Color-Critical Graphs on a Fixed Surface

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k-color-critical graphs

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- k-color-critical graphs are connected

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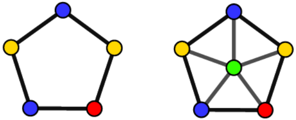
Facts:

- k-color-critical graphs are connected
- every vertex v in k-color-critical graph has $deg(v) \quad k \quad 1$



orientable surfaces

A closed and connected surface is orientable if the direction does not change after one sails around the surface by any path.

The genus of a connected, orientable surface is an integer representing the maximum number of cuttings along non-intersecting closed simple curves without rendering the resultant manifold disconnected.

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3. k=3 - odd cycles
4. k=4 - e.g. odd cycles with additional vertex connected with all the other vertices
5. k=5 - infinitely many (Fisk) (S other than sphere)
6. k=6 - finitely many
7. k 8 - finitely many (Dirac)
8. k 7 - finitely many (Gallani)

Lemma

Let G be a 4-colored graph in which each edge is contained in precisely two triangles. Then for any i, j *2* [1, 2, 3, 4], the number of vertices of odd degree and of color i has the same parity as the number of vertices of odd degree and of color j. In particular, if G has precisely two vertices of odd degree, they have the same color.

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If S is a surface other than the sphere and q is a natural number, then there is a triangulation of S of edge-width at least q and of chromatic number at least 5.

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Lemma

If S is a surface other than the sphere and q is a natural number, then there is a triangulation of S of edge-width at least q and of chromatic number at least 5.

Theorem

For any surface S other than the sphere there are infinitely many 5-color-critical graphs on S.

(k, q)-critical graphs

Let k, q be natural numbers and let G be a graph where a subgraph H with q vertices is k-colored (using colors in [1, 2, ..., k]). Then G is (k+1, q)-critical if the coloring of H cannot be extended to a k-coloring of G but, for every vertex or edge x in G, the k-coloring of $H n x$ can be extended to a k-coloring of $G n x$.

When we are able to extend the coloring of a fragment on the whole graph?

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Theorem

Let d be any natural number. There exists a 4-connected planar triangulation G having two vertices x, y of distance at least d apart such that, in any 4-coloring of G, x and y must have the same color. In particular, there are infinitely many (5, 2)-critical planar graphs with no separating triangle.

Theorem

For any non-negative integers g, q there exists a natural number (g, q) satisfying the following: If G is a graph on S_g and H is a 5-colored subgraph of G with at most q vertices, then the 5-coloring of H can be extended to a 5-coloring of G unless there is a graph H' with at most (g, q) vertices such that $H \quad H' \quad G$ and the 5-coloring of H cannot be extended to a 5-coloring of H'.

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Corollary

For every (6, q)-critical graph G on S_g $|V(G)|$ (g, q).

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Corollary

For every (6, q)-critical graph G on S_g $|V(G)|$ (g, q).

Corollary

For any surface S_g there are finitely many (6, q)-critical graphs on S_g .

list-critical graphs

G is list-critical if G is not list colorable, but every subgraph is.

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G is list-critical if G is not list colorable, but every subgraph is.

Fact

Any vertex v in a list-critical graph has degree at least $|L(v)|$ and v is called small if equality holds

Theorem

For every fixed surface S, there are only finitely many list-critical graphs on S having all lists of cardinality at least 6.

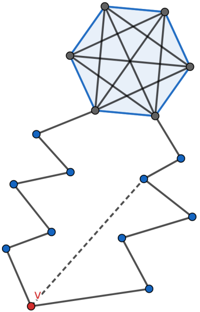
Corollary

For every fixed surface S and natural number k 7, there are only finitely many k-color-critical graphs on S.

Lemma (Gallani)

Let G be a 2-connected graph with the following property: If C is an even cycle in G and v 2 V(C), then C has a chord incident with v. Then G is either an odd cycle or a complete graph.

Induction on |V(G)|



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Let H be a list-critical graph and let H' be the subgraph of H induced by the small vertices. Then each block of H' is a complete graph or an odd cycle.

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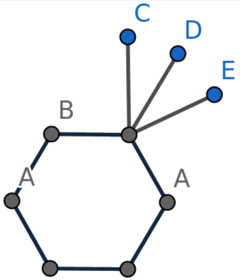

Theorem

Let G be a graph with maximum degree d, and let L be a list assignment such that $|L(v)|$ d for each vertex v of G. Then G has a list-coloring unless G has a component which is a K_{d+1} or and odd cycle if d=2.

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Theorem

Let q, g be natural numbers. Let G be a graph on S_g . Let L be a list assignment of G and let S be a set of at most q vertices in G such that |L(v)| 6 for each v *2* V(G)*n*S. If G is list-critical, then |V(G)| 150(g+q).

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For each fixed surface and for any natural numbers k, q where k 7, there are only finitely many (k, q)-critical graphs on the surface.

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Corollary

Let g be a fixed natural number. There exists a polynomially bounded algorithm for deciding if there exists a list coloring of a graph G when all lists have at least 6 colors and G can be drawn on S_g. Moreover, the algorithm gives the list coloring if it exists.

Thank you!