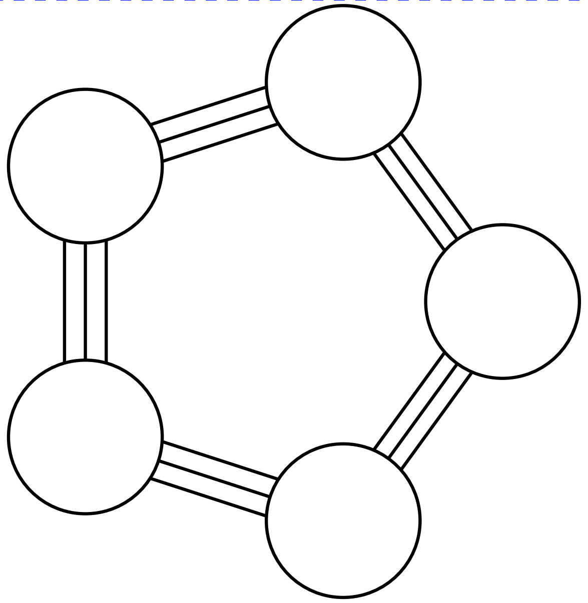
K_4-free graphs have sparse halves

Ignacy Buczek

State of the art

Does every triangle-free graph contain a subset of $\frac{n}{2}$ vertices with at most $\frac{n^2}{50}$ edges?



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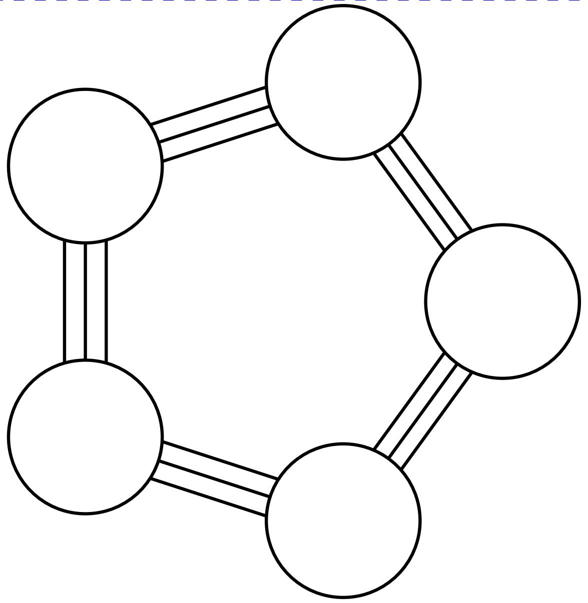

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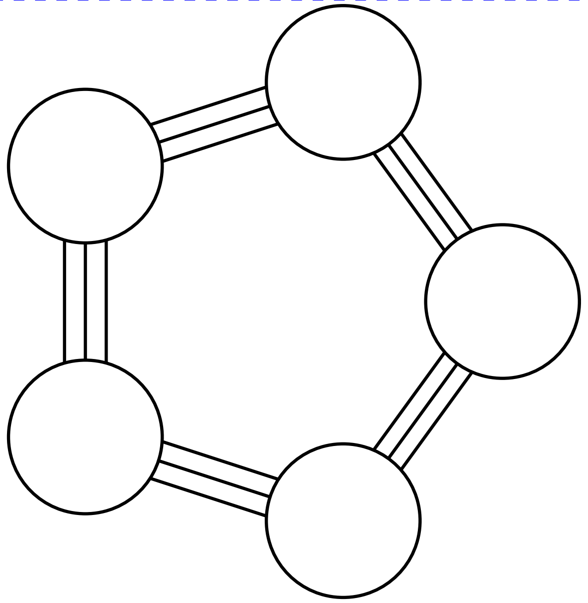

Can every triangle-free graph be made bipartite by removing at most $\frac{n^2}{25}$ edges?

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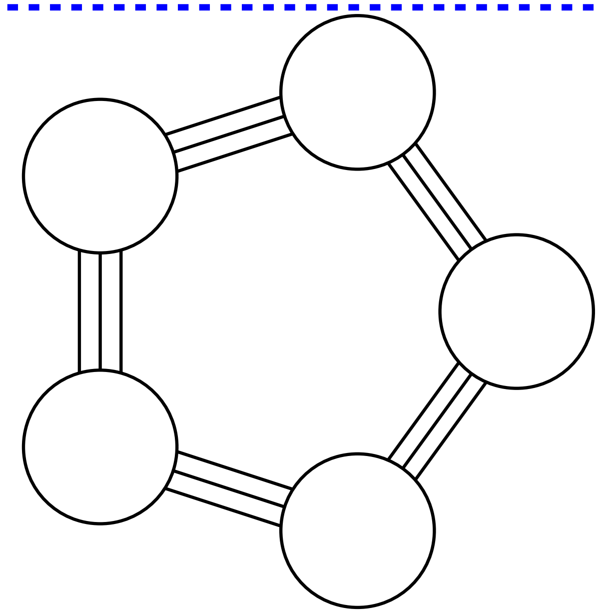

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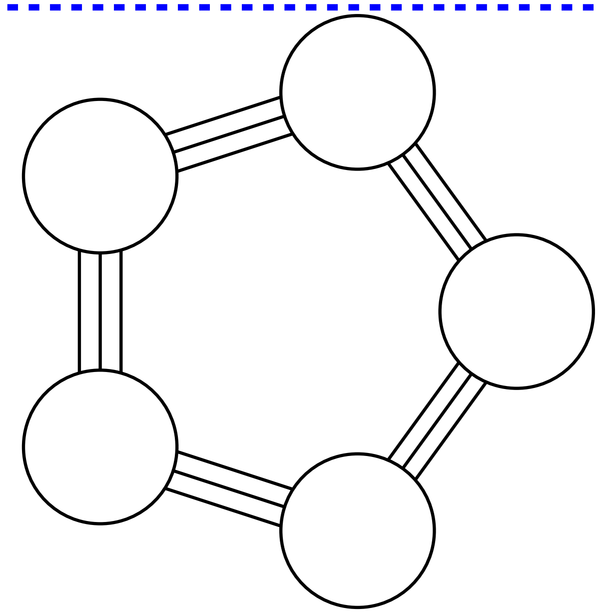

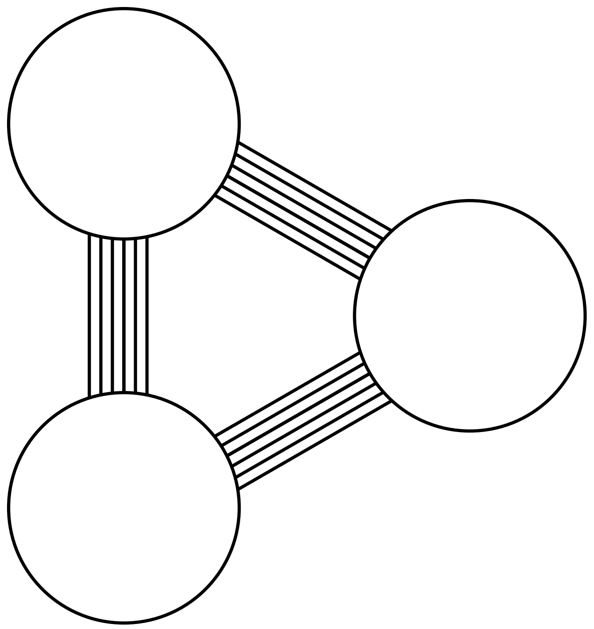

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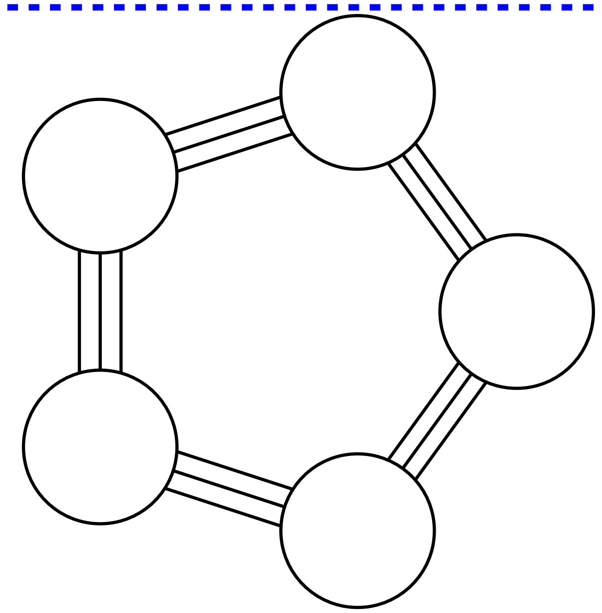

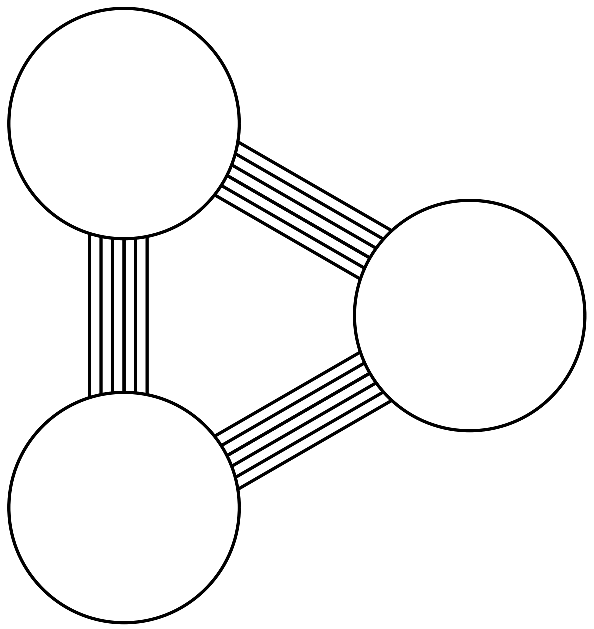

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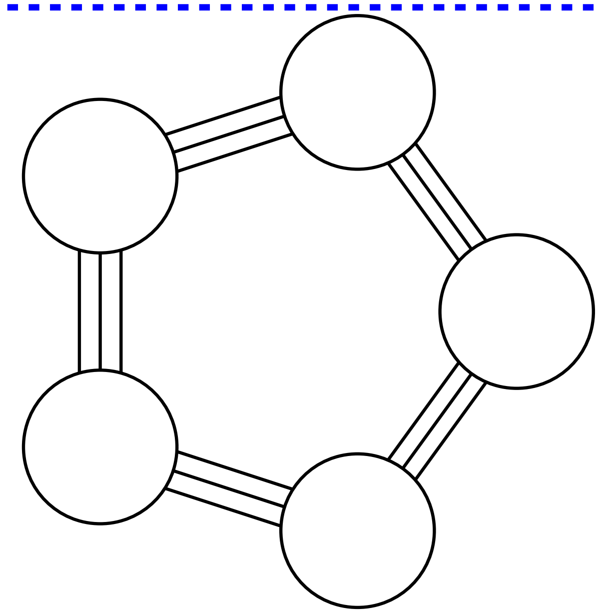

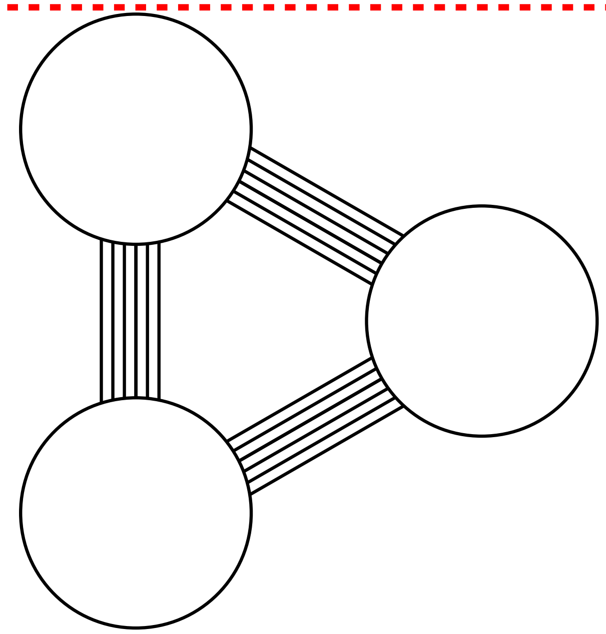

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Reiher, 2022
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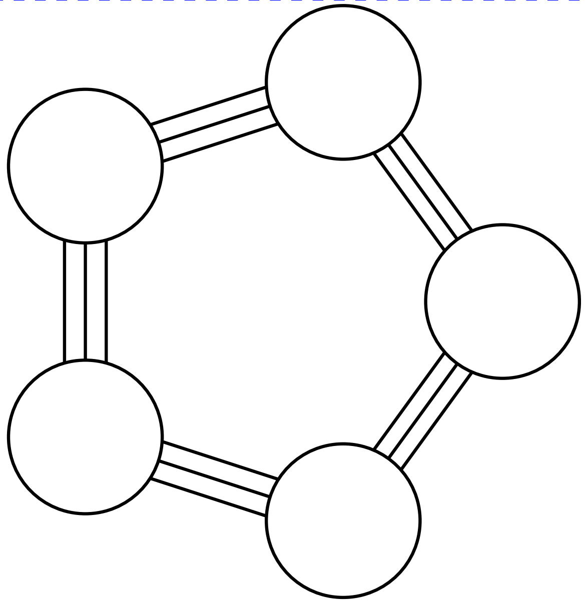

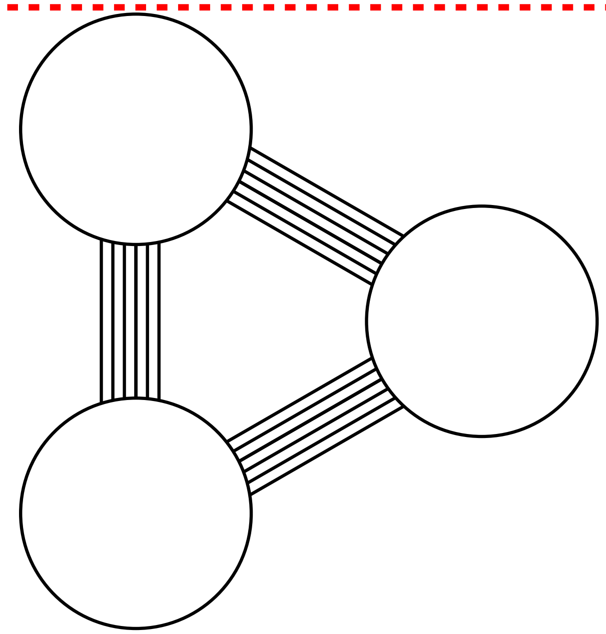

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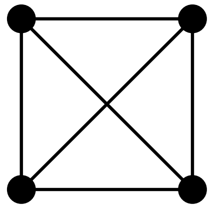
K_4-free graphs



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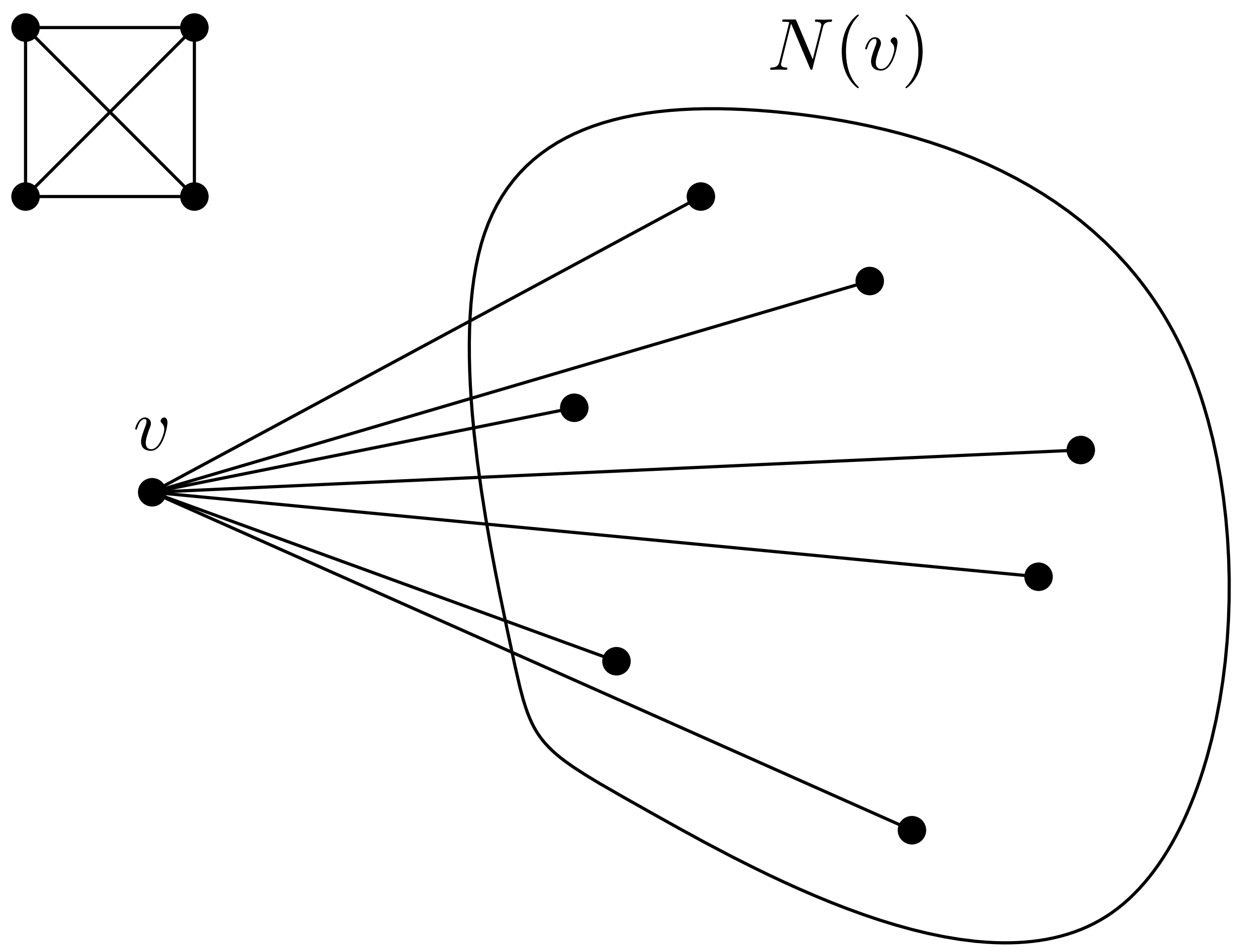

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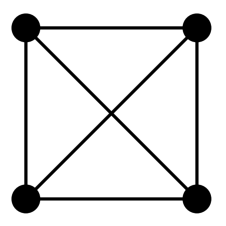

