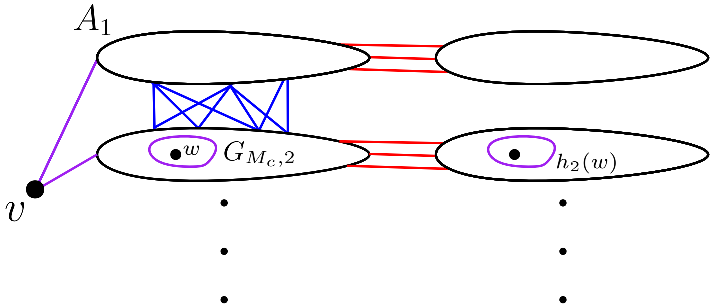
$|M| = k$

For the sake of contradiction lets assume that for $i = 1$ we have

$$f(v) + \sum_{u \in A_1} f(u) = c \neq 0.$$

For every $w \in G_{M_c,2}$ we have:

$$N[w] = N[h_2(w)] \cup \{v\} \cup A_1.$$



$|M| = k$

For every $w \in G_{M_c,2}$ we have:

$$N[w] = N[h_2(w)] \cup \{v\} \cup A_1.$$

We also know that:

$$\sum_{u \in N[w]} f(u) \in M \quad \text{and} \quad \sum_{u \in N[h_2(w)]} f(u) \in M.$$

$|M| = k$

For every $w \in G_{M_c,2}$ we have:

$$N[w] = N[h_2(w)] \cup \{v\} \cup A_1.$$

We also know that:

$$\sum_{u \in N[w]} f(u) \in M \quad \text{and} \quad \sum_{u \in N[h_2(w)]} f(u) \in M.$$

So:

$$\sum_{u \in N[h_2(w)]} f(u) \in M \cap (M - c) = M_c.$$

$|M| = k$

For every $w \in G_{M_c,2}$ we have:

$$N[w] = N[h_2(w)] \cup \{v\} \cup A_1.$$

We also know that:

$$\sum_{u \in N[w]} f(u) \in M \quad \text{and} \quad \sum_{u \in N[h_2(w)]} f(u) \in M.$$

So:

$$\sum_{u \in N[h_2(w)]} f(u) \in M \cap (M - c) = M_c.$$

For each $w‘ \in h_2(G_{M_c,2})$ we have:

$$\sum_{u \in N[w‘]} f(u) \in M_c.$$

$|M| = k$

For each $w‘ \in h_2(G_{M_c,2})$ we have:

$$\sum_{u \in N[w‘]} f(u) \in M_c.$$

Let g be a function $h_2(G_{M_c,2}) \to \mathbb{Z}_p$ such that $g(w‘) = f(w) + f(w‘)$, where $w‘ = h_2(w)$

$|M| = k$

For each $w` \in h_2(G_{M_c,2})$ we have:

$$\sum_{u \in N[w`]} f(u) \in M_c.$$

Let g be a function $h_2(G_{M_c,2}) \to \mathbb{Z}_p$ such that $g(w`) = f(w) + f(w`)$, where $w` = h_2(w)$

Then

$$\sum_{u` \in N[w`] \cap h_2(G_{M_c,2})} g(u`) = \sum_{u` \in N[w`] \cap h_2(G_{M_c,2})} f(u) + f(u`)$$
$$= \sum_{u \in N[w']} f(u) \in M_c$$

$|M| = k$

For each $w` \in h_2(G_{M_c,2})$ we have:

$$\sum_{u \in N[w`]} f(u) \in M_c.$$

Let g be a function $h_2(G_{M_c,2}) \to \mathbb{Z}_p$ such that $g(w`) = f(w) + f(w`)$, where $w` = h_2(w)$

Then

$$\sum_{u` \in N[w`] \cap h_2(G_{M_c,2})} g(u`) = \sum_{u` \in N[w`] \cap h_2(G_{M_c,2})} f(u) + f(u`)$$
$$= \sum_{u \in N[w']} f(u) \in M_c$$

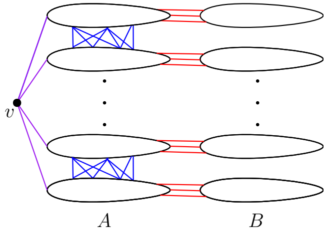
Which is contradiction with construction of G_{M_c}. $\square$

Proof summary

So we proved that for every $1 \leq i \leq p+1$, we have

$$f(v) + \sum_{u \in A_i} f(u) = 0,$$

which stands in contradiction with existence of function $f : V_M \to \mathbb{Z}_p$ such that $\sum_{u \in N[w]} f(u) \in M$ for each $w \in V_M$, proving that G_M is counterexample for the set $M \subseteq \mathbb{Z}_p \backslash \{0\}$ and this finished the proof of the induction step.



Summary

What else we know:

- Size of counterexample is at most $2^p p^{2p}$

Summary

What else we know:

- Size of counterexample is at most $2^p p^{2p}$
- If $(p-1)^2 \geq n \log n$, then there exist nowhere 0 mod p dominating function.

Summary

What else we know:

- Size of counterexample is at most $2^p p^{2p}$
- If $(p-1)^2 \geq n \log n$, then there exist nowhere 0 mod p dominating function.

Next questions:

- How small can the counterexample be?

Summary

What else we know:

- Size of counterexample is at most $2^p p^{2p}$
- If $(p-1)^2 \geq n \log n$, then there exist nowhere 0 mod p dominating function.

Next questions:

- How small can the counterexample be?
- Under what conditions nowhere 0 mod p dominating set(function) exists?

Summary

What else we know:

- Size of counterexample is at most $2^p p^{2p}$
- If $(p-1)^2 \geq n \log n$, then there exist nowhere 0 mod p dominating function.

Next questions:

- How small can the counterexample be?
- Under what conditions nowhere 0 mod p dominating set(function) exists?
- What about nowhere a mod p dominating sets(functions)?

THANKS FOR THE ATTENTION