The 3-flow conjecture, factors modulo k and the 1-2-3 conjecture

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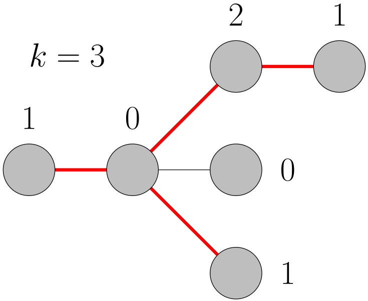
f-factor modulo k

Definition

Let G be a graph, let $k \geqslant 2$ be a number and let $f : V(G) \to \mathbb{Z}_k$ be a function. Then a spanning subpgraph H of G is called *f-factor modulo k* of G, if for each $v \in V(G)$

$$f(v) \equiv \deg_H(v) \pmod{k}$$



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There are some obvious necessary conditions for the existance of f-factor modulo k:

- $\sum_{v \in V(G)} f(v) \in 2\mathbb{Z}_k$
- If G is bipartite with bipartition $\{V_1, V_2\}$ then

$$\sum_{v \in V_1} f(v) - \sum_{v \in V_2} f(v) \equiv 0 \pmod{k}$$

About edge-connectivity

Theorem (Thomassen, 2008)

Let $k \geqslant 3$ be an odd integer, let G be a bipartite, $(3k-3)$-edge-connected graph with bipartition $\{V_1, V_2\}$ and $f : V(G) \to \mathbb{Z}_k$ such that

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Lemma (Thomassen, 2008)

Let q be a positive integer, let G be a $(2q-1)$-edge-connected. Then the bipartite subgraph of G induced by maximum cut is q-edge-connected.

Bipartite index

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- $bi(G) = |E(G)| - |\text{max-cut}(G)|$

Existance of f-factor modulo k

Theorem

Let $k \geqslant 3$ be an odd integer, G be a $(6k-7)$-edge-connected graph with $bi(G) \geqslant k-1$, then for any function $f : V(G) \to \mathbb{Z}_k$ G has an f-factor modulo k.

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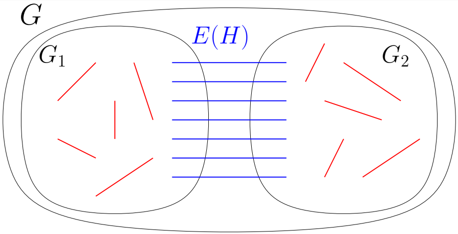

Let H be a subgraph induced by maximum cut and G_1, G_2 are graphs induced by partitions. By previous lemma, H is $(3k-3)$-edge-connected.

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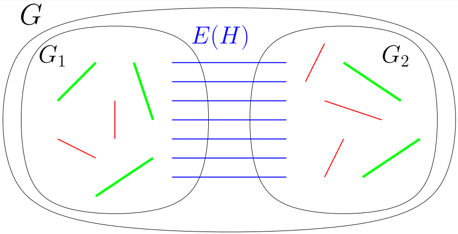

Let M be some subgraph of $G_1 \cup G_2$. Define $f'(v) = f(v) - \deg_M(v)$. If

$$\sum_{v \in V(G_1)} f'(v) - \sum_{v \in V(G_2)} f'(v) \equiv 0 \pmod{k}$$

Then, by previous theorem, H has f'-factor modulo k.

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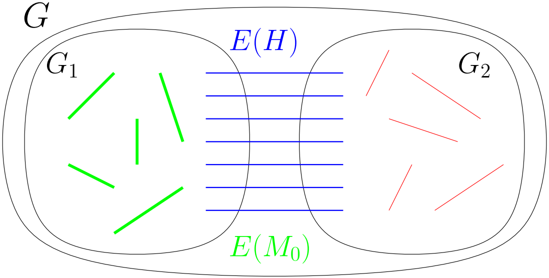