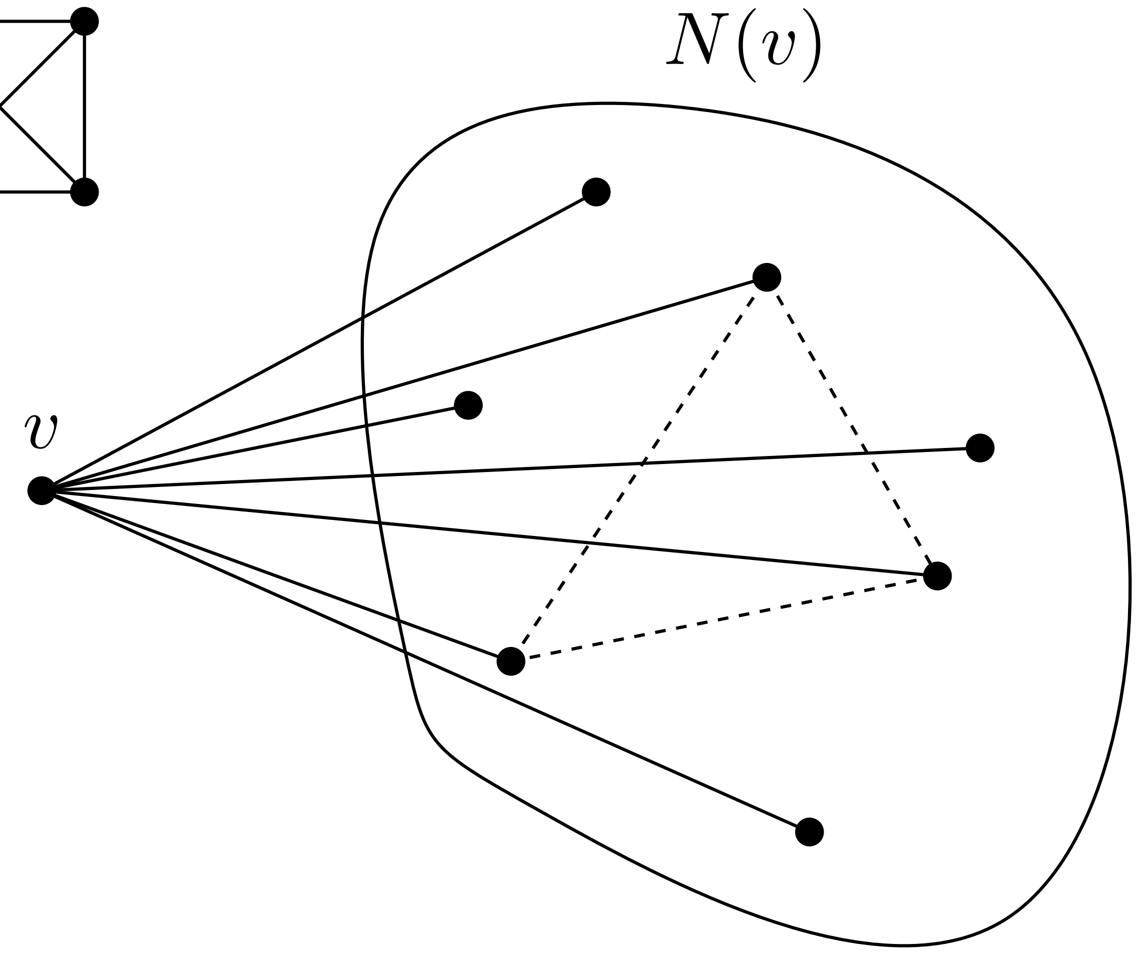

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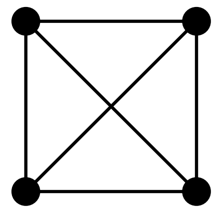
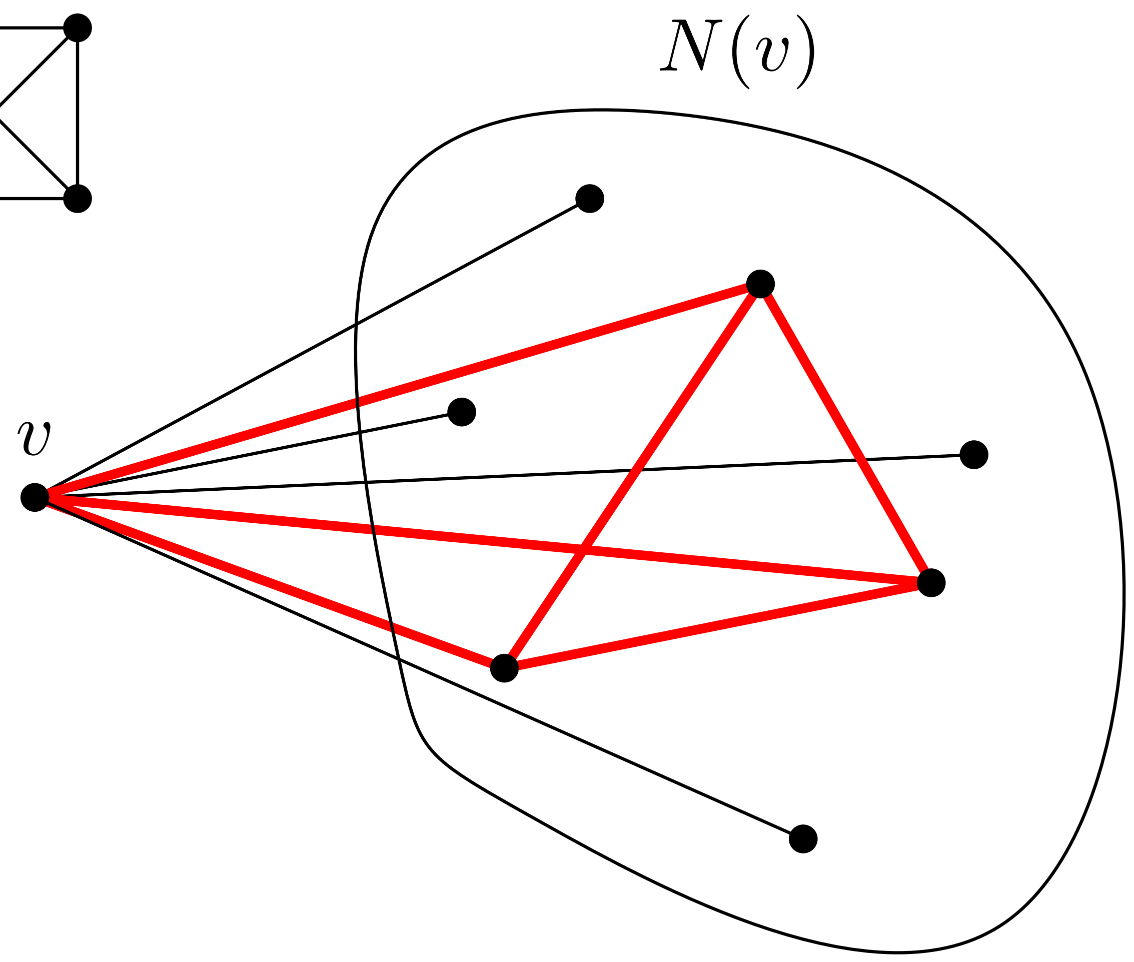
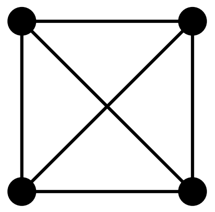
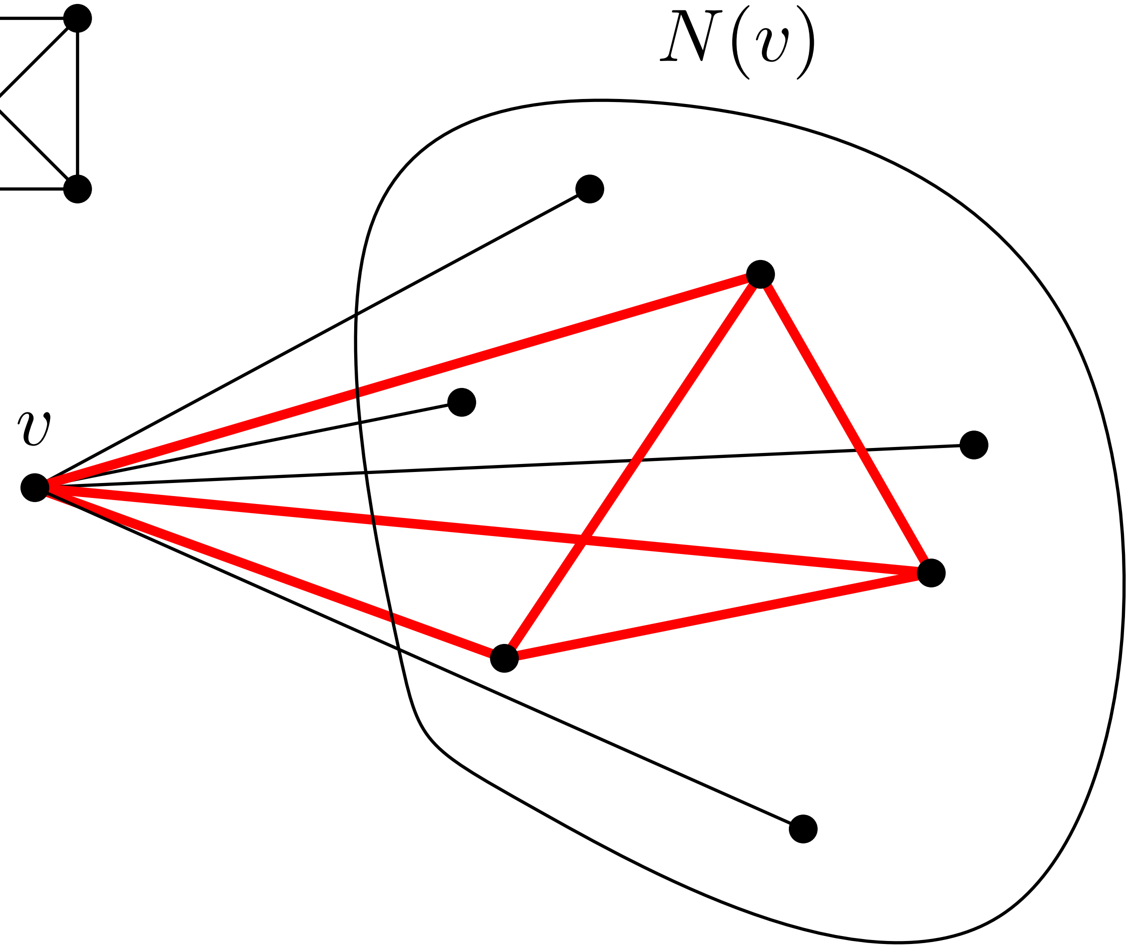


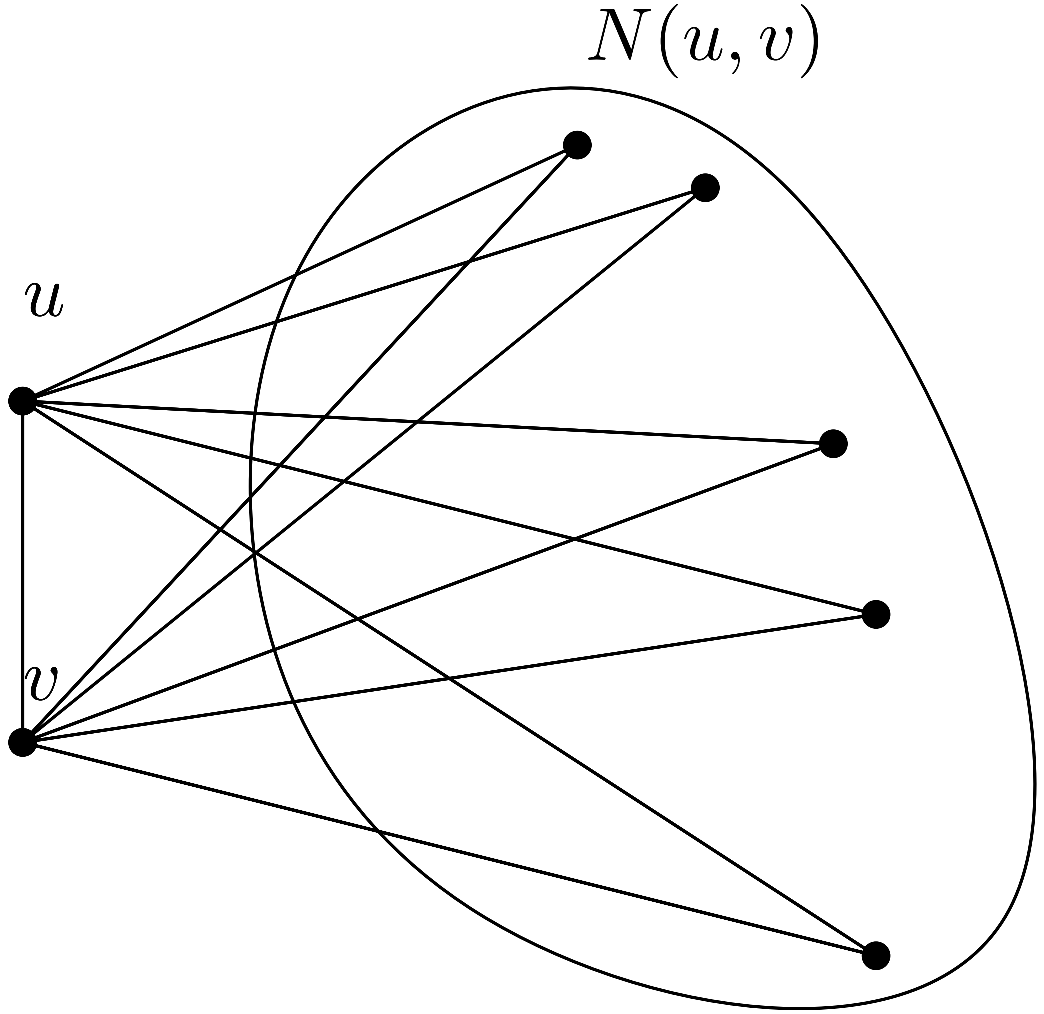


$N(v)$ is always triangle-free

K_4-free graphs







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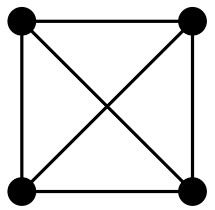

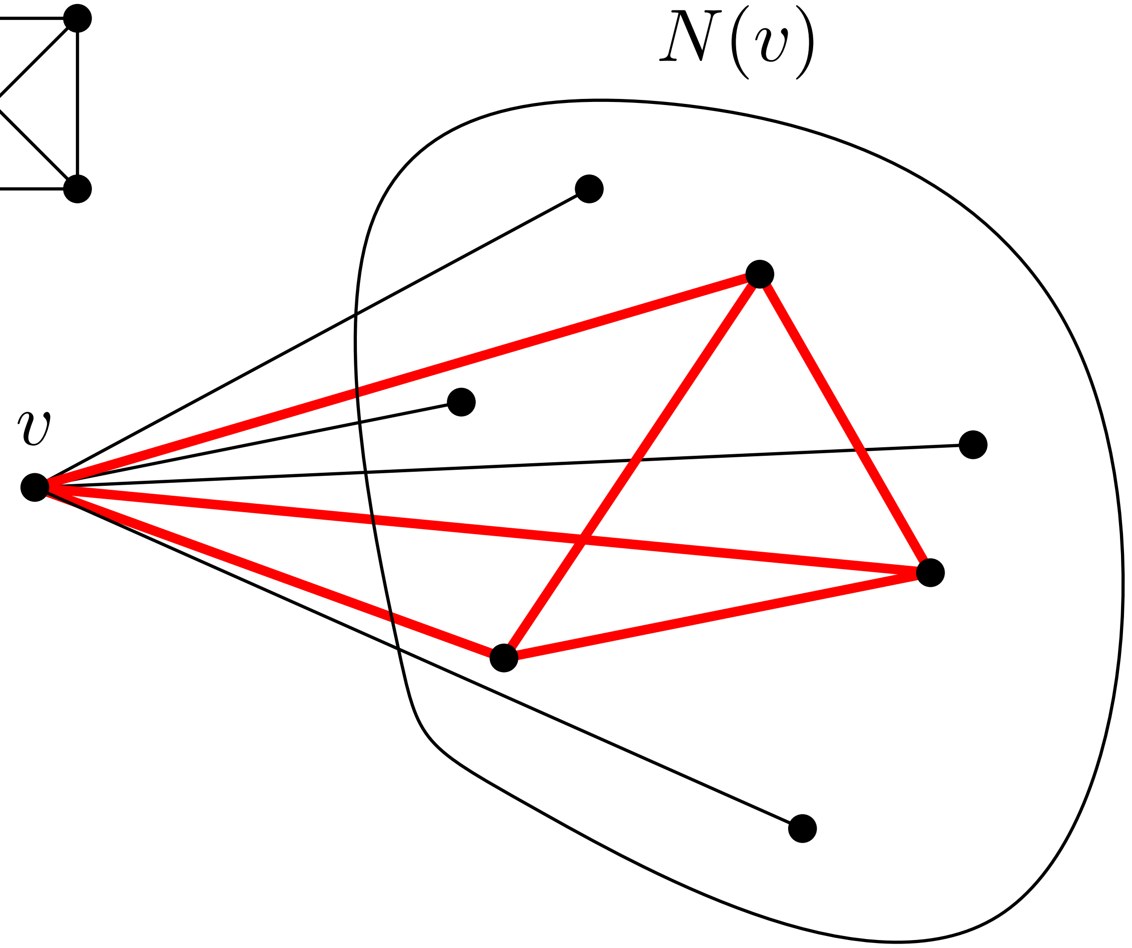

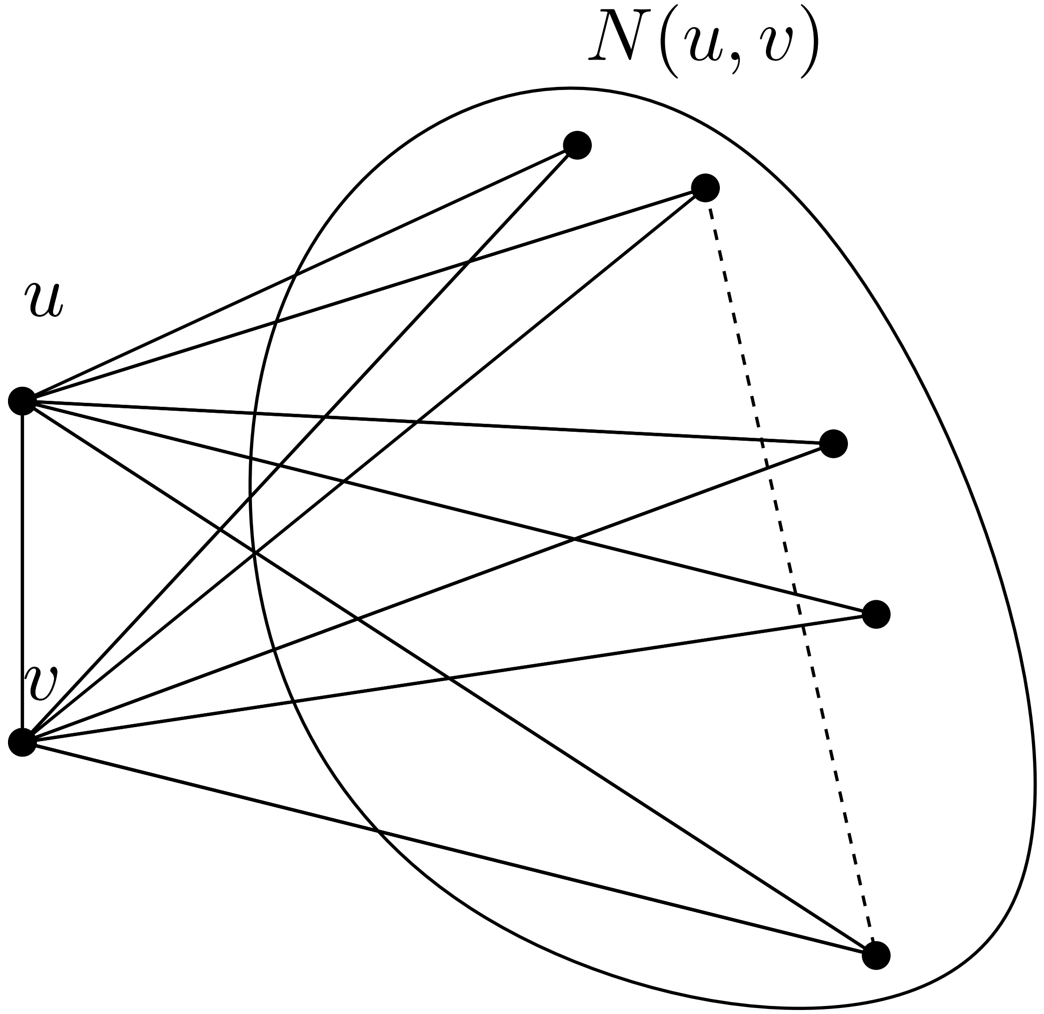

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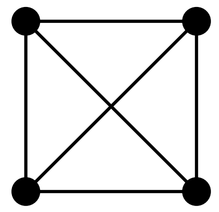

