# Improved Lower Bound for the List Chromatic Number of Graphs with no $K_t$-minor

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# Roadmap

1. Definitions
2. Problem setting
3. Main results

# Roadmap

# Definitions

## List assignment

A *list assignment* of a given undirected graph $G$ is an assignment $L : V(G) \to \mathcal{P}(\mathbb{N})$ of finite sets $L(v)$ (called lists) to vertices $v \in V(G)$.

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An *L-coloring* of an undirected graph $G$ and list assignment $L$ is a function $c : V(G) \to \mathbb{N}$, such that $c(v) \in L(v)$ for every $v \in V(G)$ and $c(u) \neq c(v)$ for every $\{u, v\} \in E(G)$.

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## List chromatic number

The *list chromatic number* $\chi_\ell(G)$ of an undirected graph $G$ is the smallest number $k \in \mathbb{N}$ such that $G$ admits an $L$-coloring for every list assignment $L$ for which $|L(v)| \geqslant k$ for every $v \in V(G)$.

# Definitions

## $K_t$-minor

For a given $t \in \mathbb{N}$, we say a graph $G$ has a $K_t$-minor if there exist pairwise disjoint, non-empty subsets $Z_1, Z_2, \dots, Z_t \subseteq V(G)$, such for each $i$ the induced subgraph $G[Z_i]$ is connected and for every $i \neq j \in [t]$ there exist $u \in Z_i, v \in Z_j$ such that $\{u, v\} \in E(G)$.

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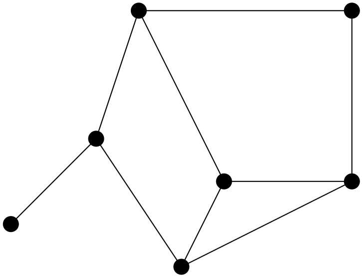

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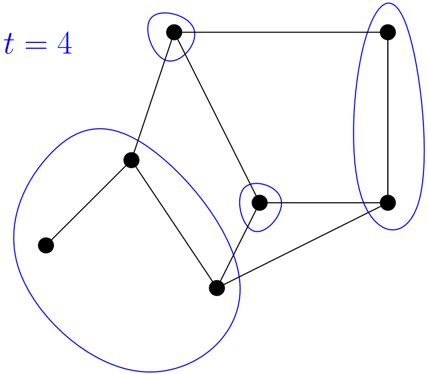

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# Hadwiger's conjecture

**Conjecture (Hadwiger - 1943)**

For every $t \in \mathbb{N}$, if $G$ does not contain a $K_t$-minor, then $\chi(G) \leqslant t - 1$, where $\chi(G)$ is the chromatic number of $G$.

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There exists an absolute constant $c > 0$ for which every $G$ not containing a $K_t$-minor, satisfies $\chi(G) \leqslant ct$.

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Current best bound is $\chi(G) \in O(t \log \log t)$ (Delcourt and Postle - 2021).

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- $\chi_\ell(G) \in O(t(\log \log t)^2)$ (Delcourt and Postle - 2021)

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- $c \geqslant 2$ **(Steiner 2021)**

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- $c \geqslant 2$ **(Steiner 2021)**
- **Question:** $c \leqslant 2$**? (Steiner 2021)**

# Roadmap

# Main Theorem

**Theorem**

For every $\varepsilon \in (0, 1)$ there is a $t_0 = t(\varepsilon)$ such that for every $t \geqslant t_0$ there exists an undirected graph with no $K_t$-minor and list chromatic number at least $(2 - \varepsilon)t$.

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**Corollary**

For every constant $c < 2$ there is a $t_0$ such that for $t \geqslant t_0$ there exists an undirected graph $G$ with no $K_t$-minor and $\chi_\ell(G) > ct$.

# Main Theorem - Construction

## Bipartite Erdős-Renyi Graph

For $n \in \mathbb{N}$ and $p \in [0, 1]$ we define $G(n, n, p)$ as a random bipartite graph $G$ with bipartition $V(G) = A \cup B, A \cap B = \emptyset$ such that $|A| = |B| = n$ and $\mathbb{P}((a, b) \in E) = p$ for every $a \in A, b \in B$ with probabilities for every pair being independent.

