Factorizing regular graphs

Filip Konieczny

Based on
Thomassen, C. (2019). *Factorizing regular graphs*

November 24, 2022

Regular graphs

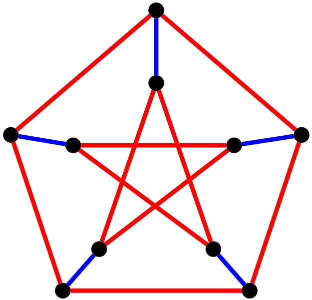
Graph $G = (V, E)$ is k-regular if for every vertex $v \in V$ $deg_G(v) = k$.

q - factor

q-factor of a graph is its spanning q-regular subgraph.

q - factorization

q-factorization is partition of edges of a graph into disjoint q-factors.

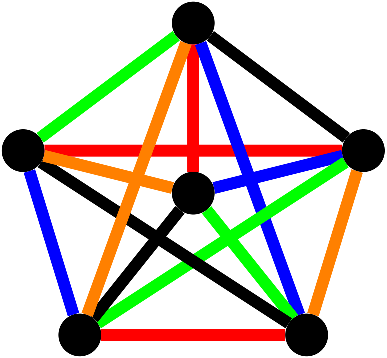


1-factorization of complete graphs

Complete graphs of even order have 1-factorization.

1-factorization of complete graphs

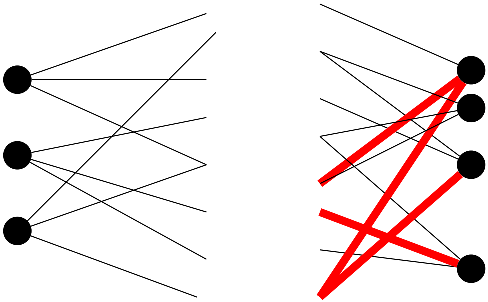
Complete graphs of even order have 1-factorization.



k-regular bipartite graphs

k-regular graphs have 1-factorization (because they have perfect matching).

Proof: Hall's theorem. There is $k \cdot |A|$ edges from the left. Every vertex from the $N(A)$ has at most k edges, so there is at least $|A|$ of them.



$k * |A|$ $\geq k * |A|$

k-factorization

$2k$-regular graphs of even order have k-factorization.

k-factorization

$2k$-regular graphs of even order have k-factorization.

Proof: Euler's cycle.

k-factorization

$2k$-regular graphs of even order have k-factorization.

Proof: Euler's cycle.

2-factorization

$2k$-regular graphs have 2-factorization.

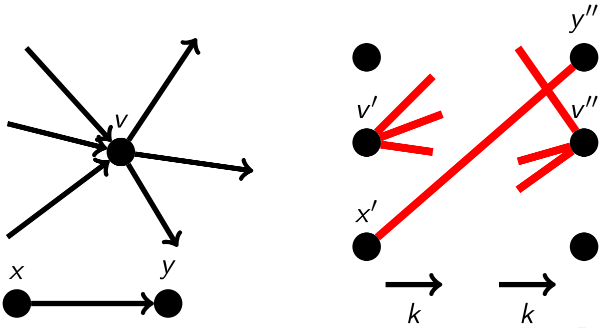
k-factorization

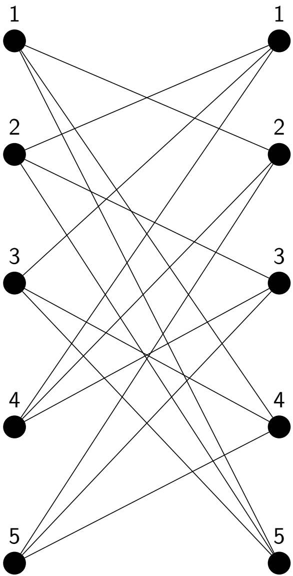
$2k$-regular graphs of even order have k-factorization.

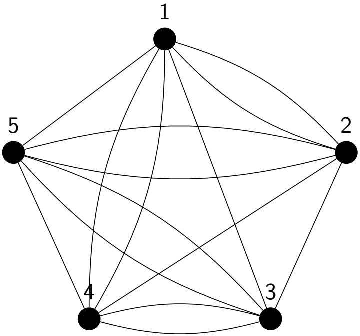
Proof: Euler's cycle.