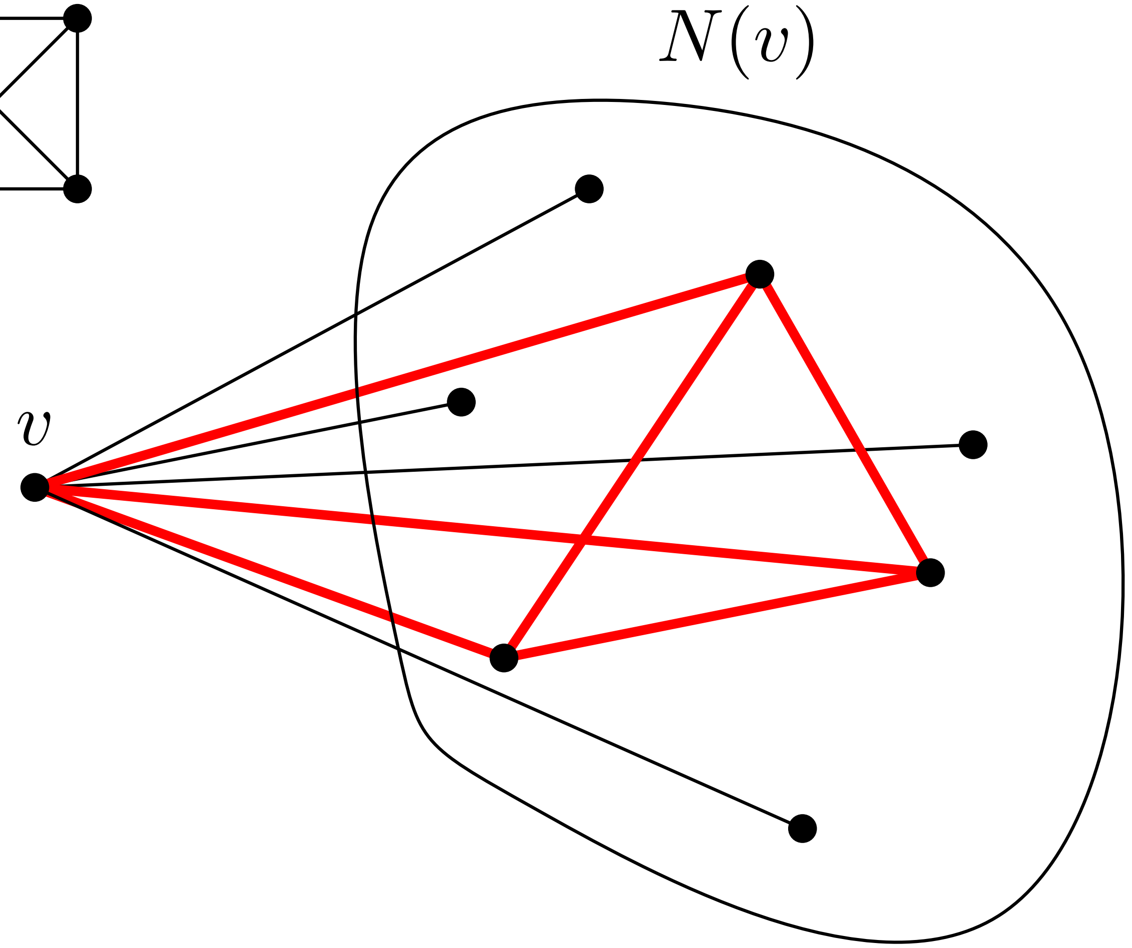

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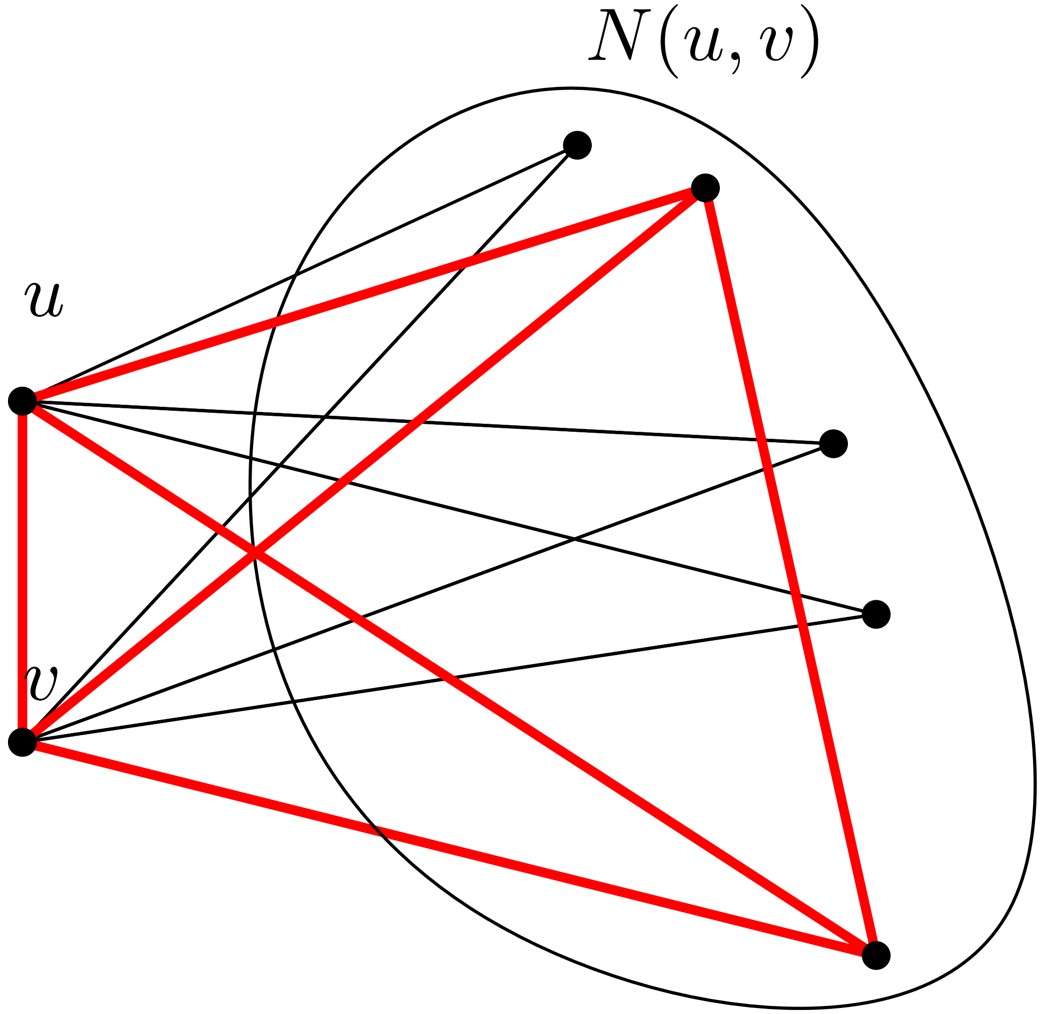
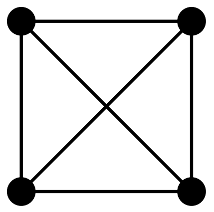
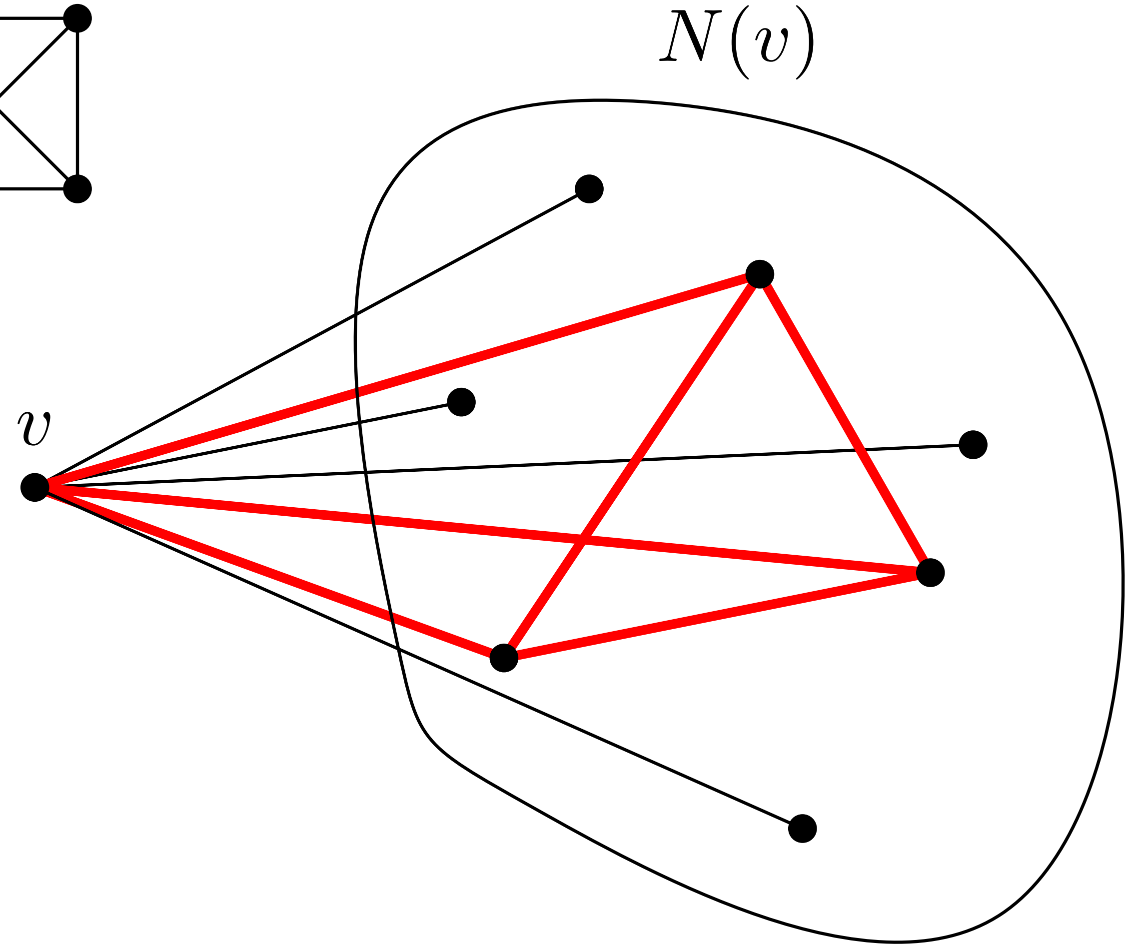
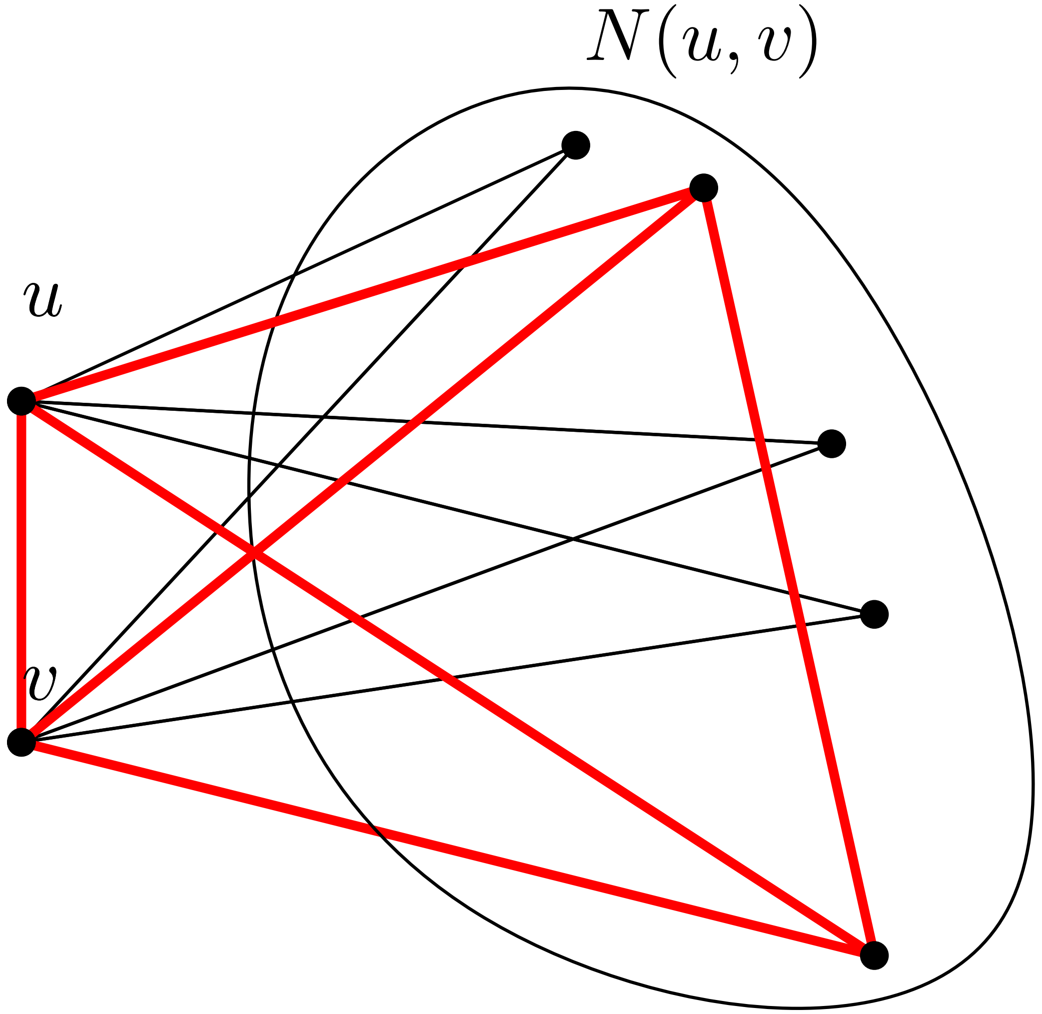


$N(u,v)$ is an independent set when $(u,v) \in E(G)$

K_4-free graphs







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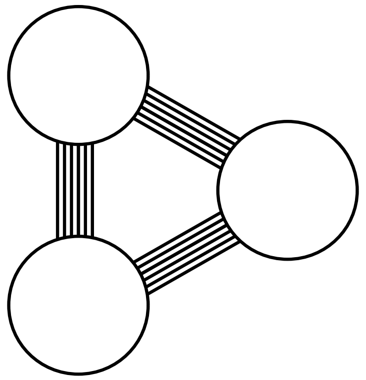
Turán's theorem

If H is K_4-free, then $e(H) \leq \frac{|H|^2}{3}$



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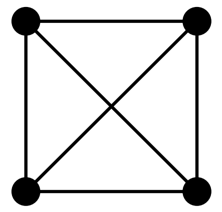

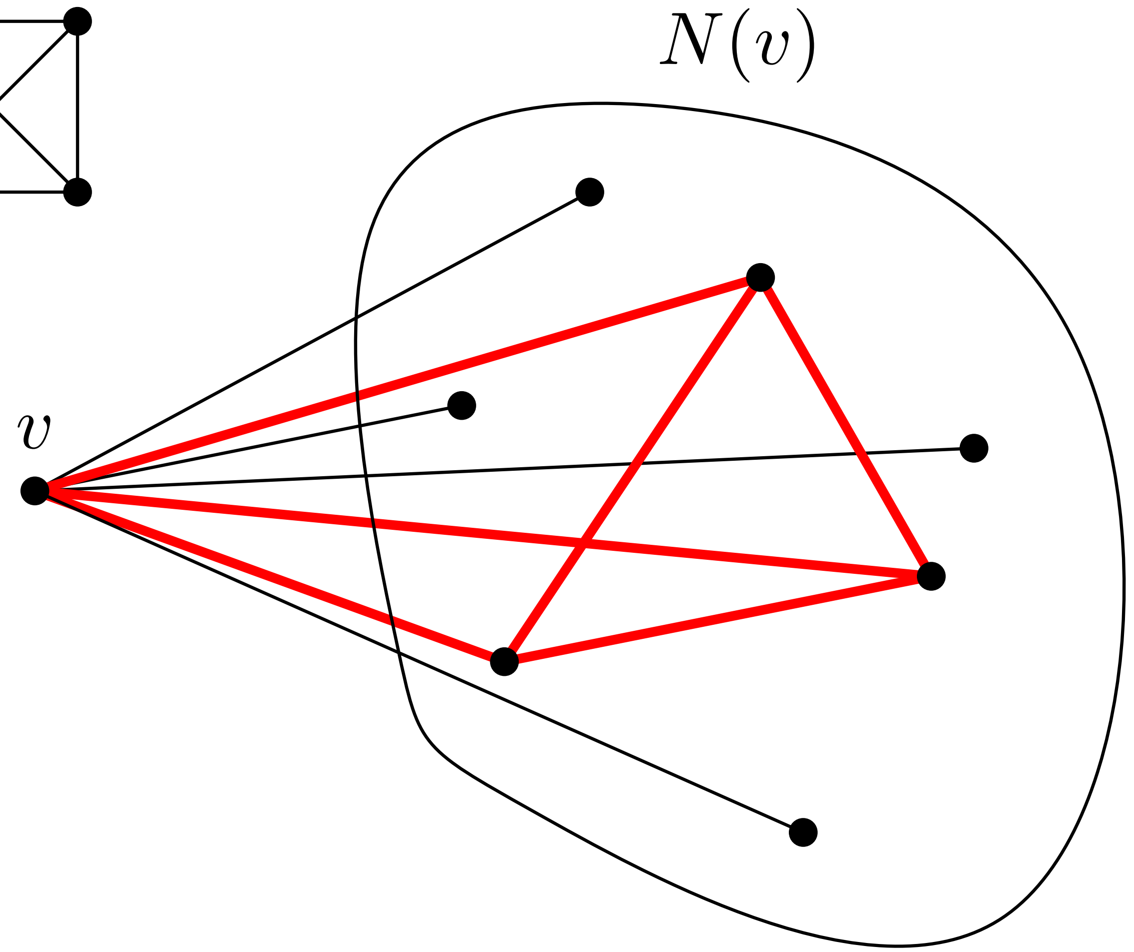

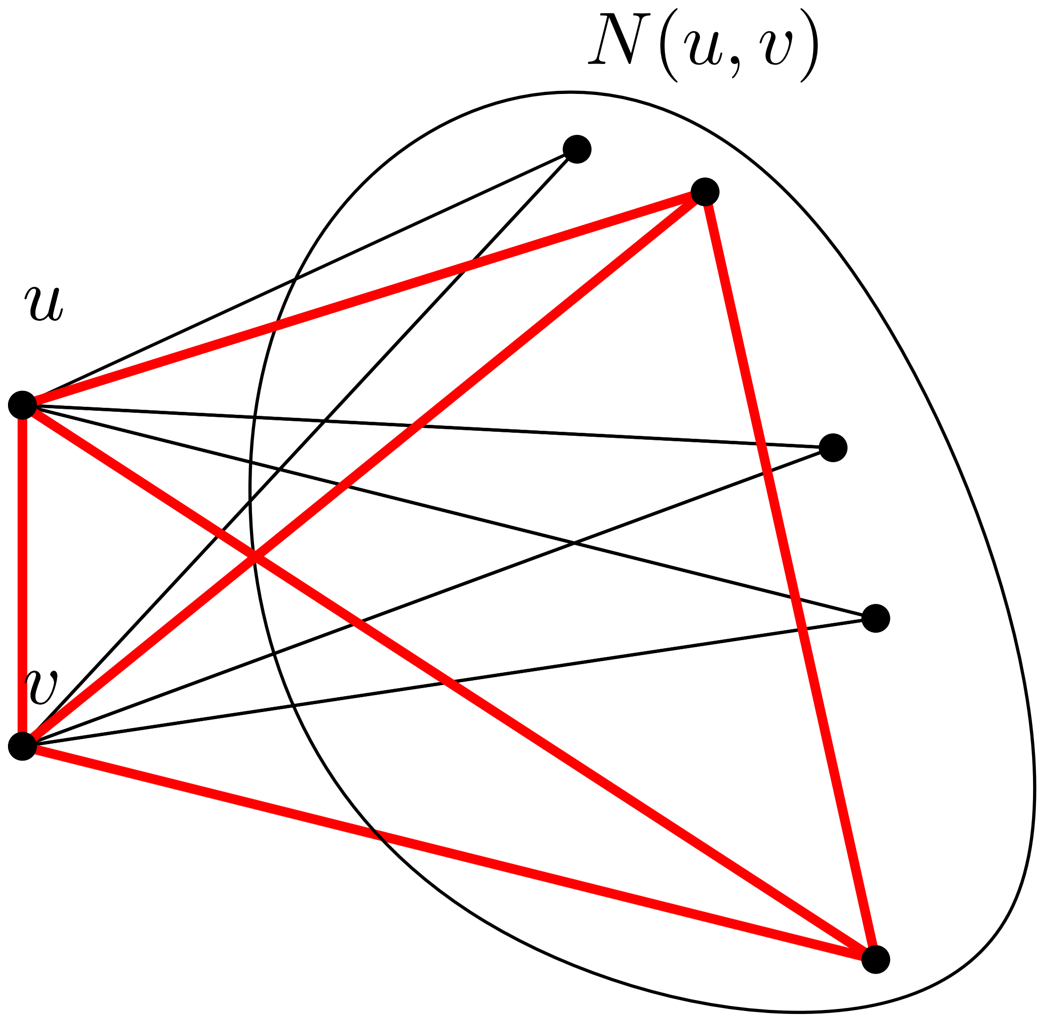

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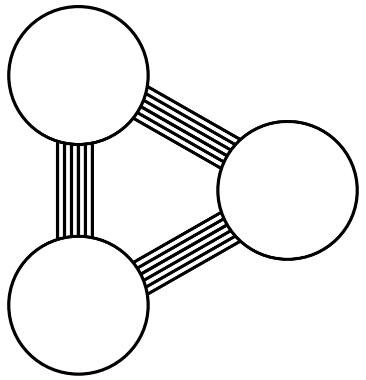

Turán's theorem

If H is K_4-free, then $e(H) \leq \frac{|H|^2}{3}$

Mantel's theorem

If $e(H) \geq \frac{|H|^2}{4}$, then H contains a triangle

Strategy

If a graph G on n vertices has the property that every set $X \subseteq V(G)$ of size $X = \lfloor \frac{n}{2} \rfloor$ spans at least $\frac{n^2}{18}$ edges, then either G contains a K_4 or n is divisible by 6 and G is a tripartite Turán graph.