# Main Theorem - Construction

## Lemma 1

Let $\varepsilon \in (0,1)$ be fixed and now let $f = f(\varepsilon) \in \mathbb{N}, \delta = \delta(\varepsilon) \in (0,1)$ be constants chosen such that $f\delta < 1$. Let $p = p(n) = n^{-\delta}$. Then $\mathcal{P} \to 1$ as $n \to \infty$ where $\mathcal{P}$ is the probability that the graph $G(n, n, p(n))$ satisfies the following properties:

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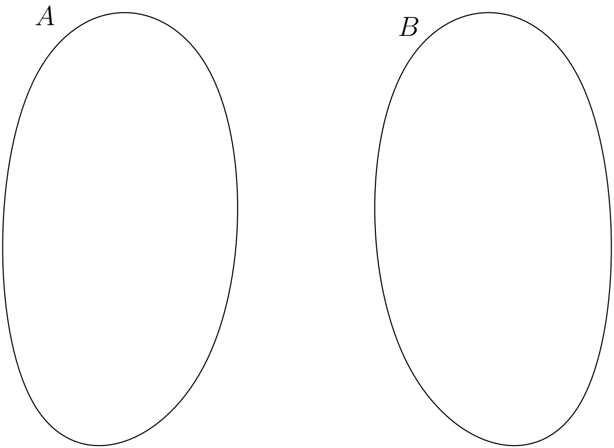

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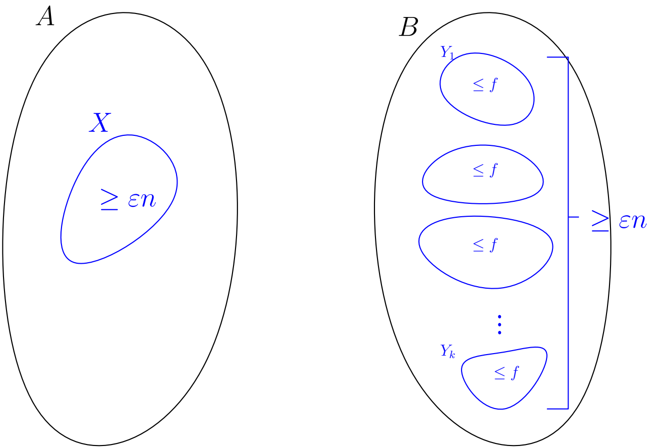

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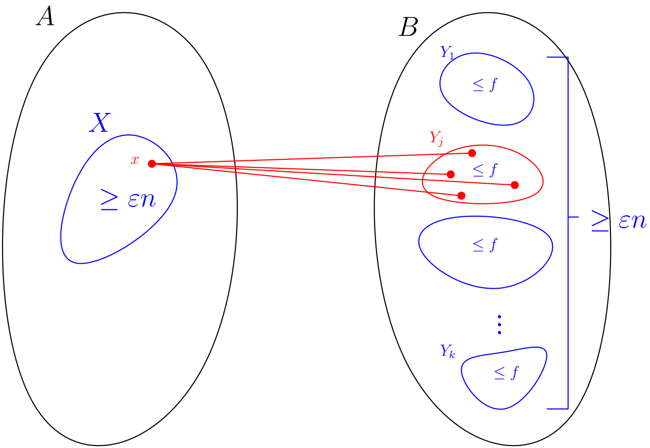

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- Chernoff bounds + union bound to prove second property.

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**Lemma 2**

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## Proof sketch

Taking $H$ to be the complement of the graph from Lemma 1 is sufficient. Only the third property is non-trivial.

# Lemma 2

**Complement of graph from Lemma 1**

For every $X \subseteq A$ such that $|X| \geqslant \varepsilon n$ and every familty of disjoint non-empty subsets $Y_1, Y_2, \ldots, Y_k \subseteq B$ such that $k \geqslant \varepsilon n$ and $|Y_i| \leqslant f$, there exists a vertex $x \in X$ and a $j \in [k]$ for which $x$ has no edges to any $y \in Y_j$.