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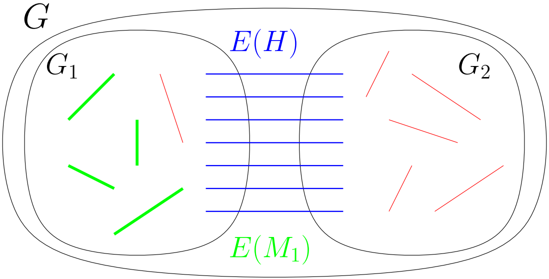

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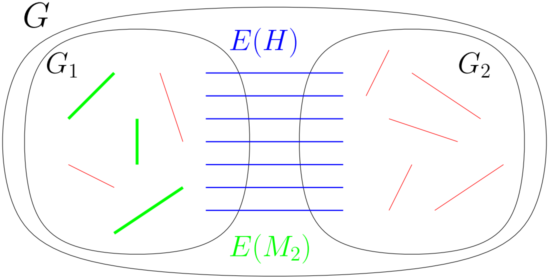

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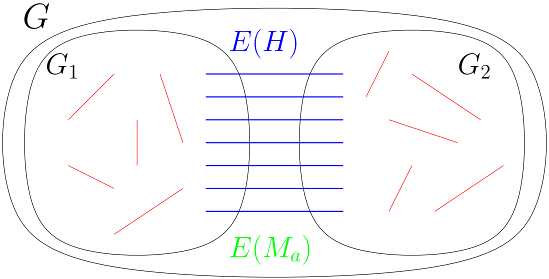

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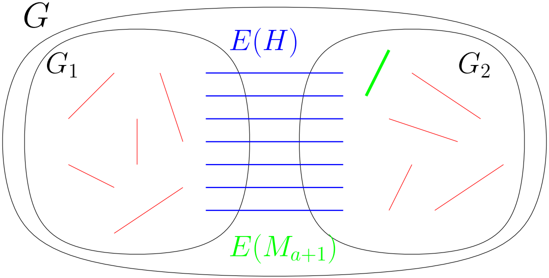

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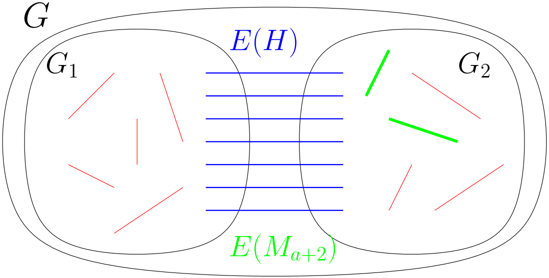

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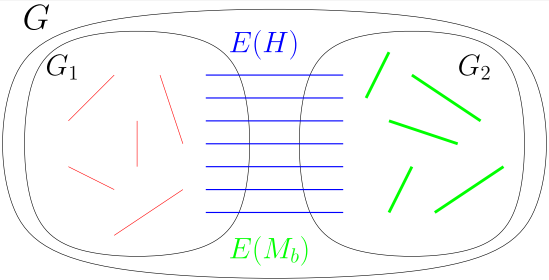

Where $b = |E(G_1)| + |E(G_2)| = bi(G)$.

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Define f_i' and s_i as follows:

$$f_i'(v) = f(v) - \deg_{M_i}(v)$$
$$s_i = \sum_{v \in V(G_1)} f_i'(v) - \sum_{v \in V(G_2)} f_i'(v)$$

Then $s_i = s_0 - 2i$ for $0 \leqslant i \leqslant bi(G)$. Since k is odd and $bi(G) \geqslant k-1$, there exists l such that $s_l \equiv 0 \pmod{k}$. We can take $M = M_l$.

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Let F be an f'-factor modulo k of graph G. Then $F \cup M$ is an f-factor modulo k.

Existance of f-factor modulo k

Theorem

Let $k \geqslant 5$ be an odd integer, G be a $(6k-7)$-edge-connected graph with $\chi(G) \geqslant k$ or (if $k \geqslant 7$) $\chi(G) \geqslant k-1$, then for any function $f : V(G) \to \mathbb{Z}_k$ G has an f-factor modulo k.