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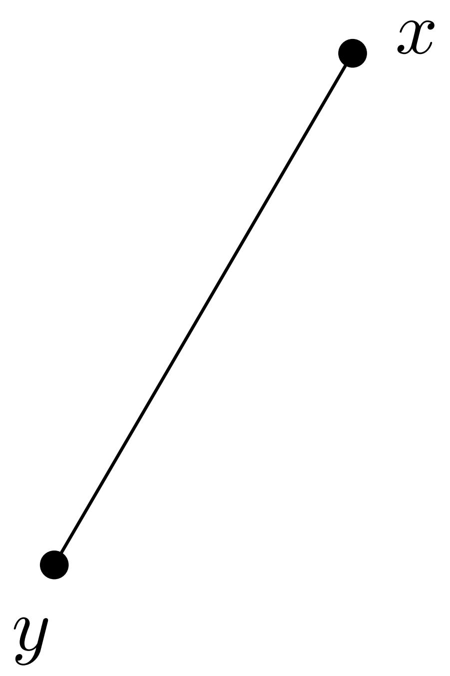

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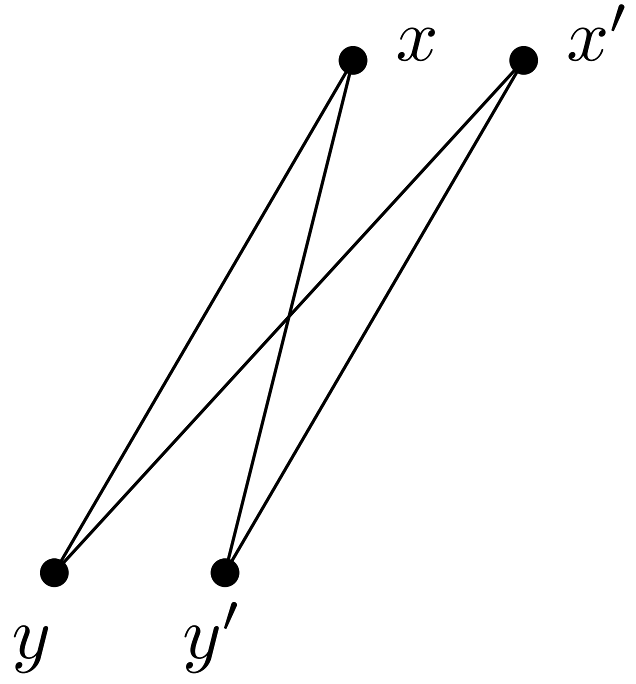

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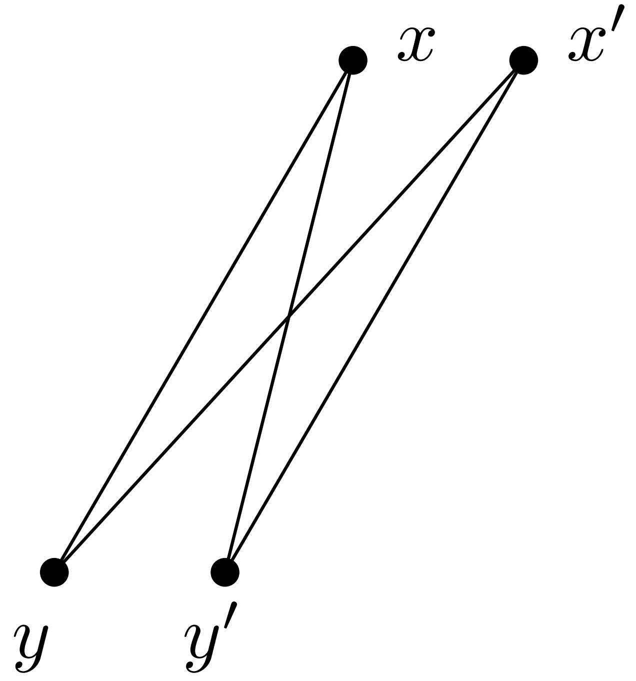

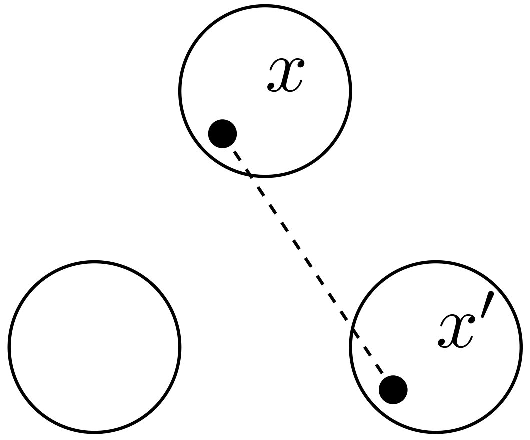

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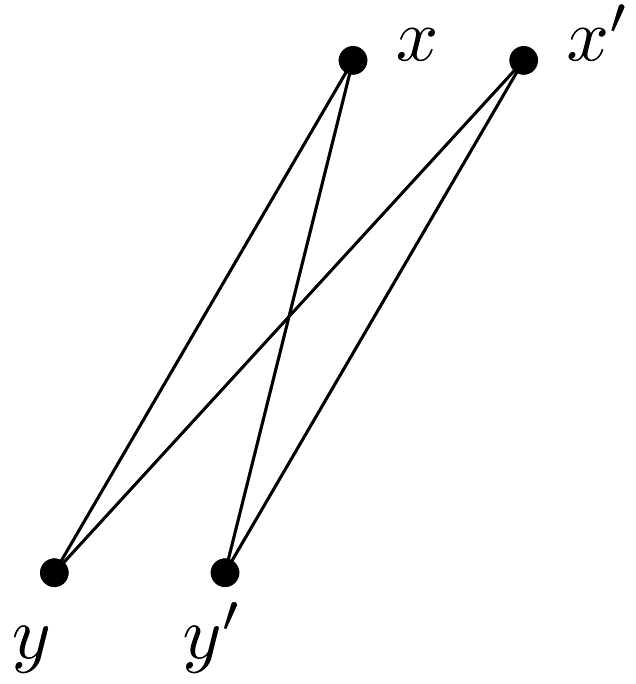

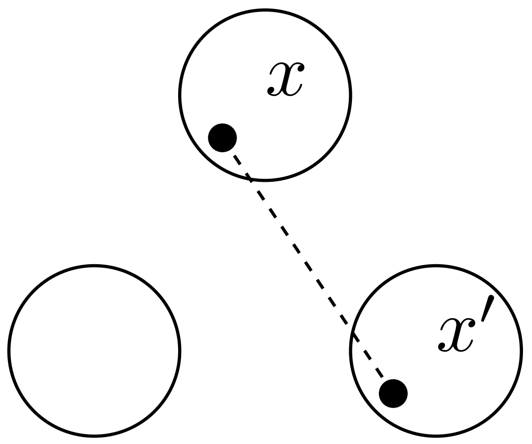

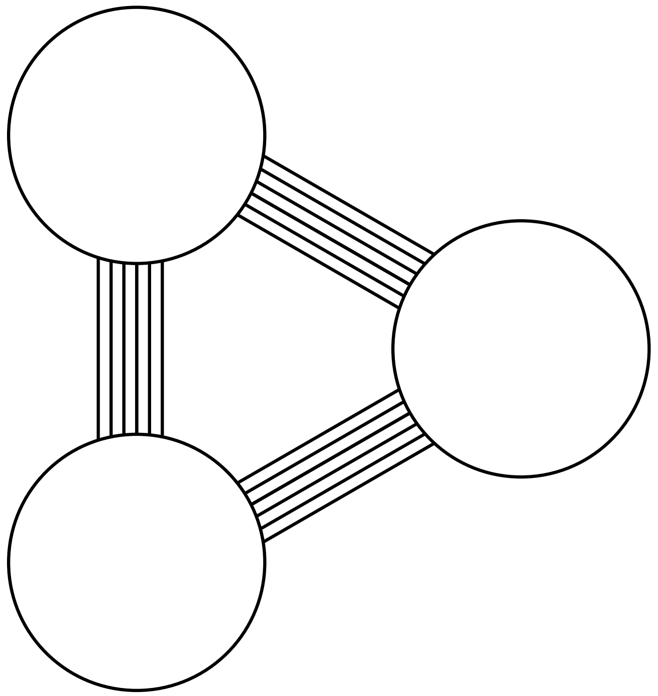

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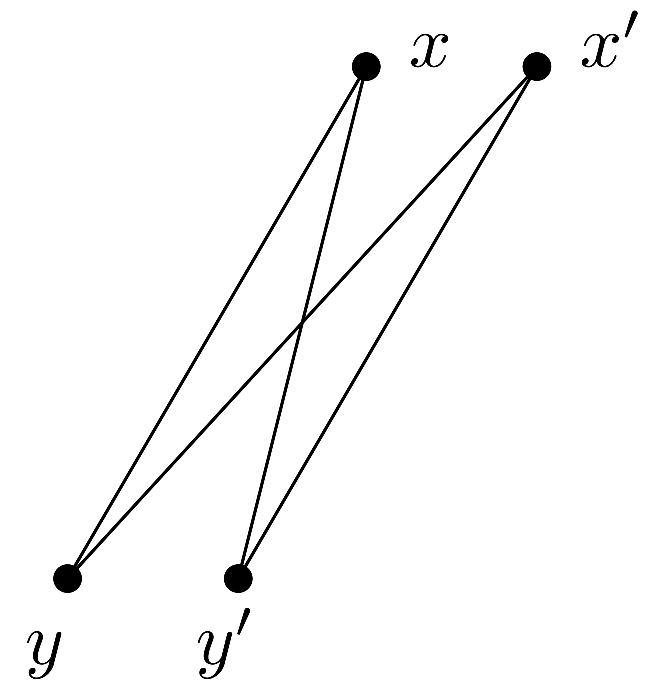

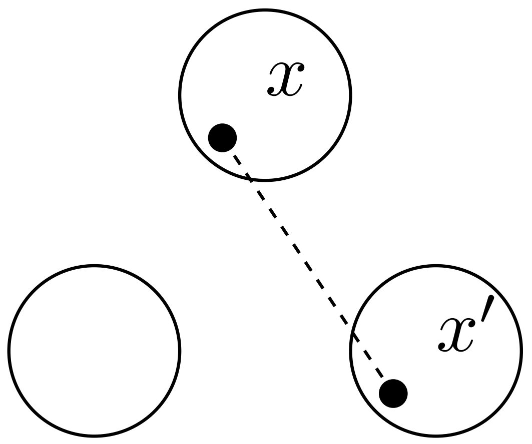

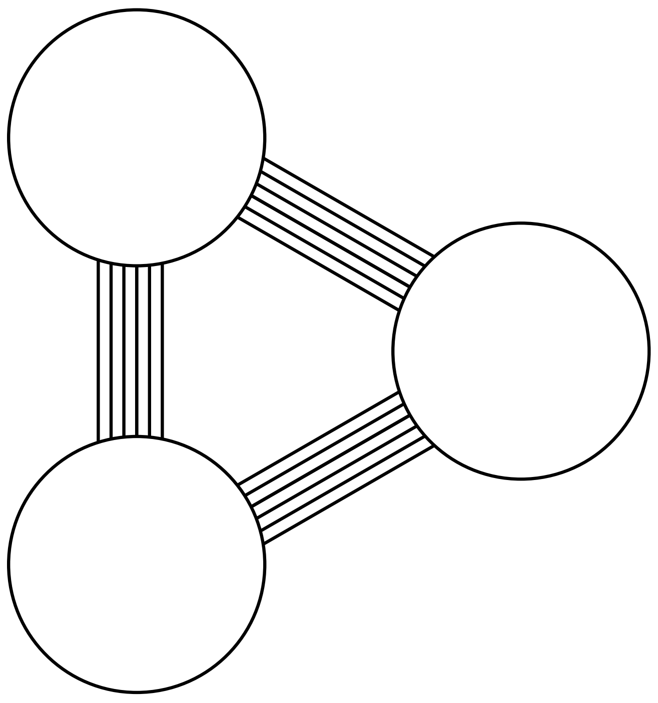

$$\frac{n}{3} \cdot \left\lfloor \frac{n}{6} \right\rfloor < \frac{n^2}{18}$$

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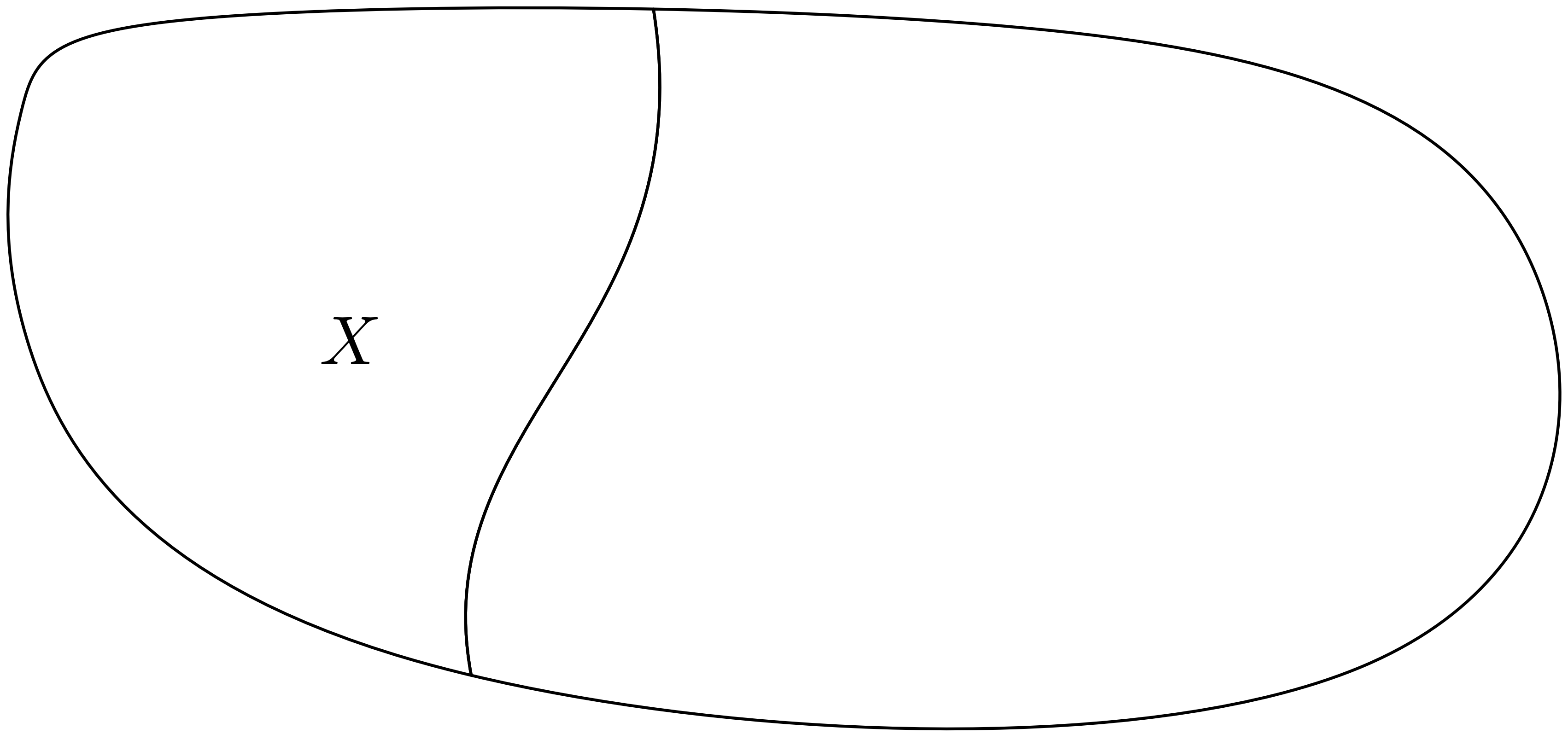
Let G be an *extremal* graph on n vertices, and n is even. Then n must be dividable by 3 and G must be a complete tripartite graph.

$\alpha(G) \leq \frac{n}{3}$

G - *extremal*
$X \subseteq V(G), |X| \in [\frac{1}{3}n; \frac{1}{2}n]$
Then $e(X) \geq \frac{1}{18}(3|X| - n)(6|X| - n)$



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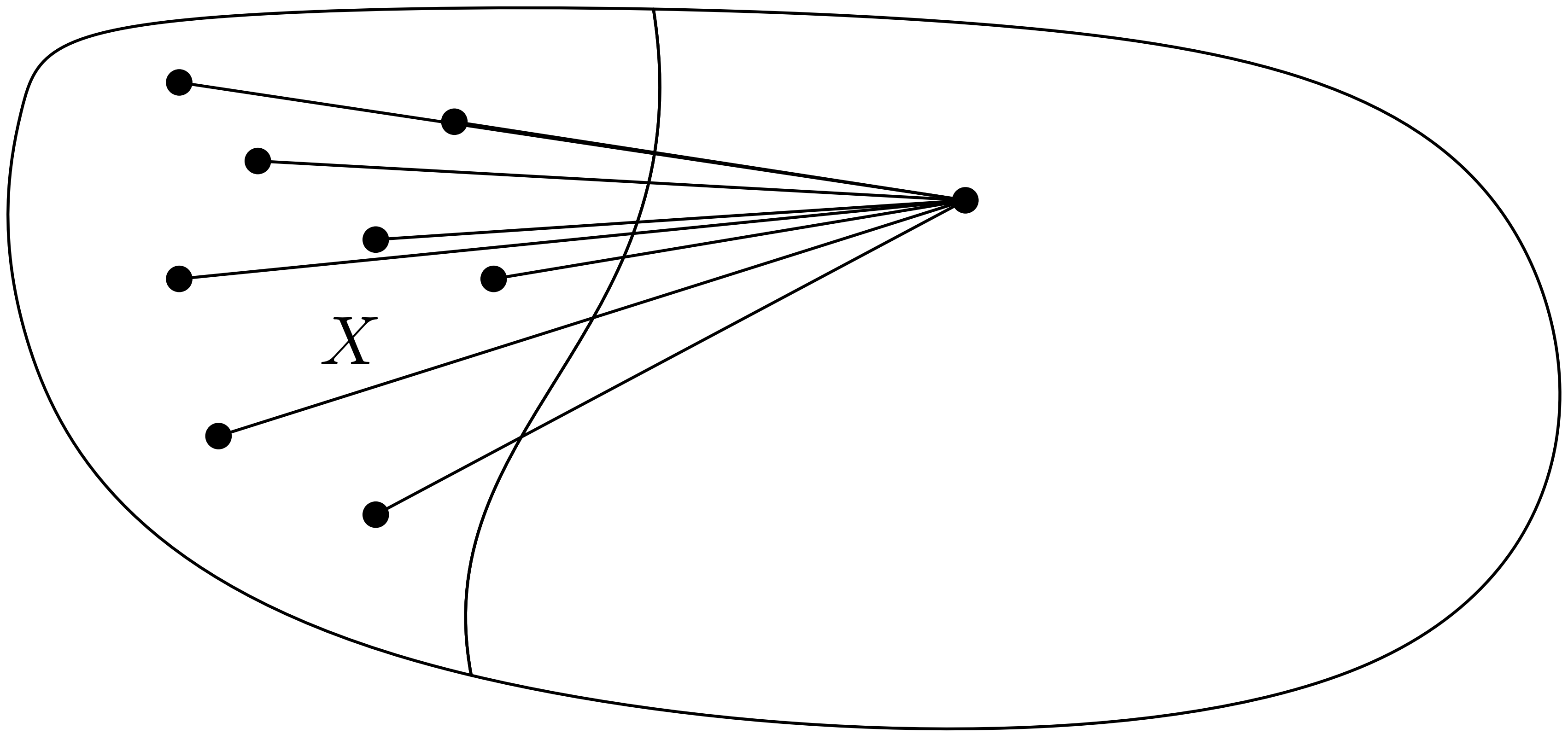

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$X \subseteq V(G), |X| \in [\frac{1}{3}n; \frac{1}{2}n]$
Then $e(X) \geq \frac{1}{18}(3|X| - n)(6|X| - n)$