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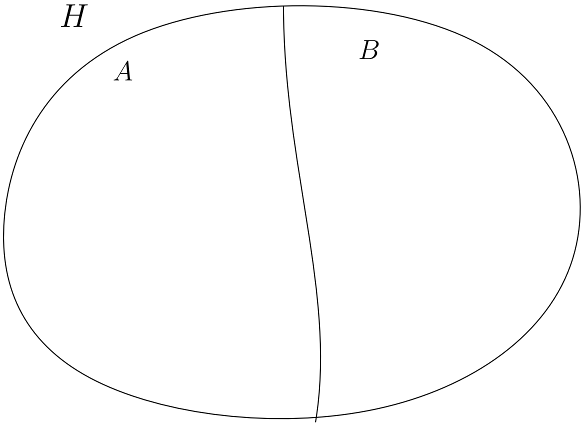

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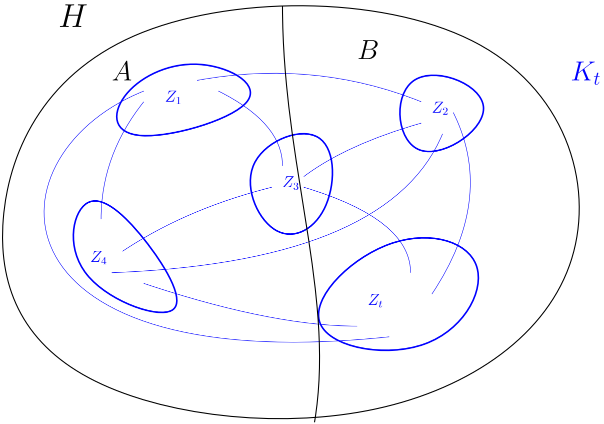

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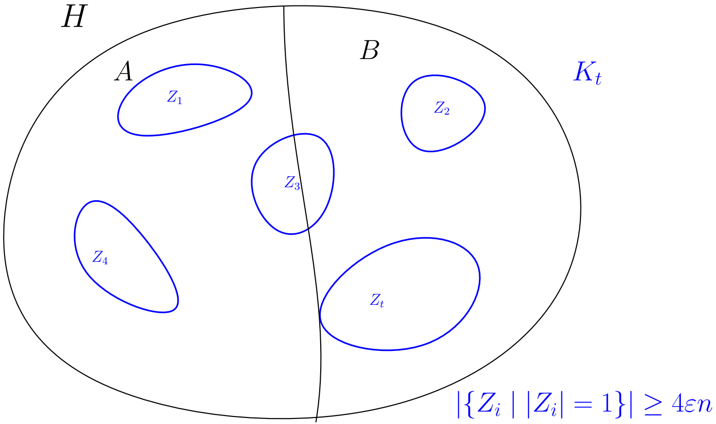

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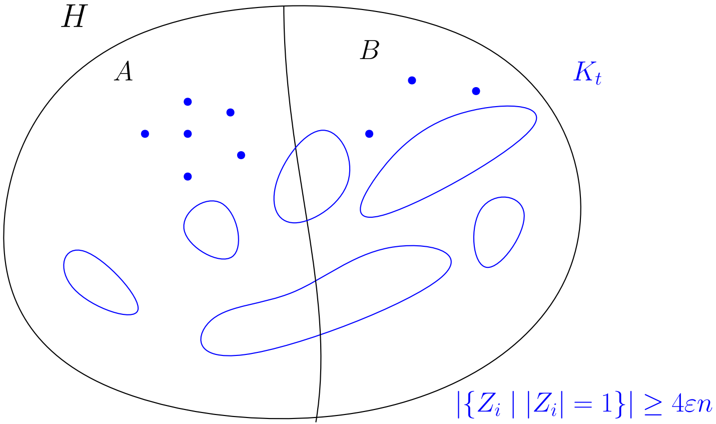