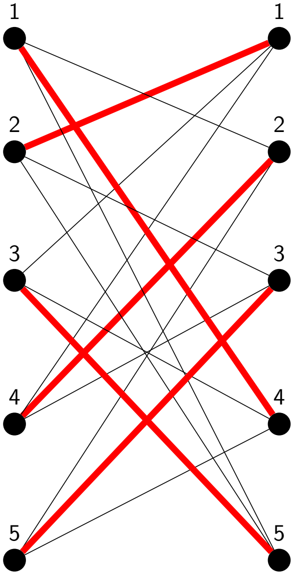
2-factorization

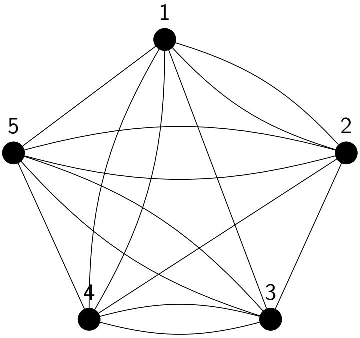
$2k$-regular graphs have 2-factorization.

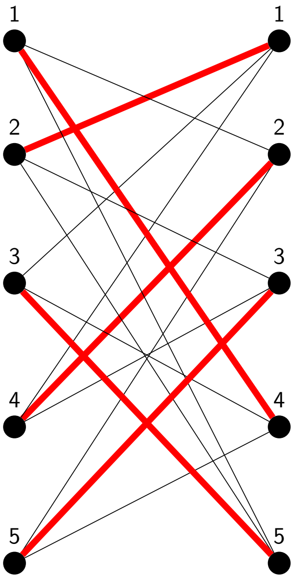
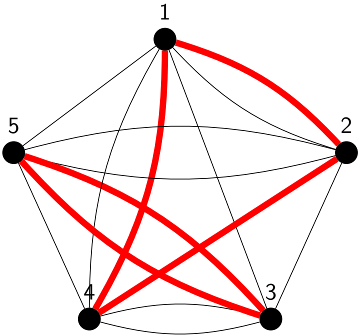



1
2
3
4
5
1
2
3
4
5


1
5
2
4
3


1
2
3
4
5
1
2
3
4
5


1
2
3
4
5


1
1
2
2
3
3
4
4
5
5

1
5
2
4
3

Conclusion

qk-regular graphs have q-factorization for even q.

Conclusion

qk-regular graphs have q-factorization for even q.

What about when q is odd?

Conclusion

qk-regular graphs have q-factorization for even q.

What about when q is odd?

Such general statement is unfortunatelly not true for odd q (e.g. triangle, Petersen graph have no 1-factorization).

Tutte's 3-flow conjecture

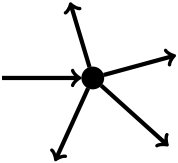
Conjecture

Every 4-connected graph has an orientation of edges such that every vertex has same in- and outdegree modulo 3.

Tutte's 3-flow conjecture

Conjecture

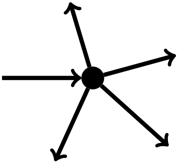
Every 4-connected graph has an orientation of edges such that every vertex has same in- and outdegree modulo 3.



Tutte's 3-flow conjecture

Conjecture

Every 4-connected graph has an orientation of edges such that every vertex has same in- and outdegree modulo 3.



Generalized version

For odd k every $(2k-2)$-connected graph has an orientation of edges such that every vertex has same in- and outdegree modulo k.

False except possibly for $k = 3, 5$.

Orientations

Proven versions

Lemma

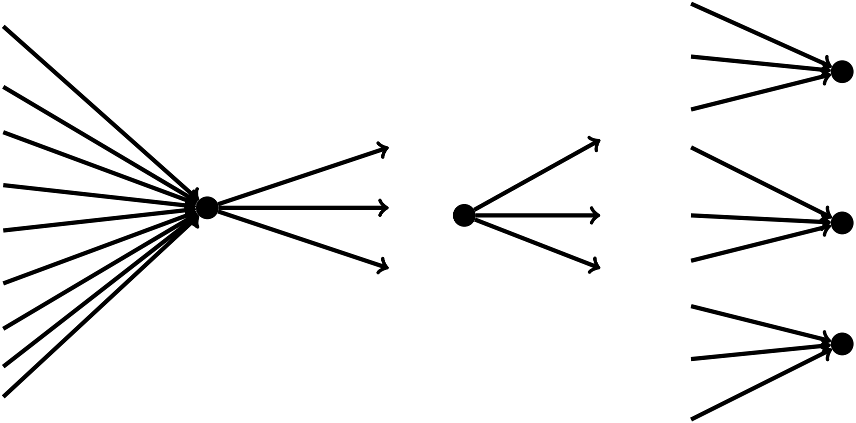
Let k be natural number and let G be $(2k^2 + k)$-edge-connected graph with vertices $v_1, \ldots, v_n$. Let d_i be integers such that $\sum d_i \equiv |E|$ modulo k. Then there is an orientation of edges such that i^{th} vertex has outdegree d_i modulo k.