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Let G_1, G_2 be as in the previous proof.

$$bi(G) = |E(G_1)| + |E(G_2)| \geqslant \binom{\chi(G_1)}{2} + \binom{\chi(G_2)}{2} \geqslant k-1$$

Thus, by the previous theorem, there exists f-factor modulo k of G.

1-2-3 conjecture

Definition

A *neighbour-distinguishing edge-weighting* is an assignment $w : E(G) \to \mathbb{Z}^+$ such that the induced labeling $w_V : V(G) \to \mathbb{Z}^+$, where $w_V(v) = \sum_{e \in E(v)} w(e)$ is a proper vertex-coloring of G.

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Every connected simple graph of order at least 3 has neighbour-distinguishing edge-weighting with weights $1, 2, 3$.

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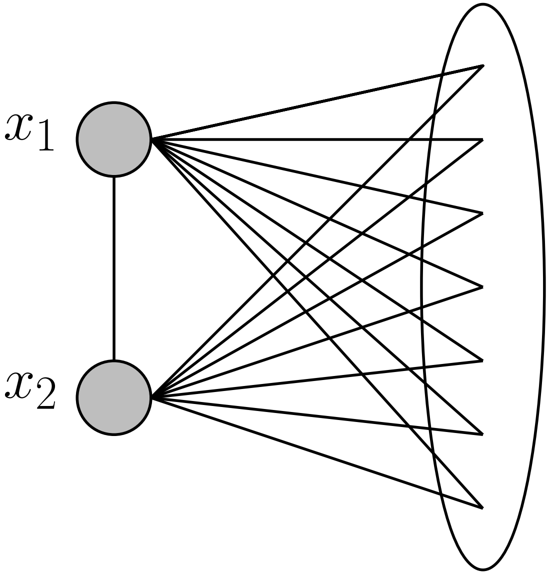
Definition

Graph G has *1-2-property* if it has neighbour-distinguishing edge-weighting with weights $1, 2$.

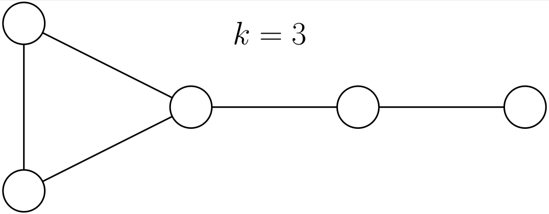
Graph $T_{2,\mu}$



Graphs with 1-2-property

Theorem

Let $k \geqslant 3$ be an odd number, and let G be a nonbipartite $(6k-7)$-edge-connected graph with chromatic number k or $k-1$. Then G has an edge-weighting with weights 1,2 such that its induced vertex-labeling reduced modulo k is a proper vertex-coloring unless G is a 3-chromatic graph of the form $T_{2,\mu}$ where $\mu \equiv 2 \pmod 3$.
Moreover, if $\chi(G) \geqslant 5$ and $c : V(G) \to \mathbb{Z}_k$, then the edge-weighting with weights 1,2 can be chosen such that its induced vertex-labeling is congruent to c modulo k.



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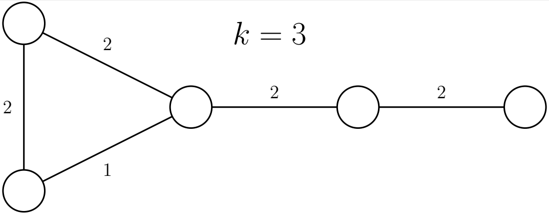

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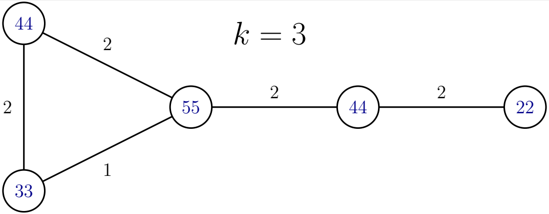