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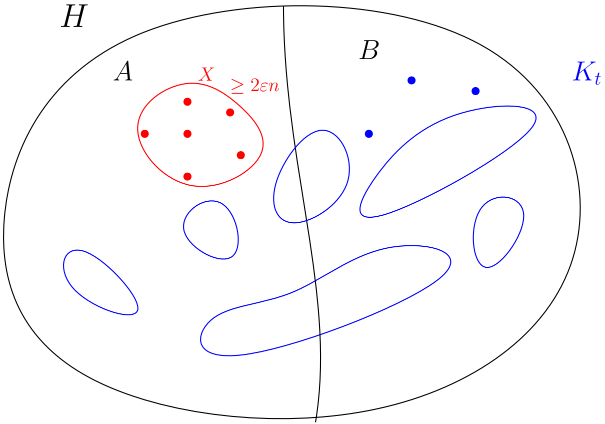

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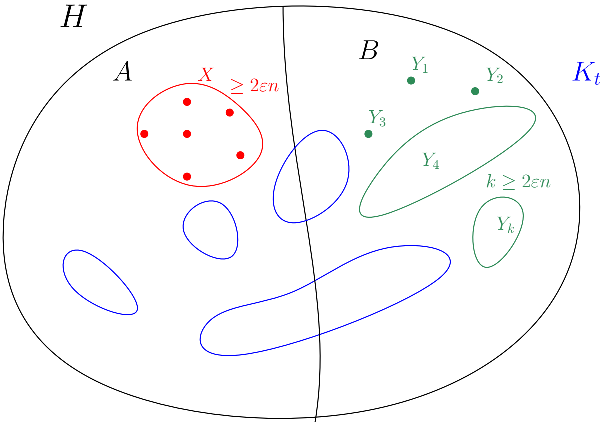


$f = \lfloor 1/\varepsilon \rfloor$

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## Lemma 2

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## Pasting Lemma

Let $G_1$ and $G_2$ be $K_t$-minor-free graphs and $V(G_1) \cap V(G_2) = C$. If both $G_1[C]$ and $G_2[C]$ are cliques, then $G_1 \cup G_2$ is also has no $K_t$-minor.

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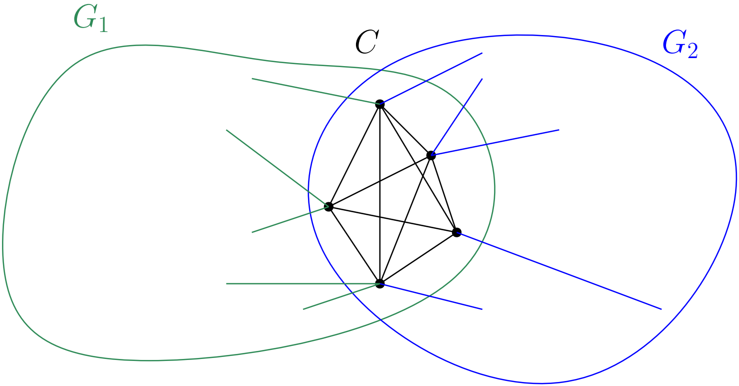

# Main Theorem - Construction

## Graph Construction

Let $\varepsilon$ be given. We know that there exists $n_0 = n_0(\varepsilon)$ such that there exists a graph which satisfies the conditions in Lemma 2. $V(G) = A \cup B$ for $A, B$ disjoint cliques of size $n$, every vertex has at most $\varepsilon n$ non-neighbors and $H$ is $K_t$-minor-free for $t \geqslant (1 + 2\varepsilon)n$.

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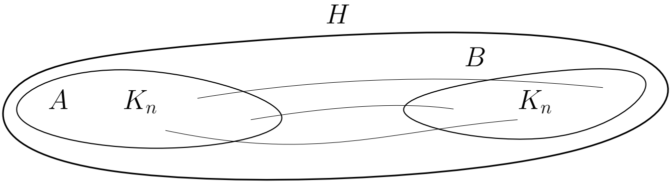
For every possible coloring $c \in [2n-1]^A$ of $A$ using colors from $[2n-1]$ we create a copy $H_c$ of $H$. Furthermore, these copies made in such a way, that they all share $A$, but have separate $B_c$ $(H_{c_1} \cap H_{c_2} = A)$.
From the Pasting Lemma we know, that the graph $\mathcal{G} = \bigcup_c H_c$ is $K_t$-minor-free for $t \geqslant (1 + 2\varepsilon)n$.