Lemma

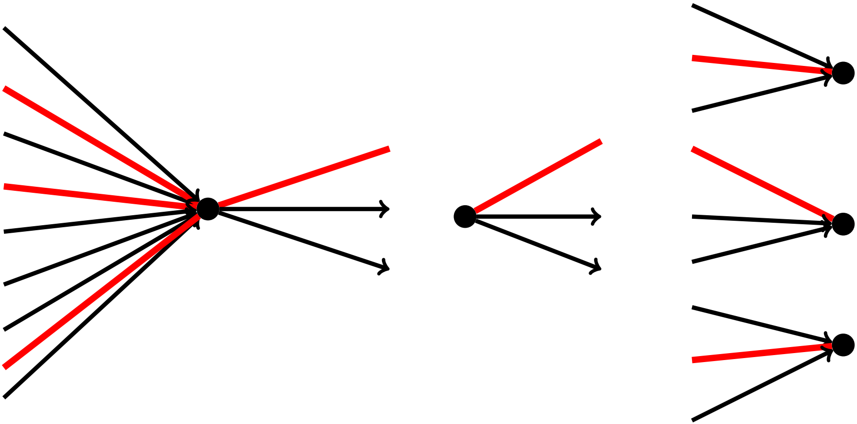
Let k be an odd natural number and let G be graph with odd-edge-connectivity at least $3k - 2$. Then there is an orientation of edges such that every vertex has same indegree and outdegree modulo k.

Theorem 1.

Let G be kq-regular graph for odd q. If k is odd and G has odd-edge-connectivity at least $3k-2$ then G has q-factorization. If k is even and $|V|$ is even and G has edge-connectivity at least k^2+2k then G has q-factorization.



$$0 \equiv kq \equiv a + a \equiv 2a \implies a \equiv 0$$



Theorem 2.

Let r be an odd natural number divisible by 3. G is an r-regular graph with odd-edge-connectivity at least $r-2$. Let $r = r_1 + r_2 + \ldots + r_m$, where $r_i \geq 2$. Then G can be decomposed into r_i-factors.

We split G into $r/3$ 3-regular graps $G_1, \ldots, G_{r/3}$ using Theorem 1. $r_1, \ldots, r_p$ are odd and $r_{p+1}, \ldots, r_m$ are even.

$$\bigcup_{i=p+1}^{m} G_i \text{ is regular of even degree,}$$

so can be decomposed into 2-factors. We can use them to construct $r_{p+1}, \ldots, r_m$-factors and extend $G_1, \ldots, G_p$ to $r_1, \ldots, r_p$ factors.

Other results

Theorem 3.

Let r be an odd natural number divisible by 3 and G be $(r-3)$ connected graph of even order and $\Delta(G) \leq r$. Let $r = r_1 + \ldots + r_m$ where $r_i \geq 2$. Then G can be edge partitioned into covering subgraphs $G_1, \ldots, G_m$ where $\Delta(G_i) \leq r_i$.

Theorem 4.

Let r be an odd natural number divisible by 3 and G be $(r-3)$ connected graph such that each vertex has odd degree at least r. Let $r = r_1 + \ldots + r_m$ where $r_i \geq 2$. Then G can be edge partitioned into covering subgraphs $G_1, \ldots, G_m$ where $\delta(G_i) \geq r_i$.

Theorem 5.

Every planar, 2-connected, $3q$-regular graph has q-factorization.

Conjectures

Conjecture 1.

Theorem 3. holds when r is not divisible by 3.

Conjectures

Conjecture 1.

Theorem 3. holds when r is not divisible by 3.

Conjecture 2.

Can every r-regular r-connected graph of even order be factorized as 1-factors and one 2-factor?

Conjectures

Weakening of Conjecture 2.

There exists r_0 such that for $r \geq r_0$ every r-regular r-connected graph has two disjoint 1-factors.

Conjectures

Weakening of Conjecture 2.

There exists r_0 such that for $r \geq r_0$ every r-regular r-connected graph has two disjoint 1-factors.

Further weakening

If r is an odd natural number ≥ 5, then every r-regular r-connected graph can be edge partitioned into 3 odd regular factors.

Thank you for your attention!