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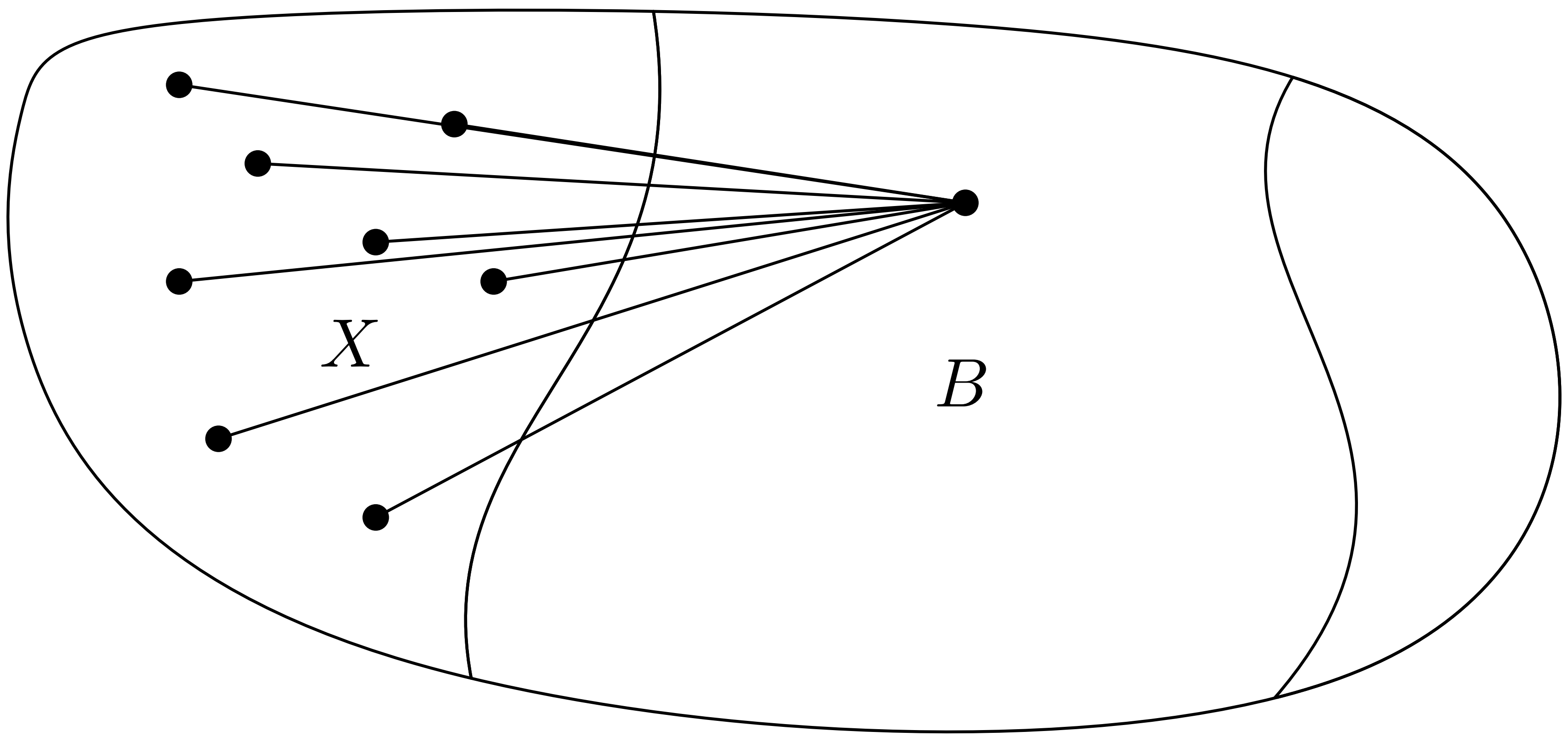

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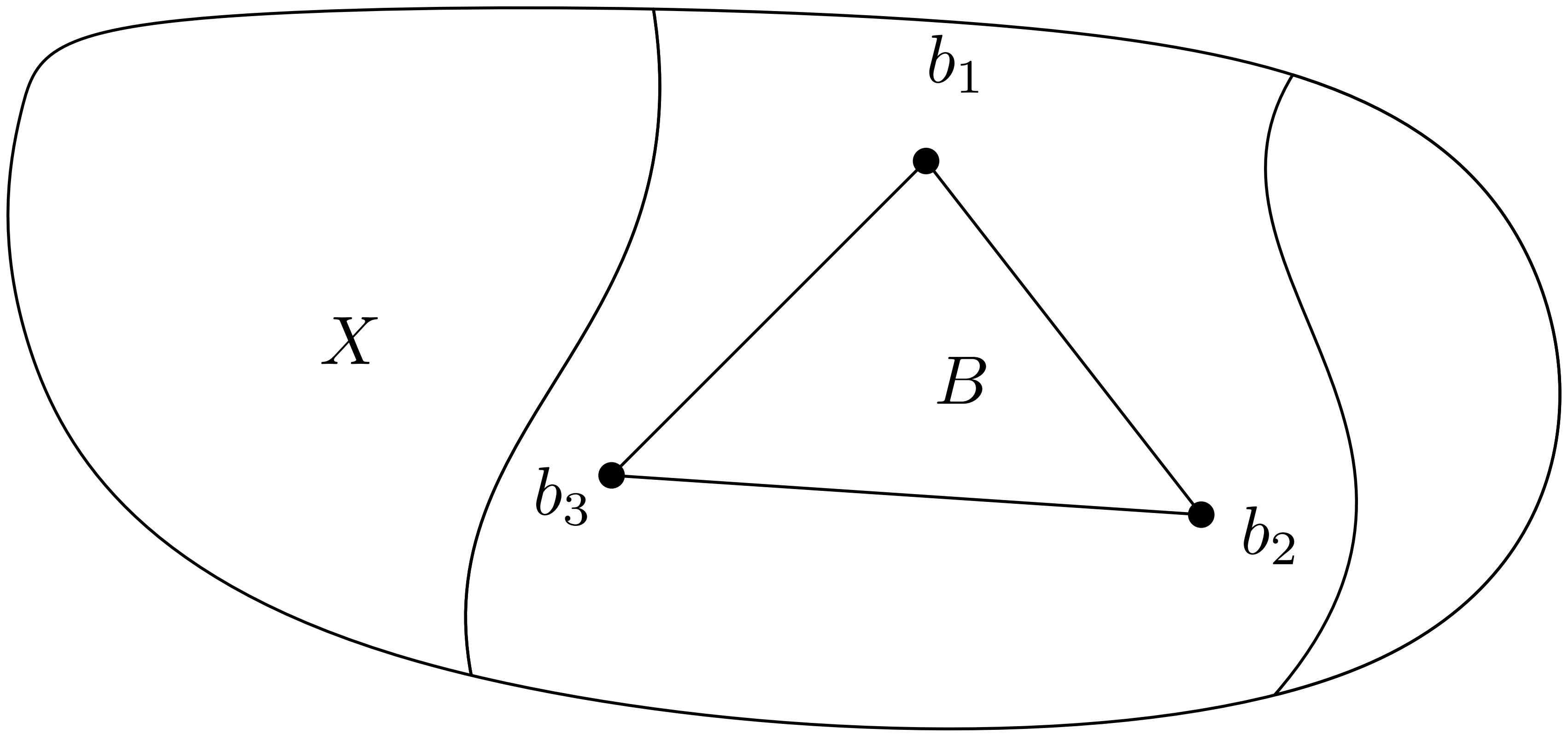

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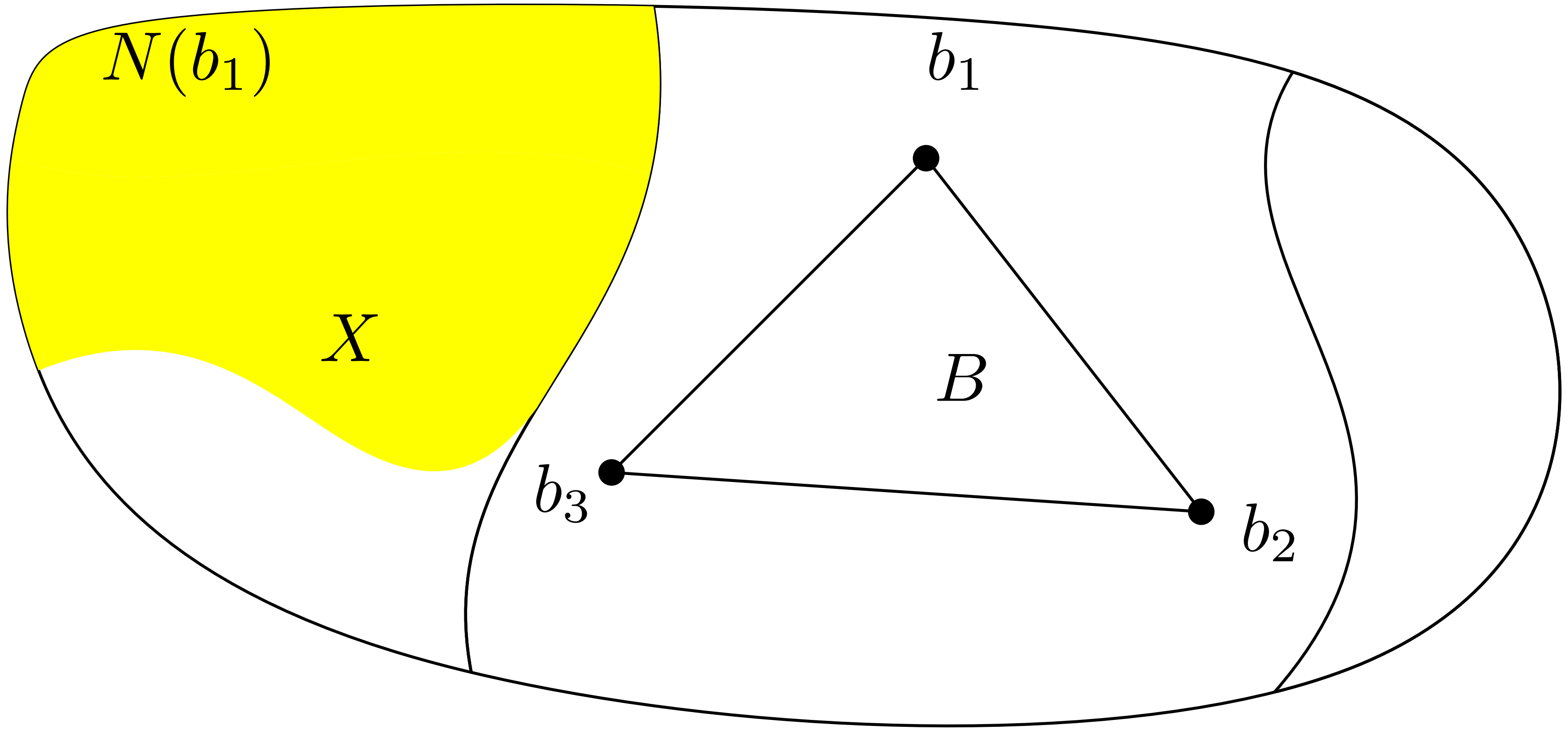

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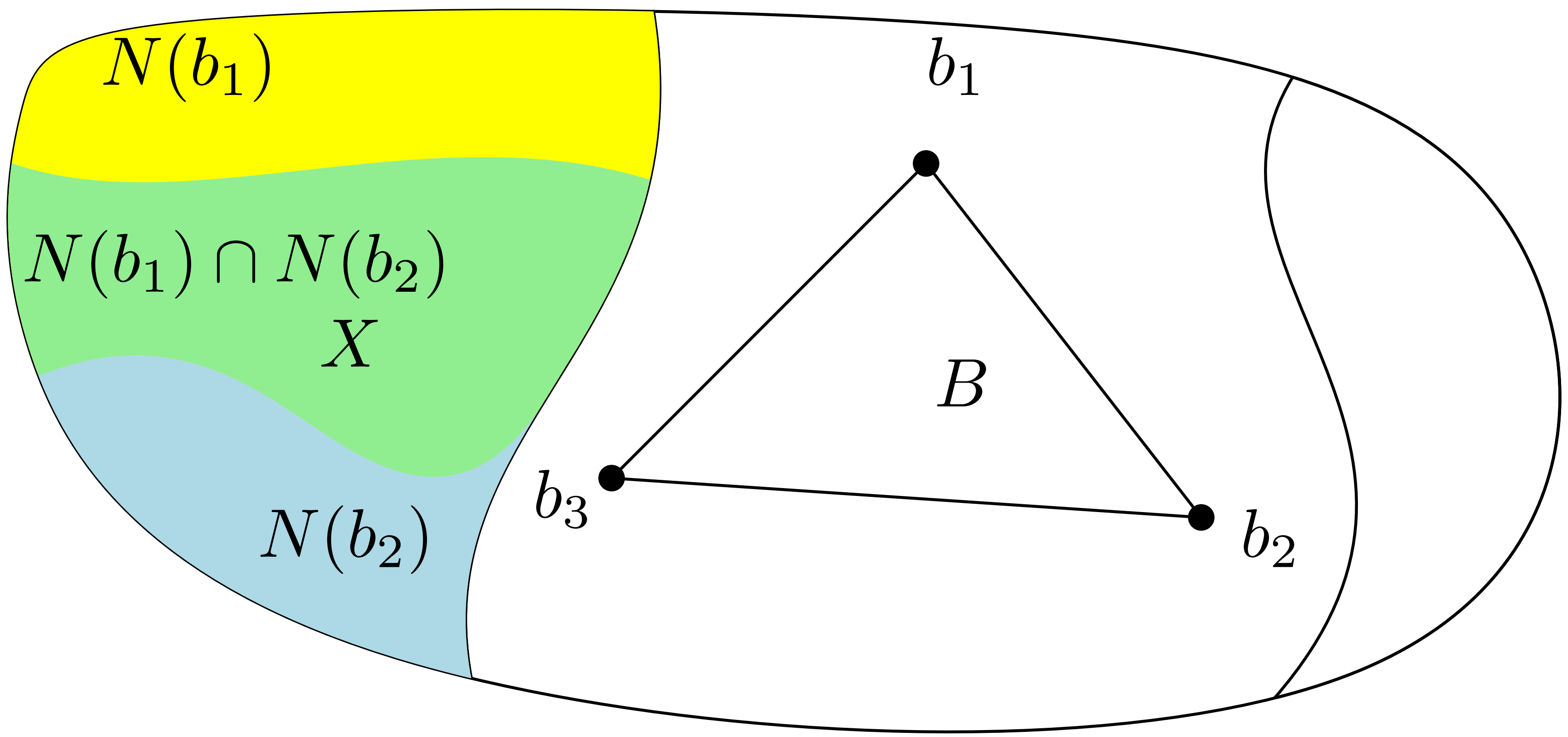

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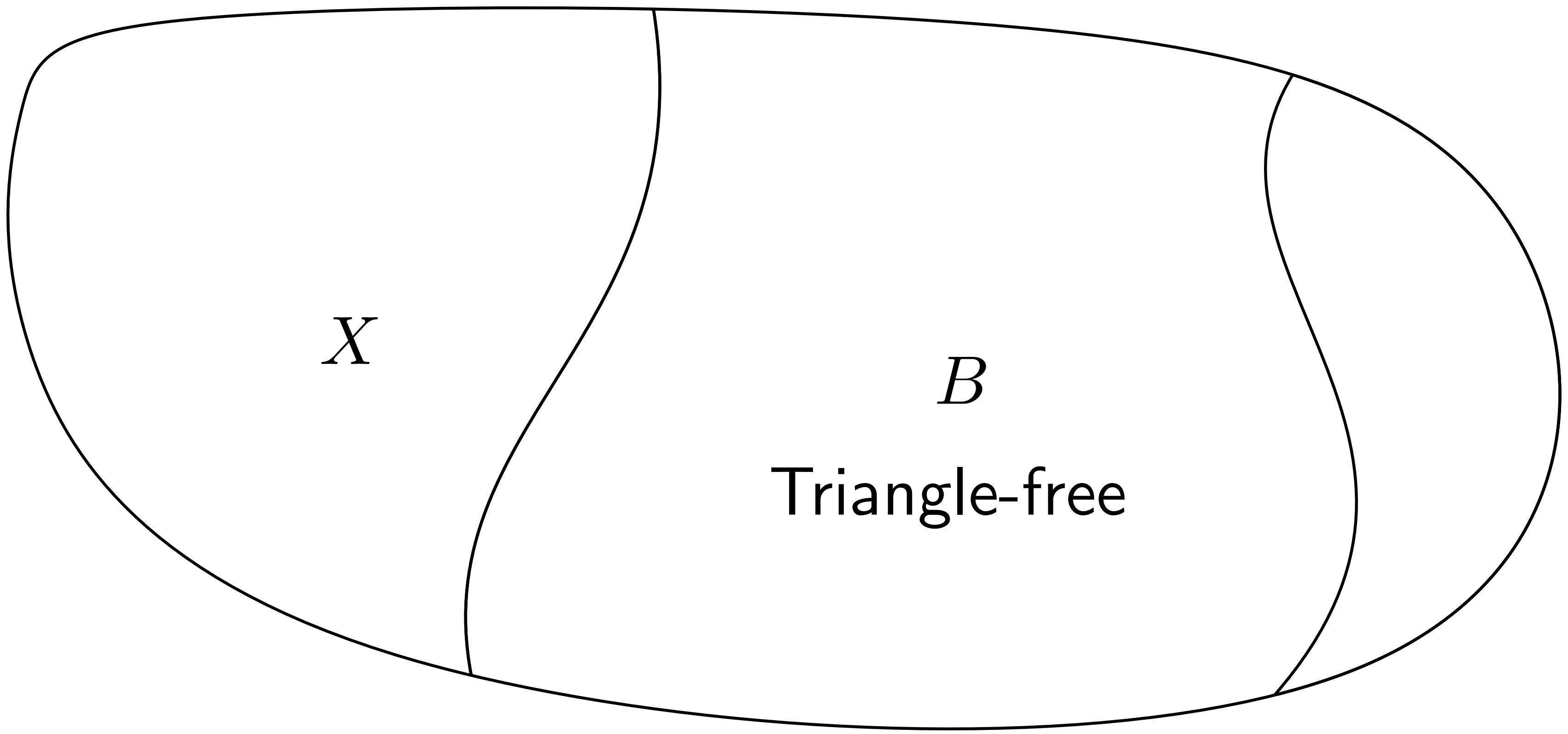

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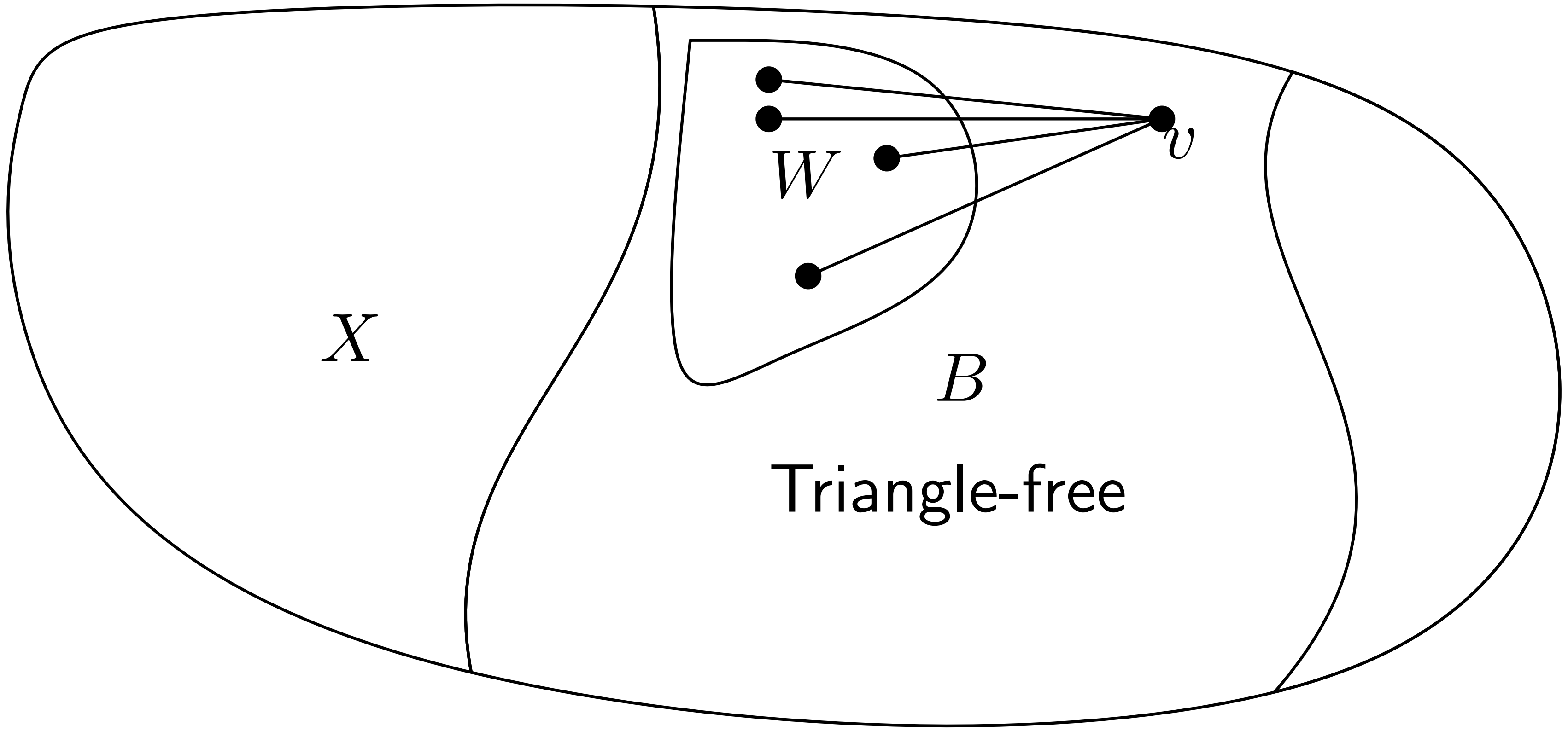