# Main Theorem - Graph $\mathcal{G}$



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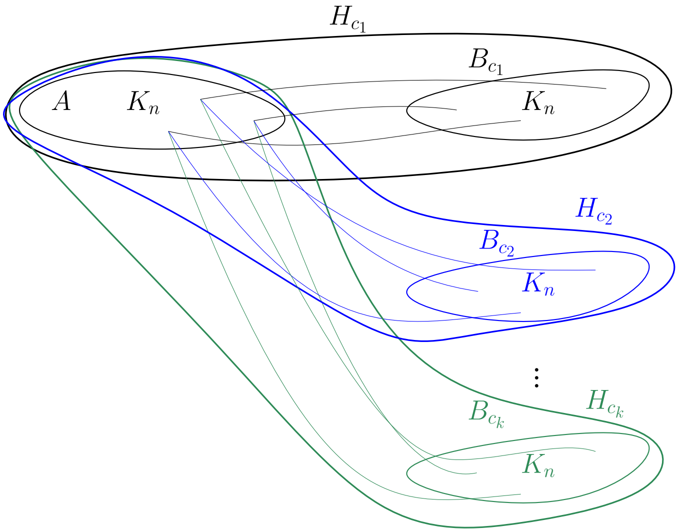

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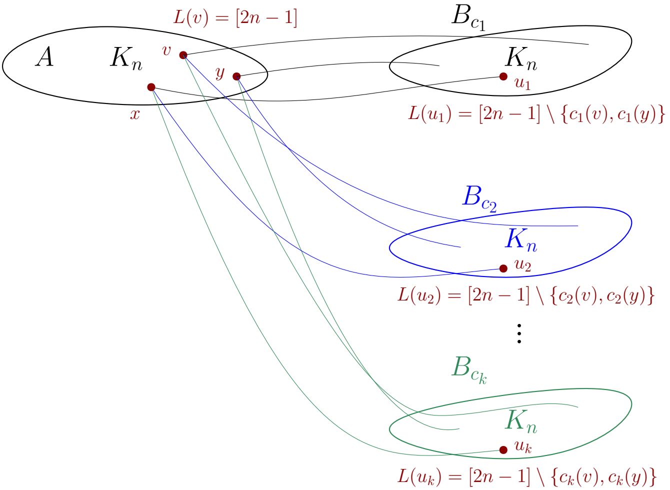

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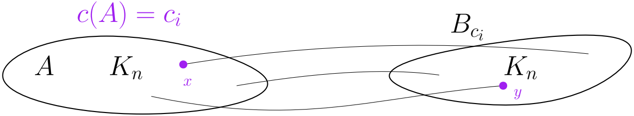


$$2n - 1 - \varepsilon n \leq |L(v)| \leq 2n - 1$$

$$|V(H_{c_i})| = 2n$$

$$c(x) = c(y) \qquad c(x) \notin L(y)$$

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## Summary

We showed that $\mathcal{G}$ is $K_t$-minor-free for $t \geqslant t_0 = (1 + 2\varepsilon)n$ and can't be colored using lists of length $\geqslant (2 - \varepsilon)n - 1$ colors. Therefore $\chi_\ell(\mathcal{G}) \geqslant (2 - \varepsilon)n$ for all $n \geqslant n_0$.

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## Proof

To achieve the bound $\chi_\ell(\mathcal{G}) \geqslant (2 - \varepsilon)t$, we substitute $\varepsilon, n_0, t_0$ in the proof with $\varepsilon', n_0', t_0'$ such that:

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Then for every $t \geqslant t_0$ and $n = \left\lfloor \frac{t}{1+2\varepsilon'} \right\rfloor$ we get that $\chi_\ell(\mathcal{G}) \geqslant (2 - \varepsilon')n$ implies $\chi_\ell(\mathcal{G}) \geqslant (2 - \varepsilon)t$, which finishes the proof of the Main Theorem.