$k = 3$

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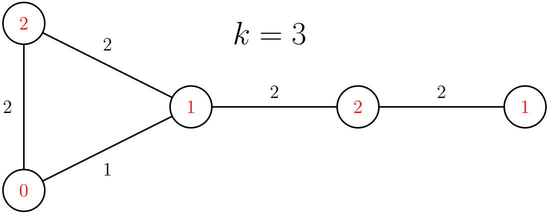

Graph $T_{2,\mu}$

Corollary

Graph $T_{2,\mu}$ *with* $\mu \equiv 2 \pmod 3$. *does not admit an edge-weightening with weights 1,2 such that its induced vertex-labeling reduced modulo 3 is a proper vertex-coloring.*
However, $T_{2,\mu}$ *has 1-2-property.*

Bipartite graphs with 1-2-property

Lemma

Let G be a connected graph and $f : V(G) \to \mathbb{Z}_2$ be a function satisying $\sum_{v \in V(G)} f(v) \equiv 0 \pmod 2$. Then G has an f-factor modulo 2.

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Corollary

Let G be a connected bipartite graph. G has 1-2-property unless each bipartite class has odd number of vertices.

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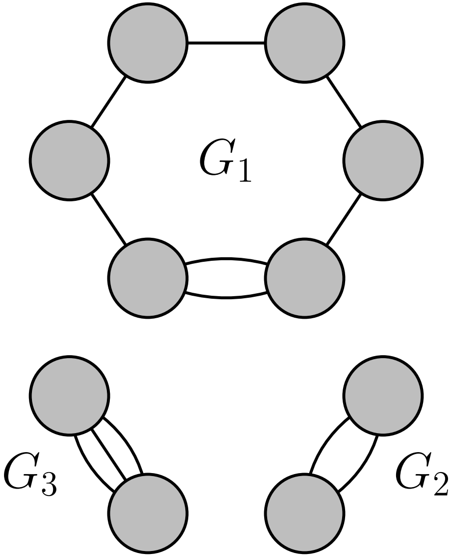
Let G be a connected bipartite graph. G has 1-2-property unless each bipartite class has odd number of vertices.

Theorem

Let G be a simple connected bipartite graph of minimum degree at least 3. Then G has 1-2-property.

Odd multi-cactus



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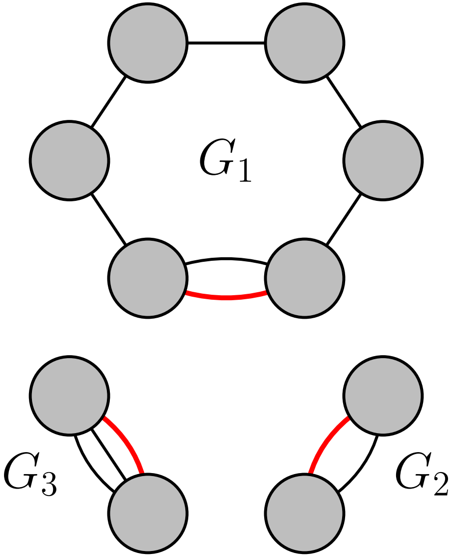

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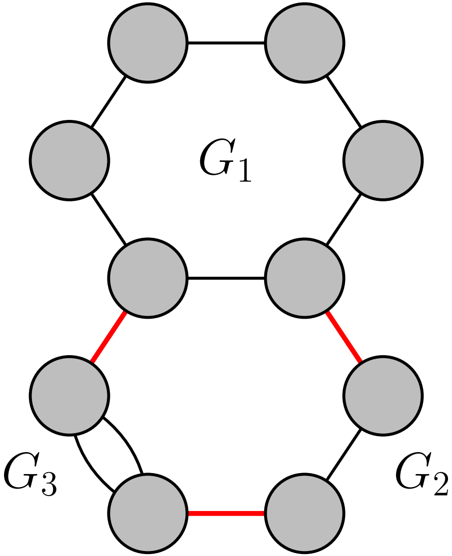

Bipartite graphs without 1-2-property

Corollary

Let G be an odd multi-cactus. Then G doesn't have 1-2-property.

Theorem

Let G be a connected bipartite graph without 1-2-property. Then G is an odd multi-cactus.

Thank you!