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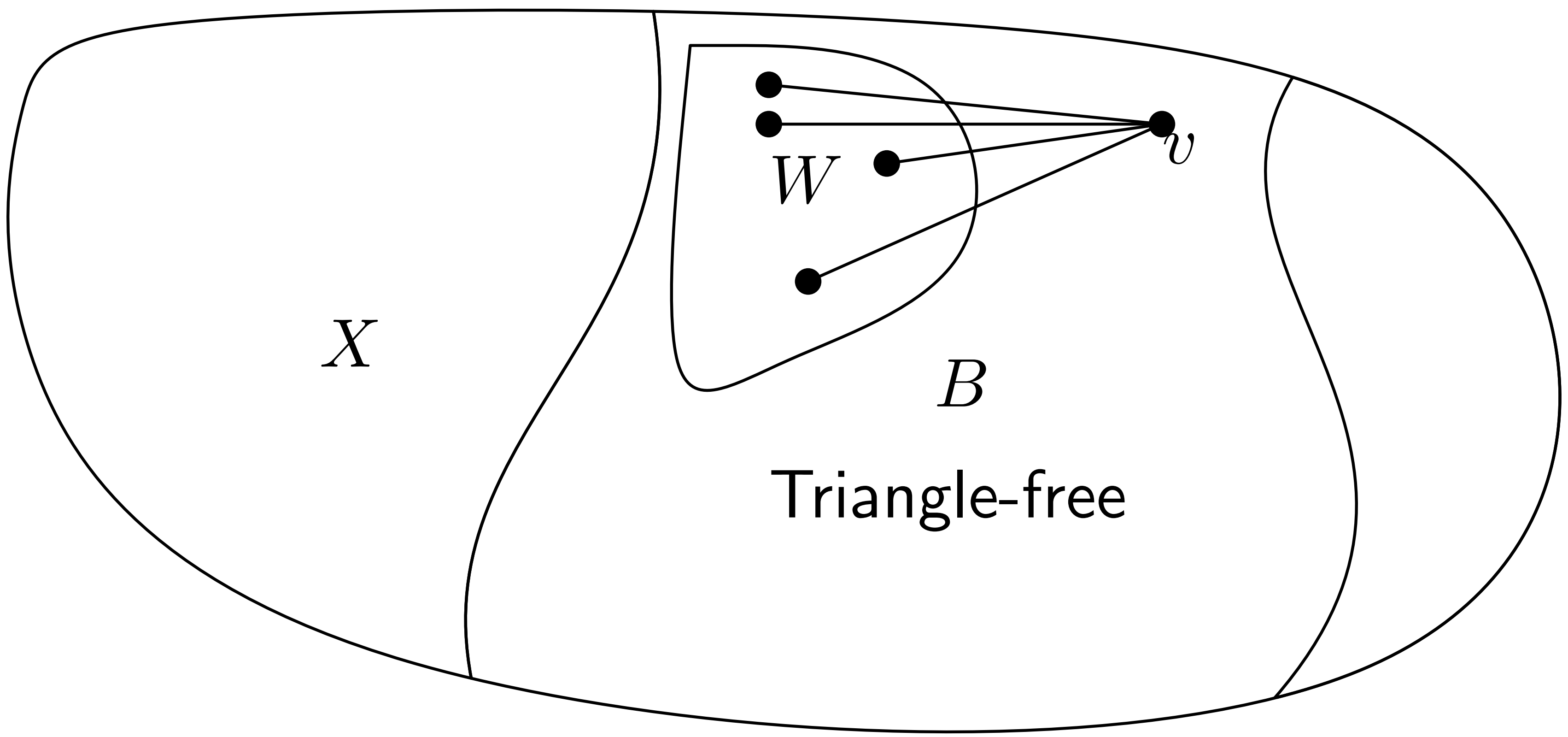

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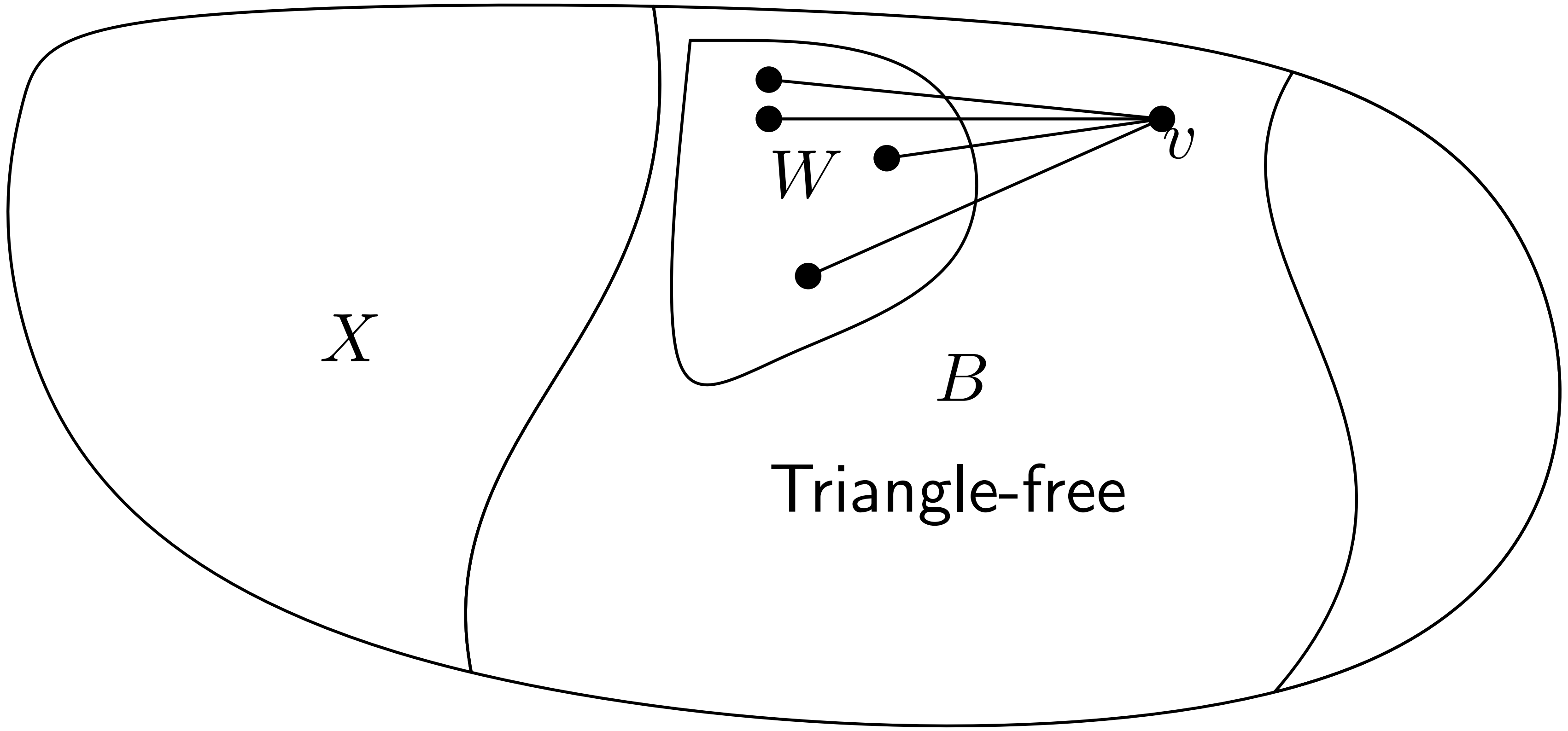

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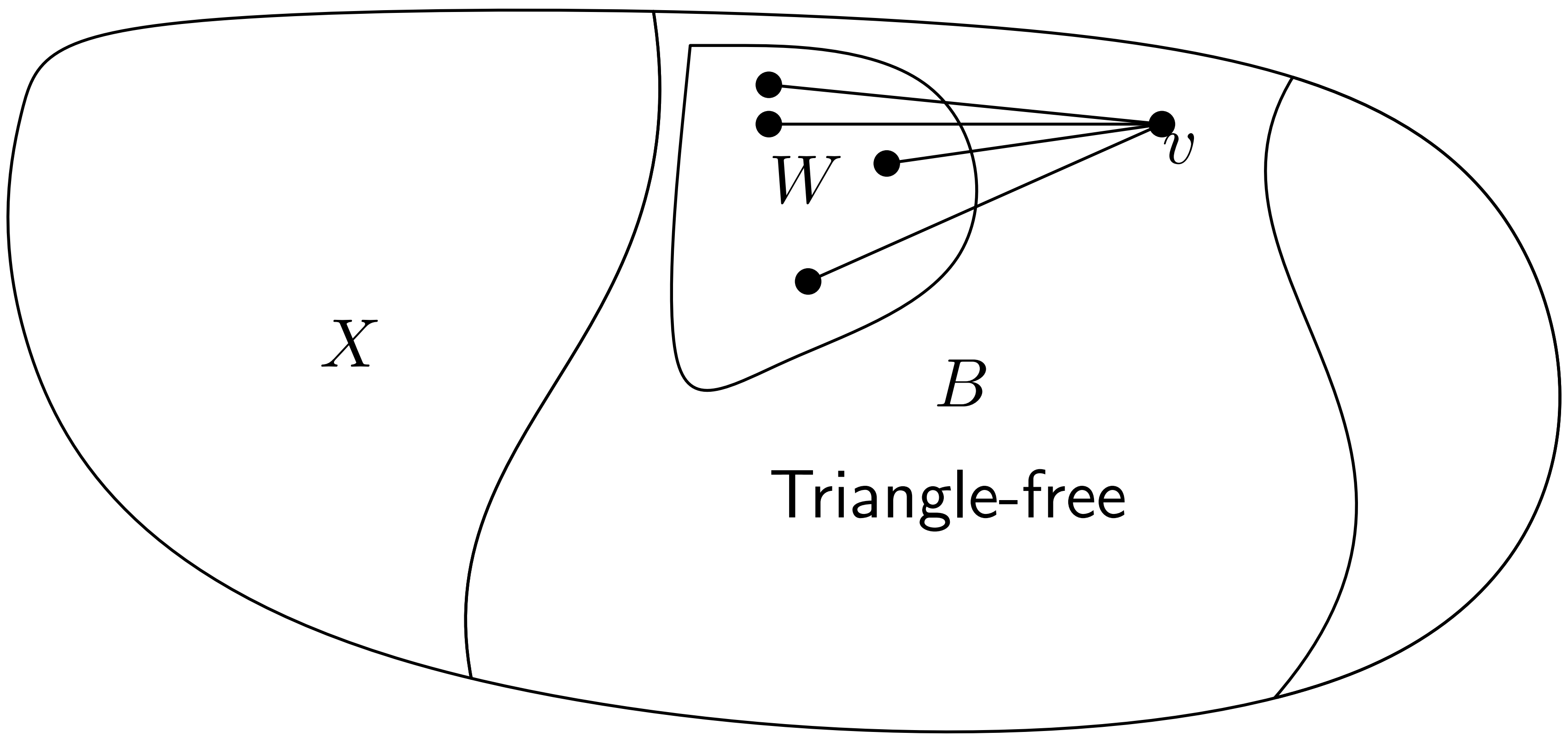

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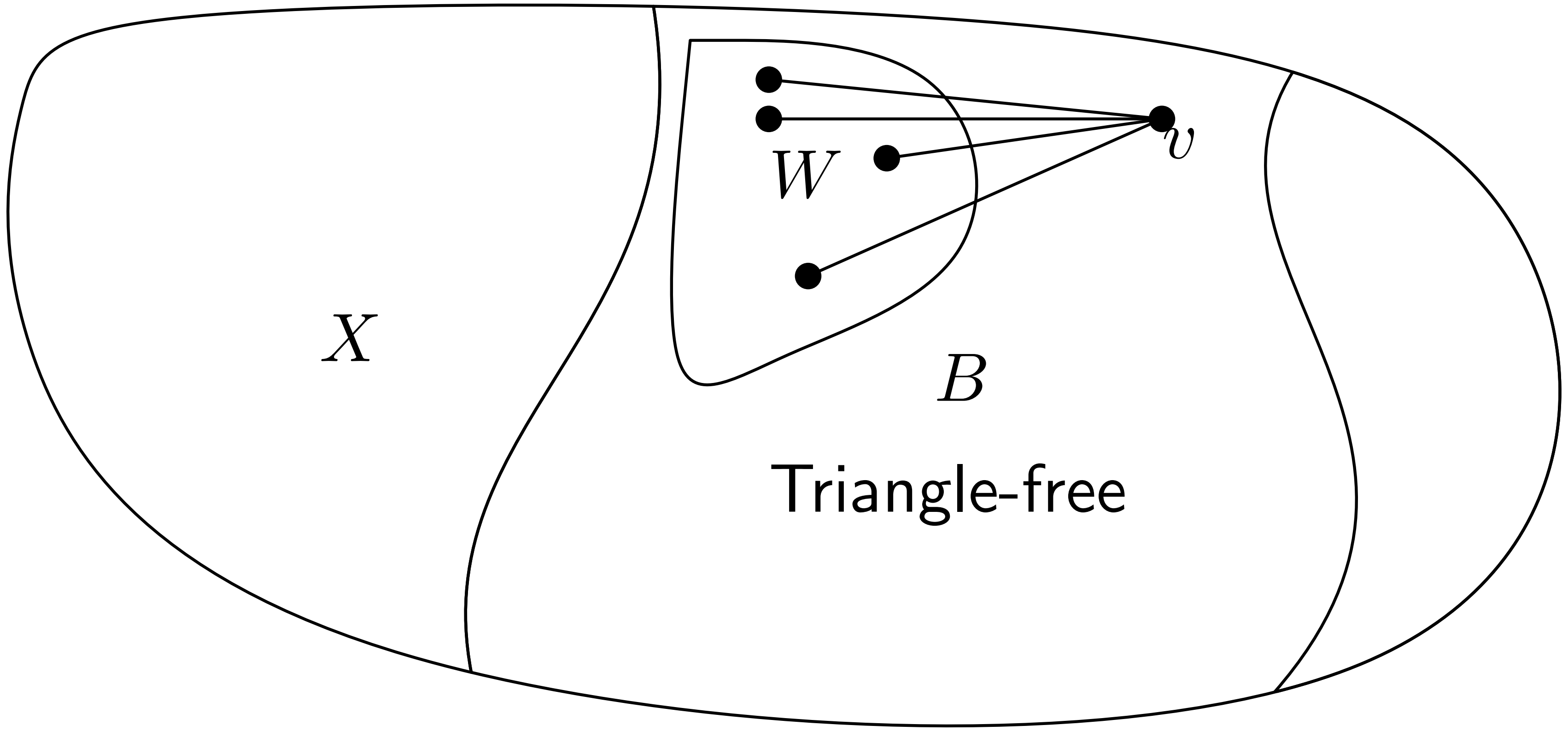

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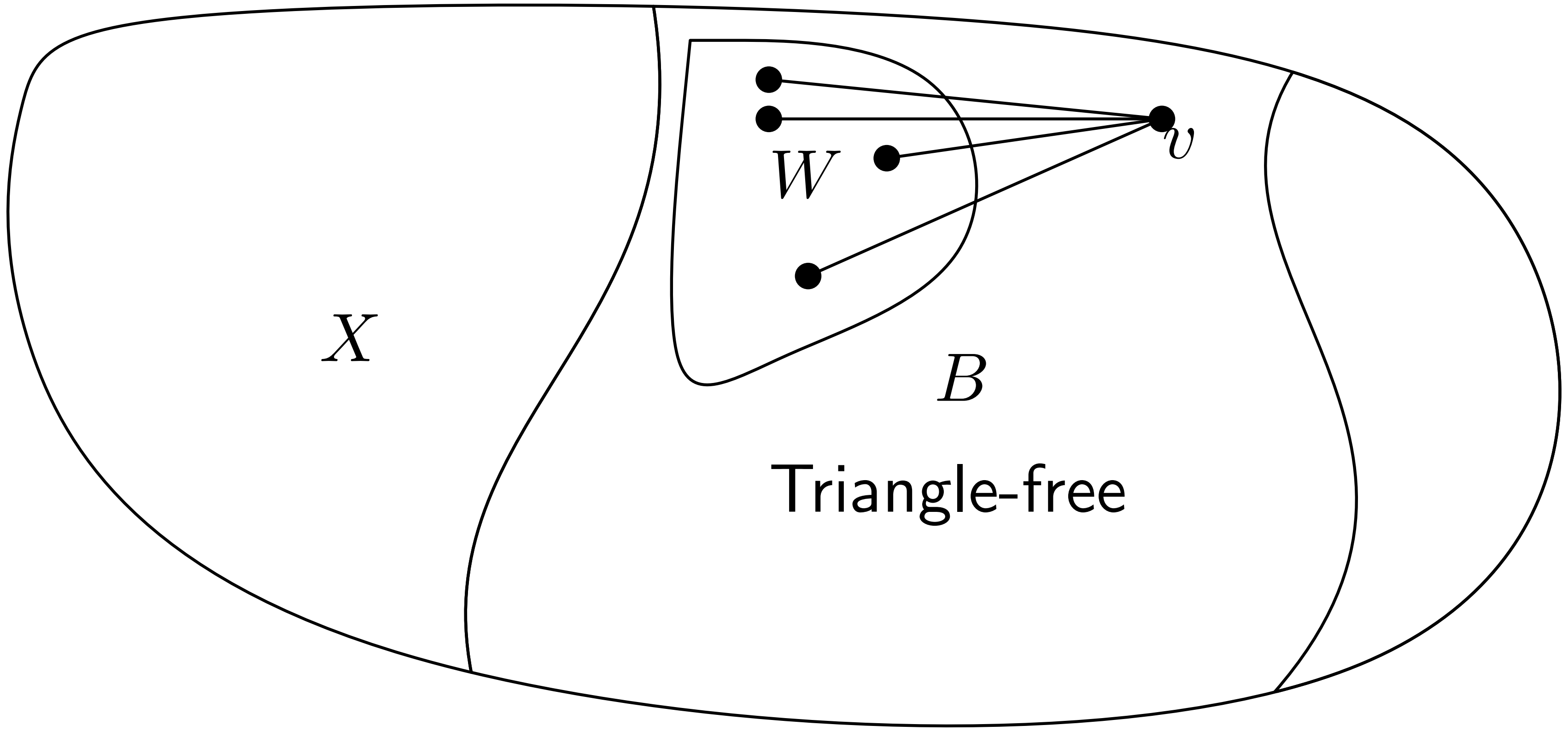

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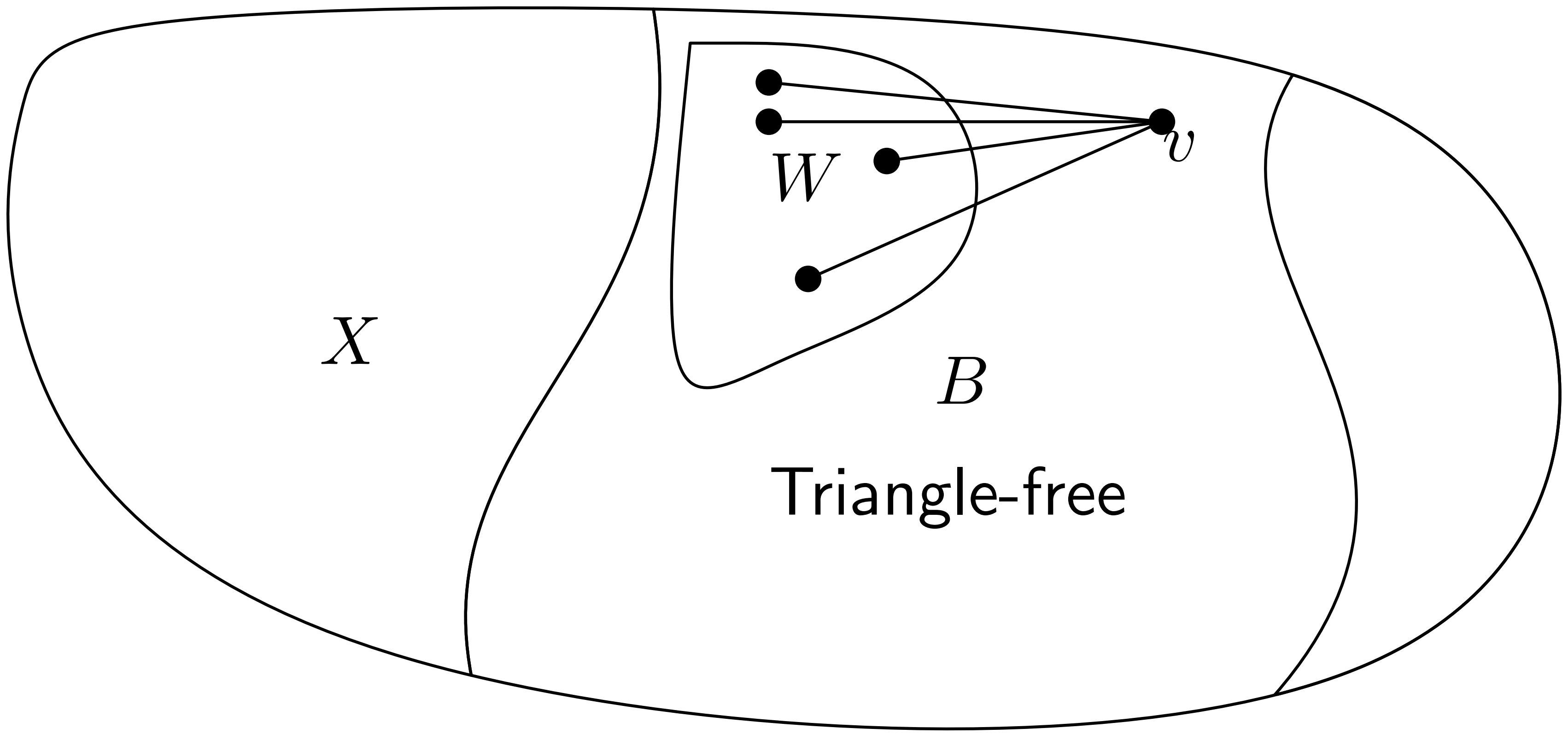

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$e(G)$ should be large, right?

