Farey sequence and Graham's conjectures

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Introduction

Notation:

- (a, b) - greatest common divisor of a and b
- $[a, b]$ - least common multiple of a and b

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- $F_6 = \left\{\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1}\right\}$

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$$\mathcal{Q}(S) = \left\{ \frac{x}{y} : x, y \in S, x \leqslant y \text{ and } y \neq 0 \right\}.$$

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$$\mathcal{Q}(S) = \left\{ 0, \frac{3}{8}, \frac{5}{12}, \frac{9}{10}, 1 \right\}$$

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We are mainly interested in sets $S \subseteq F_n$, such that $\mathcal{Q}(S) \subseteq F_n$ for some n.

Graham's Conjectures

Conjecture 1.

Let $a_1, a_2, \dots, a_n$ be distinct positive integers, we have

$$\max_{i,j} \frac{a_i}{(a_i, a_j)} \geqslant n$$

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For example, for $a = (4, 6, 8, 12)$ we have

$$\max_{i,j} \frac{a_i}{(a_i, a_j)} = \frac{8}{(8,6)} = \frac{8}{2} = 4 \geqslant 4$$

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- $\exists_i \, a_i$ is prime - Winterle (1970)
- $n = p + 1$ - Vélez (1977)
- $n = p$ - Szemerédi (1977)
- $\exists_{i, p > (n-1)/2} \, p \mid a_i$ - Boyle (1977)
- sufficiently large n - Szegedy, Zaharescu (1986-87)
- general case - Balasubramanian, Soundararajan (1996)

Graham's Conjectures

Let $M_n = lcm(1, 2, \ldots, n)$.

Conjecture 2.

Let $a_1 < a_2 < \cdots < a_n$ be distinct positive integers,

$$gcd(a_1, a_2, \ldots, a_n) = 1$$

and

$$\max_{i,j} \frac{a_i}{(a_i, a_j)} = n.$$

Then $\{a_1, a_2, \ldots, a_n\}$ can only be $\{1, 2, \ldots, n\}$ or $\left\{\frac{M_n}{n}, \frac{M_n}{n-1}, \ldots, \frac{M_n}{1}\right\}$ except for $n = 4$, where we have the additional sequence $\{2, 3, 4, 6\}$.

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- sufficiently large n - Szegedy, Zaharescu (1986-87)
- $n > 10^{50000}$ - Cheng, Pomerance (1994)
- general case - Balasubramanian, Soundararajan (1996)

Conjecture 1.

Let $a_1, a_2, \ldots, a_n$ be distinct positive integers, we have

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We get the following result from Conjecture 1.

Theorem 1.

Suppose $S \subseteq F_n$, if $\mathcal{Q}(S) \subseteq F_n$, the S has at most $n+1$ elements.

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We get the following result from Conjecture 1.

Theorem 1.

Suppose $S \subseteq F_n$, if $\mathcal{Q}(S) \subseteq F_n$, the S has at most $n+1$ elements.

In fact, the above theorem is equivalent to Conjecture 1.

Theorem 2.

Conjecture 1 is equivalent to Theorem 1.

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We prove by contradiction.
Suppose there is a subset $S \subseteq F_n$ such that $\mathcal{Q}(S) \subseteq F_n$, but $|S| \geqslant n+2$.
Then $S' = S \backslash \{0\}$ has at least $n+1$ distinct elements x_k/y_k with $(x_k, y_k) = 1$.

$$\frac{x_1}{y_1} < \frac{x_2}{y_2} < \cdots < \frac{x_n}{y_n} < \frac{1}{1}$$

Theorem 1.