G - $\triangle$-free, m, q - integers such that $q \geq \frac{2}{9}m^2$ and $n \geq m$
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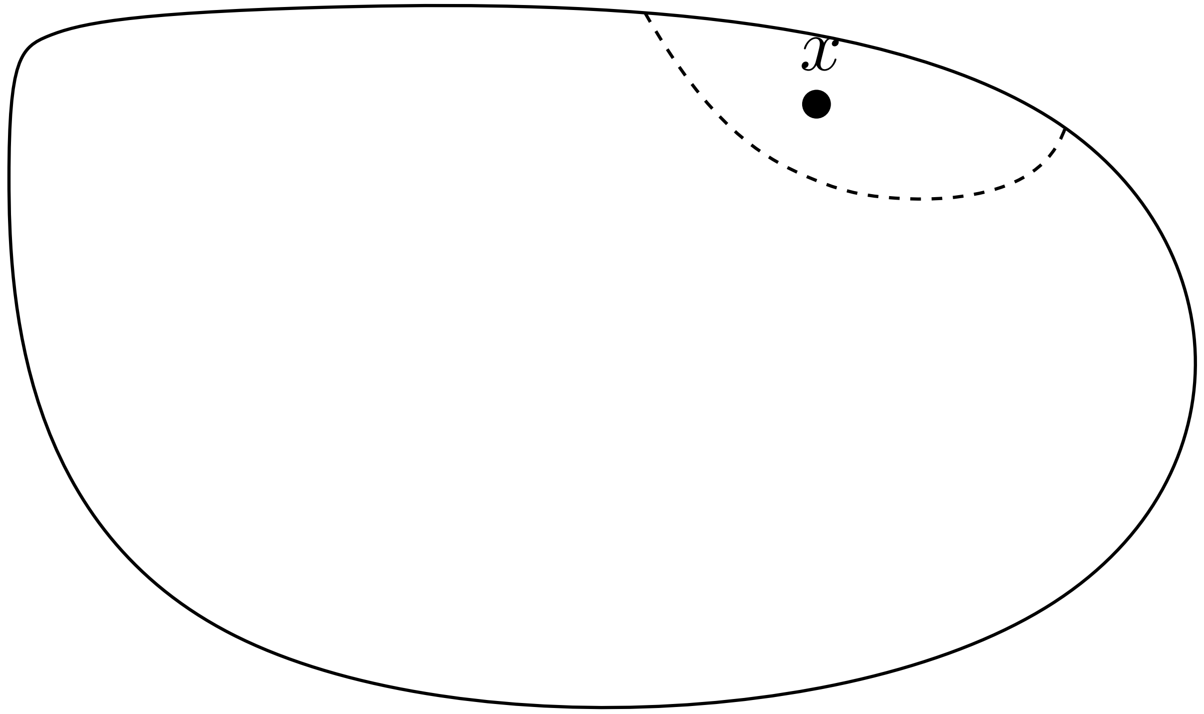

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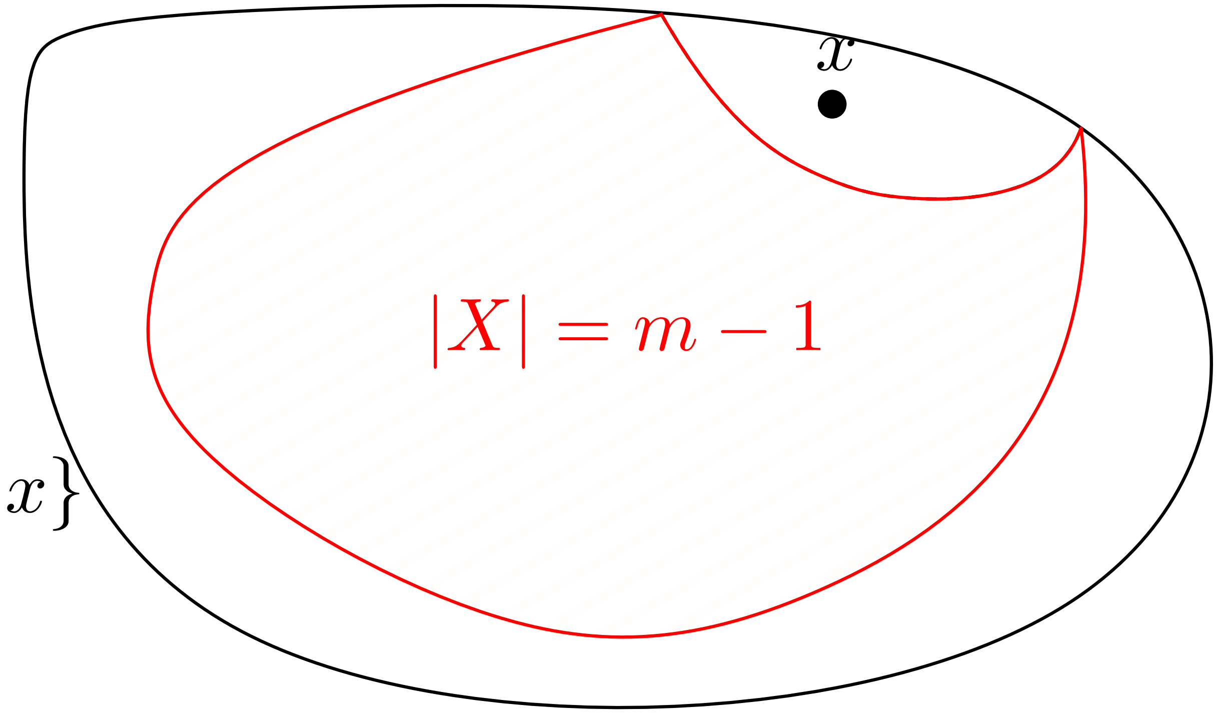

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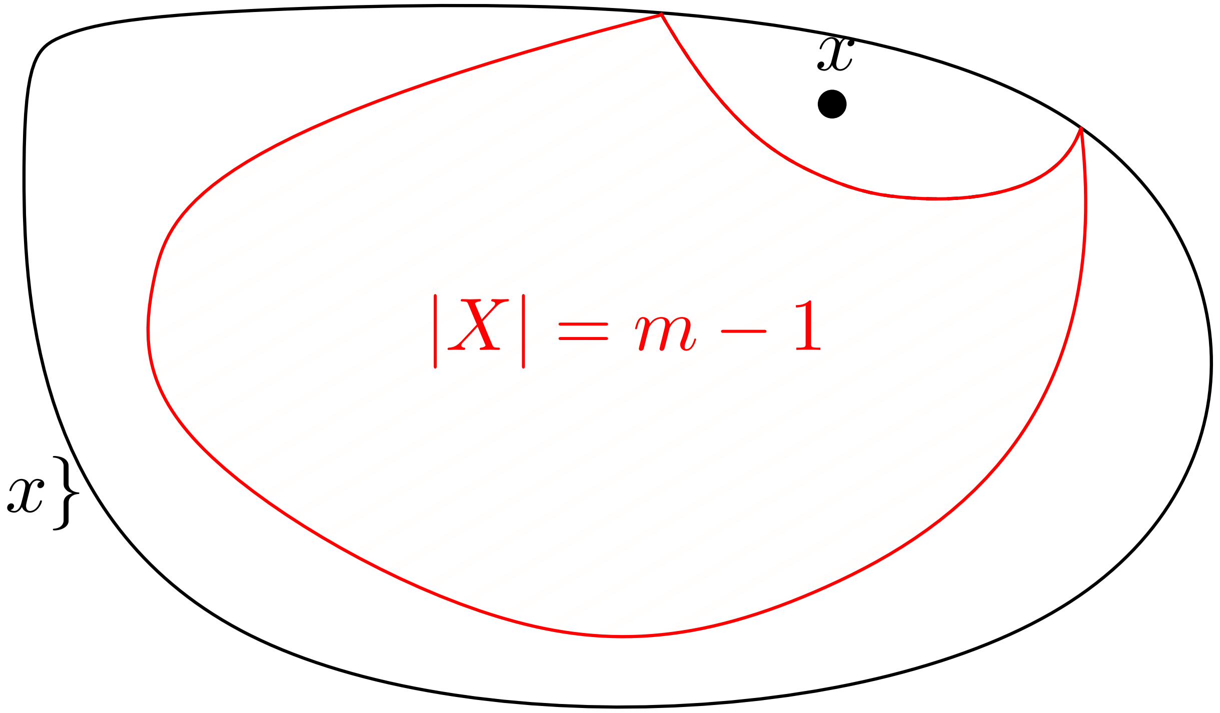

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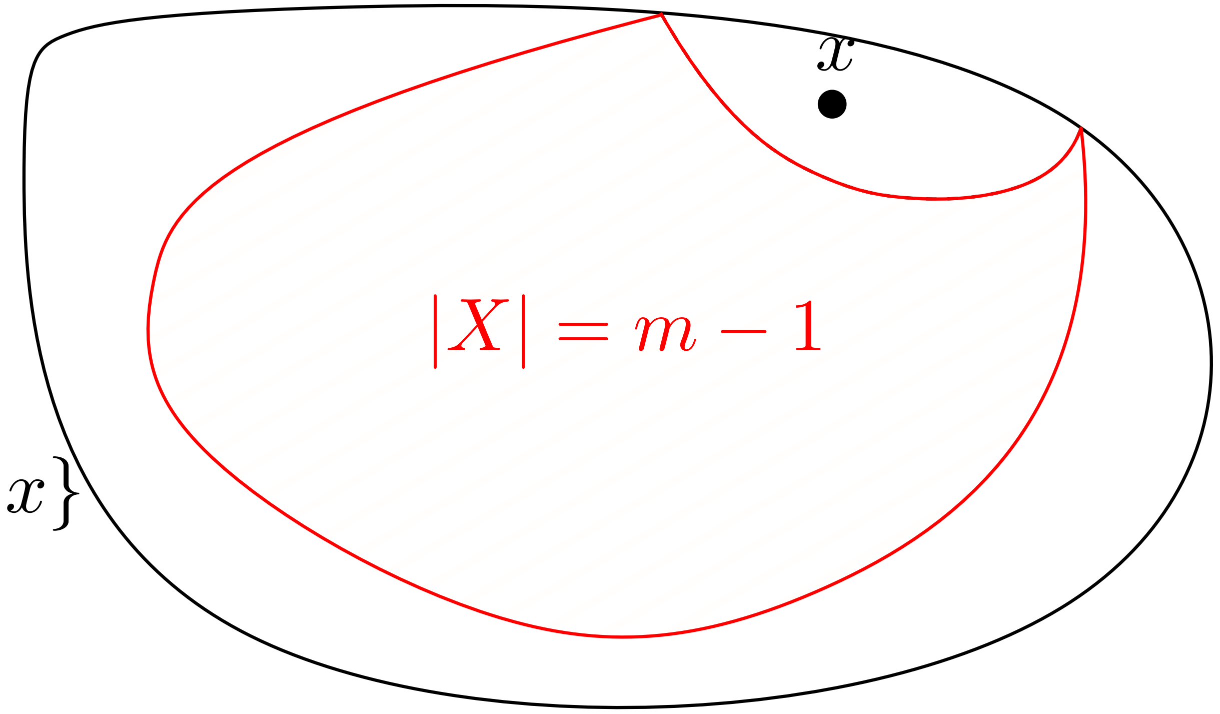

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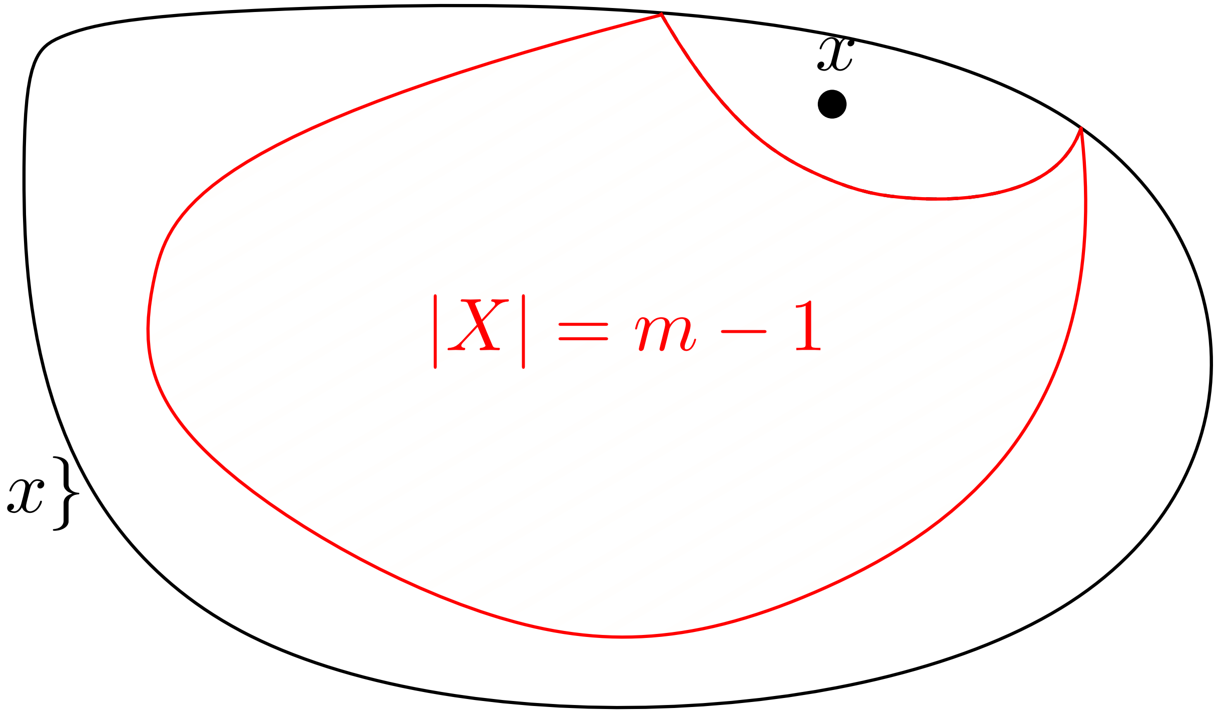

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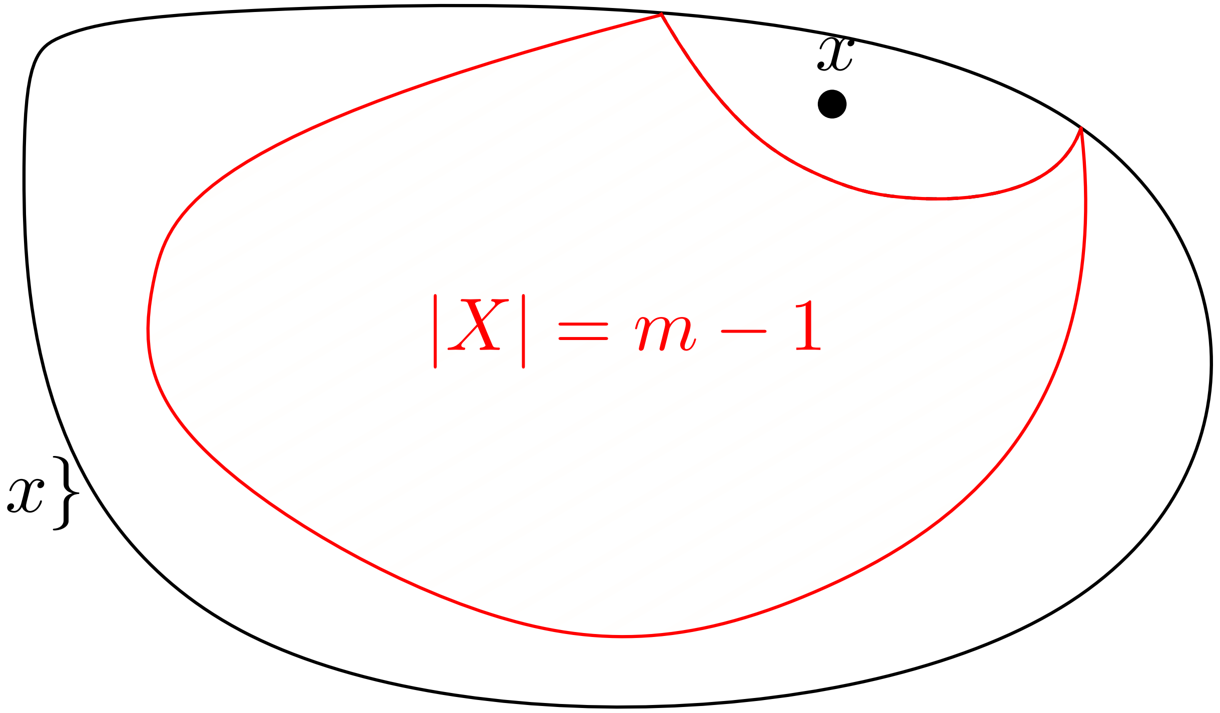

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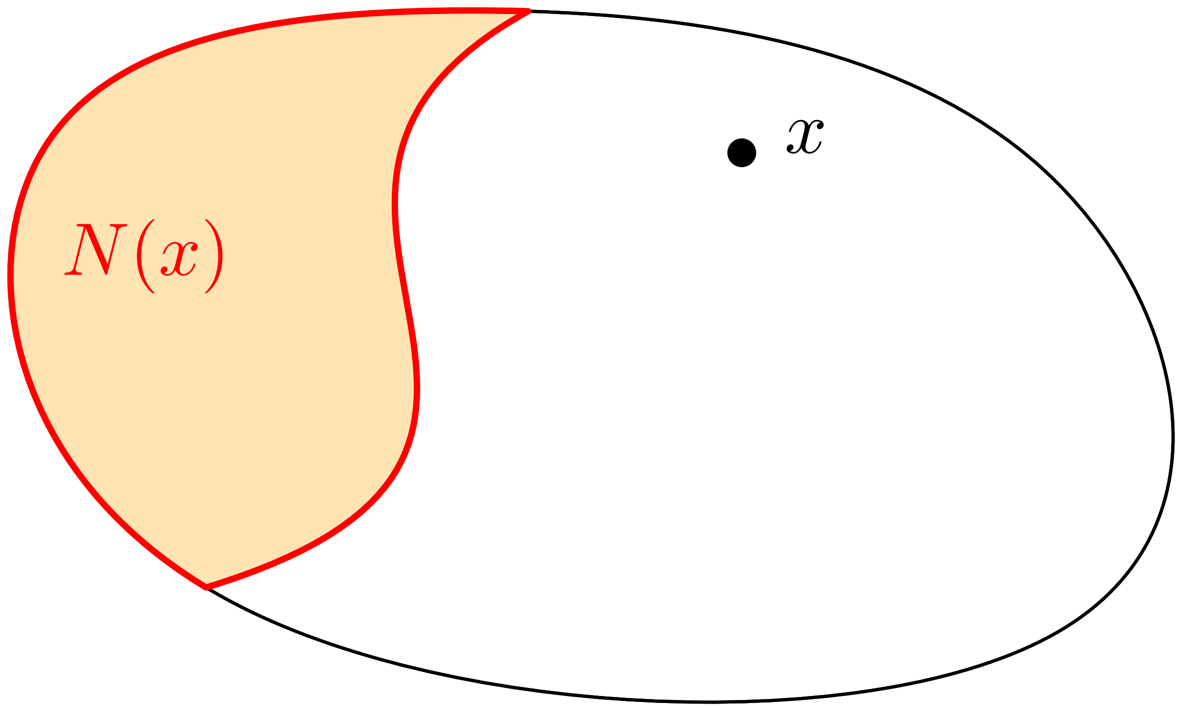