Suppose $S \subseteq F_n$, if $\mathcal{Q}(S) \subseteq F_n$, the S has at most $n+1$ elements.

$$\frac{x_1}{y_1} < \frac{x_2}{y_2} < \cdots < \frac{x_n}{y_n} < \frac{1}{1}$$

Let $a_{n+1} = [x_1, \ldots, x_n]$, $a_k = a_{n+1} \cdot \frac{y_k}{x_k}$.

$$a_{n+1} \cdot \frac{y_1}{x_1} > a_{n+1} \cdot \frac{y_2}{x_2} > \cdots > a_{n+1} \cdot \frac{y_n}{x_n} > a_{n+1}$$

$$a_1 > a_2 > \cdots > a_n > a_{n+1}$$

For $1 \leqslant i < j \leqslant n$:

$$\begin{aligned}(a_i, a_j) &= \left(a_{n+1} \cdot \frac{y_i}{x_i}, a_{n+1} \cdot \frac{y_j}{x_j}\right) \\ &= \frac{a_{n+1}}{[x_i, x_j]} \cdot (y_i, y_j) \left(\frac{[x_i, x_j]}{x_i} \cdot \frac{y_i}{(y_i, y_j)}, \frac{[x_i, x_j]}{x_j} \cdot \frac{y_j}{(y_i, y_j)}\right)\end{aligned}$$

For $1 \leqslant i < j \leqslant n$:

$$(a_i, a_j) = \frac{a_{n+1}}{[x_i, x_j]} \cdot (y_i, y_j) \left(\frac{[x_i, x_j]}{x_i} \cdot \frac{y_i}{(y_i, y_j)}, \frac{[x_i, x_j]}{x_j} \cdot \frac{y_j}{(y_i, y_j)} \right)$$

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Note that the g.c.d. part equals 1, because

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- $\frac{[x_i,x_j]}{x_i} \mid x_j \wedge \frac{y_j}{(y_i,y_j)} \mid y_j \wedge (x_j, y_j) = 1 \implies \left(\frac{[x_i,x_j]}{x_i}, \frac{y_j}{(y_i,y_j)} \right) = 1$

For $1 \leqslant i < j \leqslant n$:

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Note that the g.c.d. part equals 1, because

- $\left(\frac{[x_i,x_j]}{x_i}, \frac{[x_i,x_j]}{x_j} \right) = 1$
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- $\frac{y_i}{(y_i,y_j)} \mid y_i \wedge \frac{[x_i,x_j]}{x_j} \mid x_i \wedge (x_i, y_i) = 1 \implies \left(\frac{y_i}{(y_i,y_j)}, \frac{[x_i,x_j]}{x_j} \right) = 1$

Hence

$$(a_i, a_j) = \frac{a_{n+1}}{[x_i, x_j]}(y_i, y_j)$$

For $j = n + 1$:

$$(a_i, a_{n+1}) = \left(\frac{a_{n+1}}{x_i} y_i, \frac{a_{n+1}}{x_i} x_i \right) = \frac{a_{n+1}}{x_i}$$

For $j = n + 1$:

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We can therefore extend previous result to $1 \leqslant i < j \leqslant n + 1$.

$$(a_i, a_j) = \frac{a_{n+1}}{[x_i, x_j]} (y_i, y_j)$$

and

$$\frac{a_i}{(a_i, a_j)} = \frac{a_i}{a_{n+1}} \cdot \frac{[x_i, x_j]}{(y_i, y_j)} = \frac{y_i}{x_i} \cdot \frac{[x_i, x_j]}{(y_i, y_j)}$$

On the other hand,

$$\frac{x_i}{y_i} \Big/ \frac{x_j}{y_j} = \frac{x_i y_j}{x_j y_i} = \frac{x_i}{(x_i, x_j)} \cdot \frac{y_j}{(y_i, y_j)} \Big/ \left(\frac{x_j}{(x_i, x_j)} \cdot \frac{y_i}{(y_i, y_j)} \right)$$

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Theorem 1.

Suppose $S \subseteq F_n$, if $\mathcal{Q}(S) \subseteq F_n$, the S has at most $n+1$ elements.

The fraction in the right side is in its lowest term and since $\mathcal{Q}(S') \subseteq F_n$, we have

$$\frac{x_j}{(x_i, x_j)} \cdot \frac{y_i}{(y_i, y_j)} \leqslant n$$

$$\frac{[x_i, x_j]}{x_i} \cdot \frac{y_i}{(y_i, y_j)} \leqslant n$$

For $1 \leqslant i < j \leqslant n+1$:

$$\frac{a_i}{(a_i, a_j)} = \frac{y_i}{x_i} \cdot \frac{[x_i, x_j]}{(y_i, y_j)}$$

$$\frac{[x_i, x_j]}{x_i} \cdot \frac{y_i}{(y_i, y_j)} \leqslant n$$

$$\frac{a_i}{(a_i, a_j)} \leqslant n$$

This contradicts Conjecture 1.

Conjecture 1.

Let $a_1, a_2, \ldots, a_n$ be distinct positive integers, we have

$$\max_{i,j} \frac{a_i}{(a_i, a_j)} \geqslant n$$

Theorem 2.

Conjecture 1 is equivalent to Theorem 1.

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We have already shown that Conjecture 1 implies Theorem 1. Now we assume Theorem 1 and give a proof by contradiction of Conjecture 1.

Conjecture 1.

Let $a_1, a_2, \ldots, a_n$ be distinct positive integers, we have

$$\max_{i,j} \frac{a_i}{(a_i, a_j)} \geqslant n$$

Suppose there are $n+1$ distinct positive integers $a_1 < a_2 < \cdots < a_{n+1}$ such that

$$\max_{i,j} \frac{a_i}{(a_i, a_j)} \leqslant n$$

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Let $x_k = \frac{a_1}{(a_1, a_k)}$, $y_k = \frac{a_k}{(a_1, a_k)}$. Then:

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Let $x_k = \frac{a_1}{(a_1, a_k)}$, $y_k = \frac{a_k}{(a_1, a_k)}$. Then:

- $(x_k, y_k) = 1$

Conjecture 1.

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Let $x_k = \frac{a_1}{(a_1, a_k)}$, $y_k = \frac{a_k}{(a_1, a_k)}$. Then:

- $(x_k, y_k) = 1$
- $\frac{x_k}{y_k} = \frac{a_1}{a_k}$ are distinct reduced fractions

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Let $x_k = \frac{a_1}{(a_1,a_k)}$, $y_k = \frac{a_k}{(a_1,a_k)}$. Then:

- $(x_k, y_k) = 1$
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- $x_k \leqslant y_k \leqslant n$

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Let $x_k = \frac{a_1}{(a_1, a_k)}$, $y_k = \frac{a_k}{(a_1, a_k)}$. Then:

- $(x_k, y_k) = 1$
- $\frac{x_k}{y_k} = \frac{a_1}{a_k}$ are distinct reduced fractions
- $x_k \leqslant y_k \leqslant n$
- $\frac{x_k}{y_k} \in F_n$

Thus $S = \left\{0, \frac{x_1}{y_1}, \frac{x_2}{y_2}, \ldots, \frac{x_{n+1}}{y_{n+1}}\right\}$ is a subset of F_n.

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But also for every $i \leqslant j$, $\frac{x_i}{y_i} \geqslant \frac{x_j}{y_j}$ we have

$$\frac{x_j}{y_j} \Big/ \frac{x_i}{y_i} = \frac{a_1/a_j}{a_1/a_i} = \frac{a_i}{a_j} = \frac{a_i/(a_i, a_j)}{a_j/(a_i, a_j)} \in F_n$$

$$\max_{i,j} \frac{a_i}{(a_i, a_j)} \leqslant n$$

$$S = \left\{0, \frac{x_1}{y_1}, \frac{x_2}{y_2}, \dots, \frac{x_{n+1}}{y_{n+1}}\right\}$$

$$\forall_{x,y\in S, x\leqslant y, y\neq 0} \frac{x}{y} \in F_n$$

Thus $\mathcal{Q}(S) \subseteq F_n$, but $|S| \geqslant n+2$, which contradicts Theorem 1.

Theorem 1.

Suppose $S \subseteq F_n$, if $\mathcal{Q}(S) \subseteq F_n$, the S has at most $n+1$ elements.

□

Theorem 3.

Suppose $S \subseteq F_n$, $|S| = n + 1$ and $\mathcal{Q}(S) \subseteq F_n$, then S can only be one of the following sets:

- $S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\right\}$
- $S = \left\{0, 1, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\right\}$
- $S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}\right\}$ for $n = 4$

The result is based on Graham's second conjecture.

Theorem 4.

Suppose $S \subseteq F_n$ and $\mathcal{Q}(S) = F_n$, then S can only be one of the following two sets:

- $S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\right\}$
- $S = \left\{0, 1, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\right\}$