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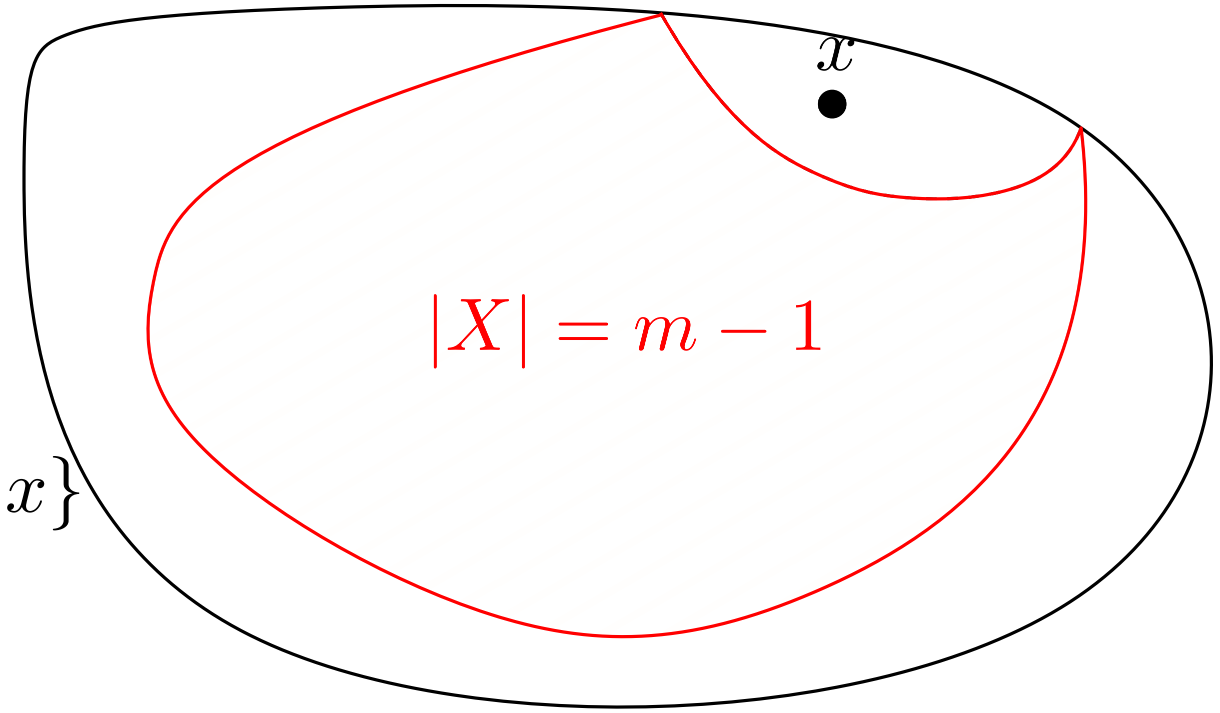

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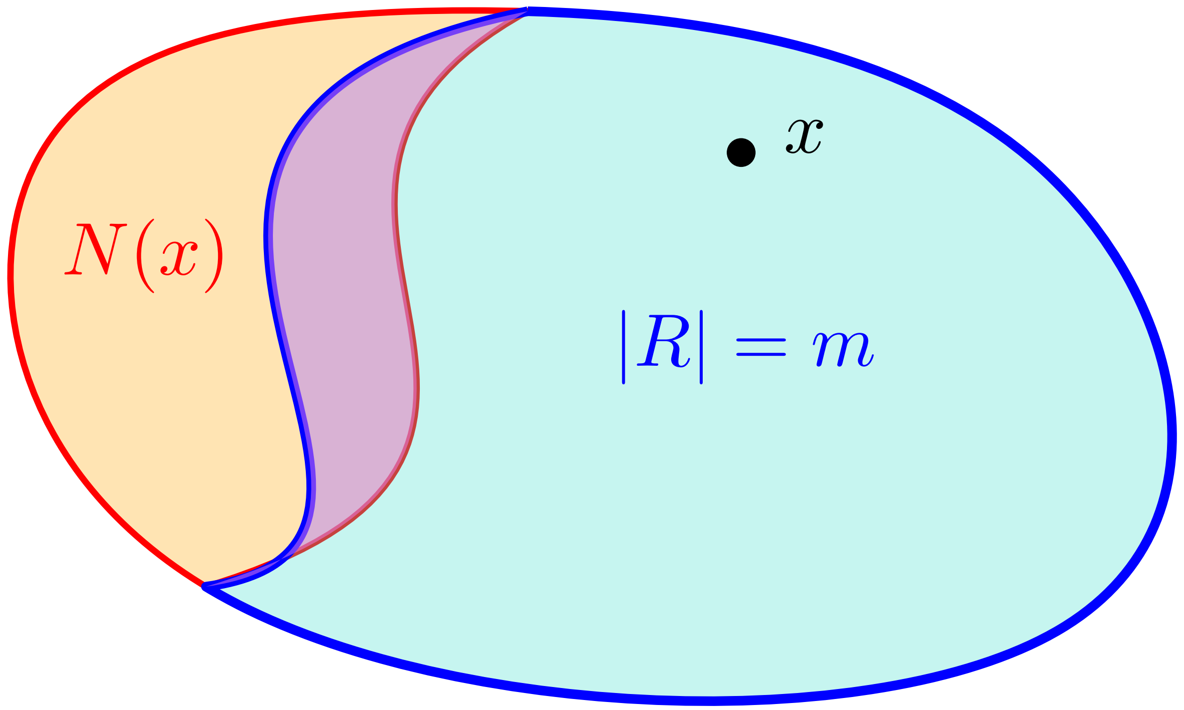

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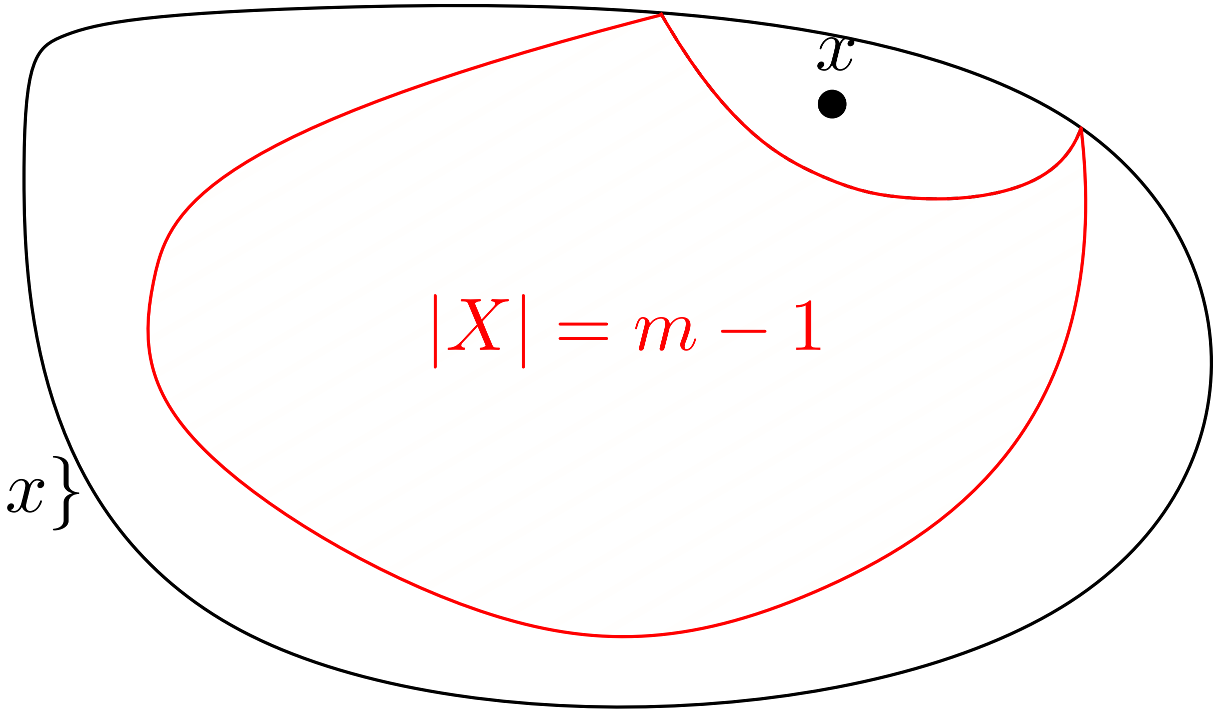

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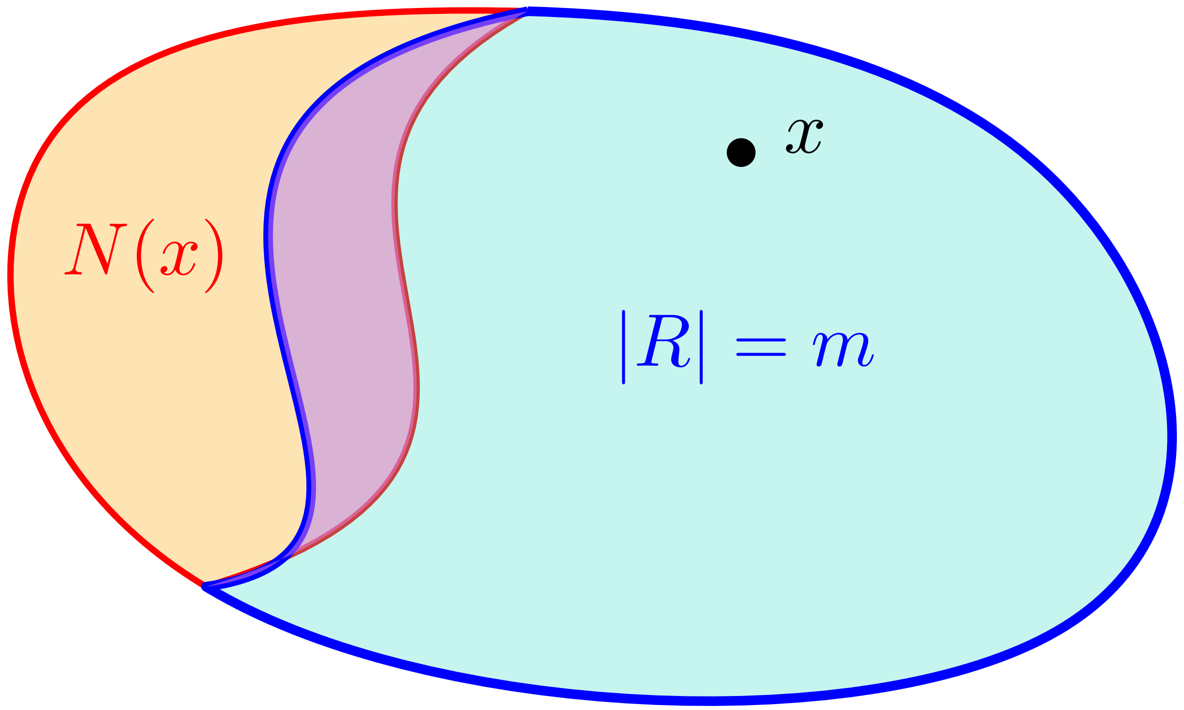

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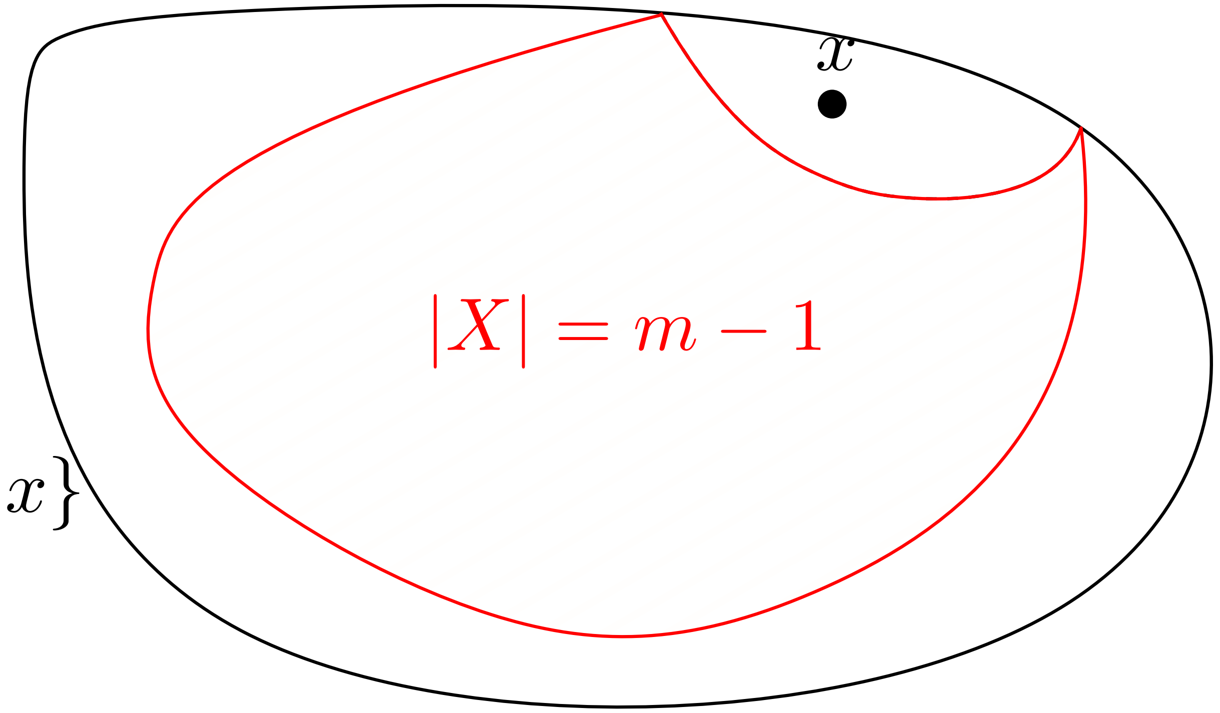

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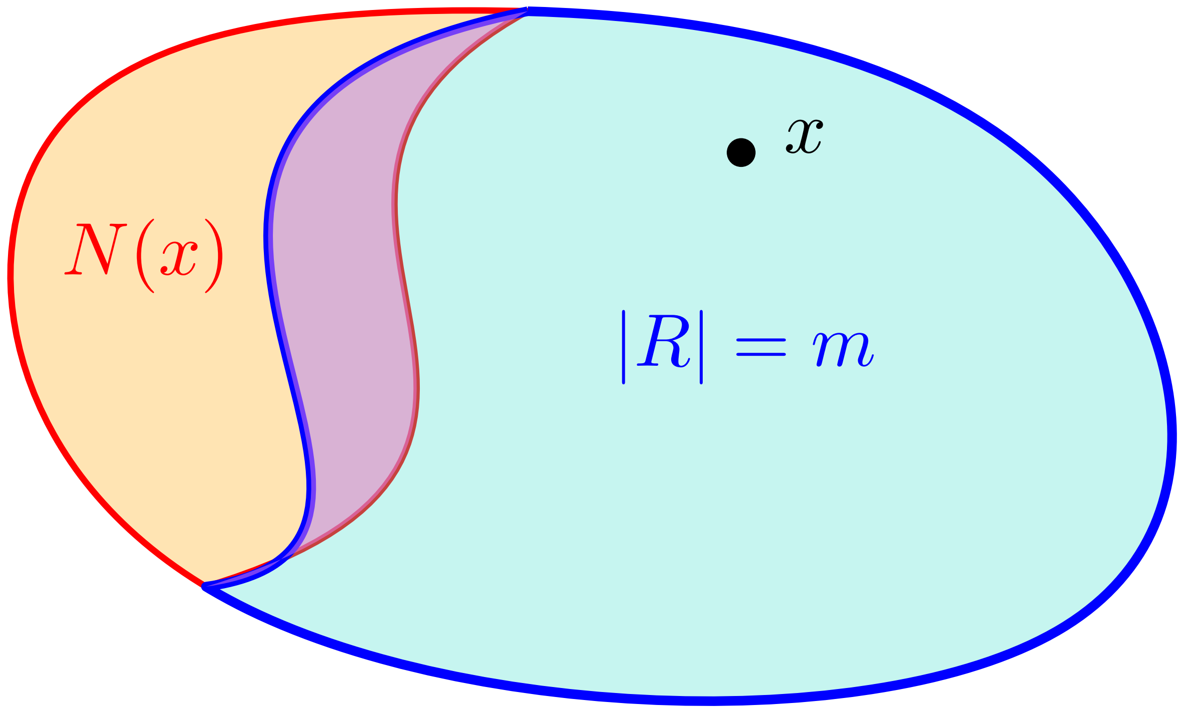

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$q \leq e(V \setminus N(x)) + \frac{d(x)-(n-m)}{d(x)} e(N(x), V \setminus N(x))$

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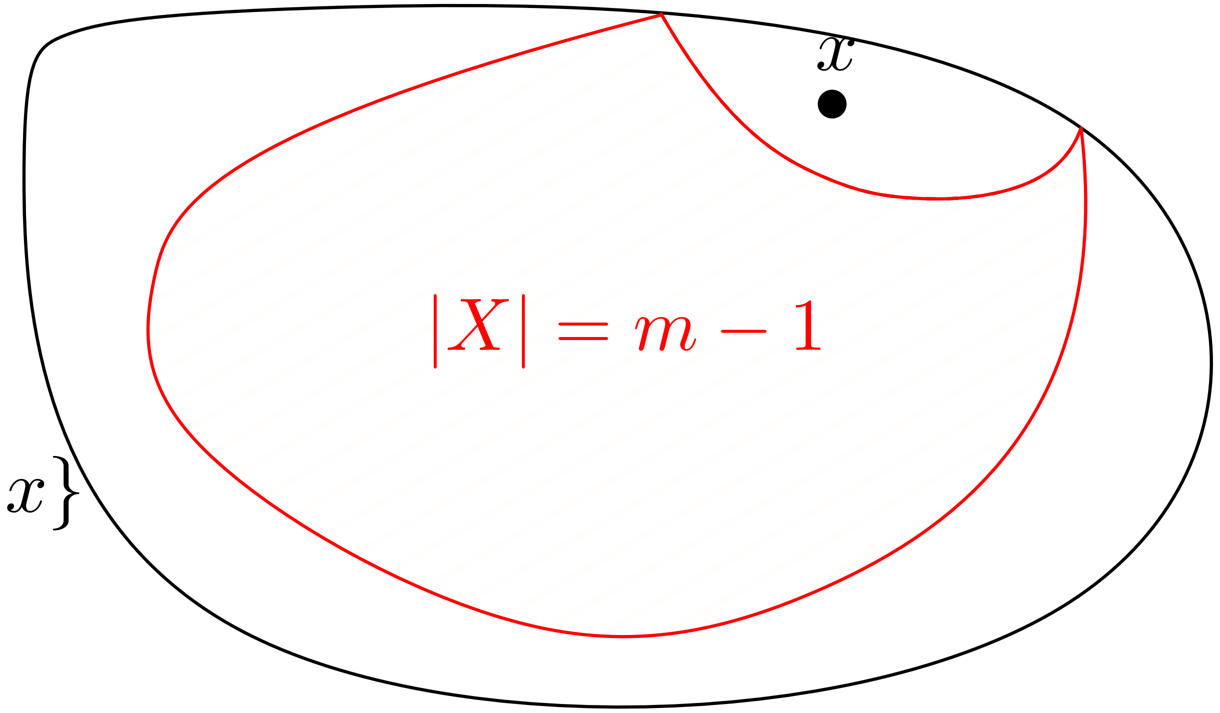
Then $e(G) \geq \frac{nq}{2m-n}$

(n, m, q, G) - minimal counterexample w. r. t. n

1. there exists x, so that $d(x) \leq n - m$



$d := d(x)$

$G' := G \setminus \{x\}$

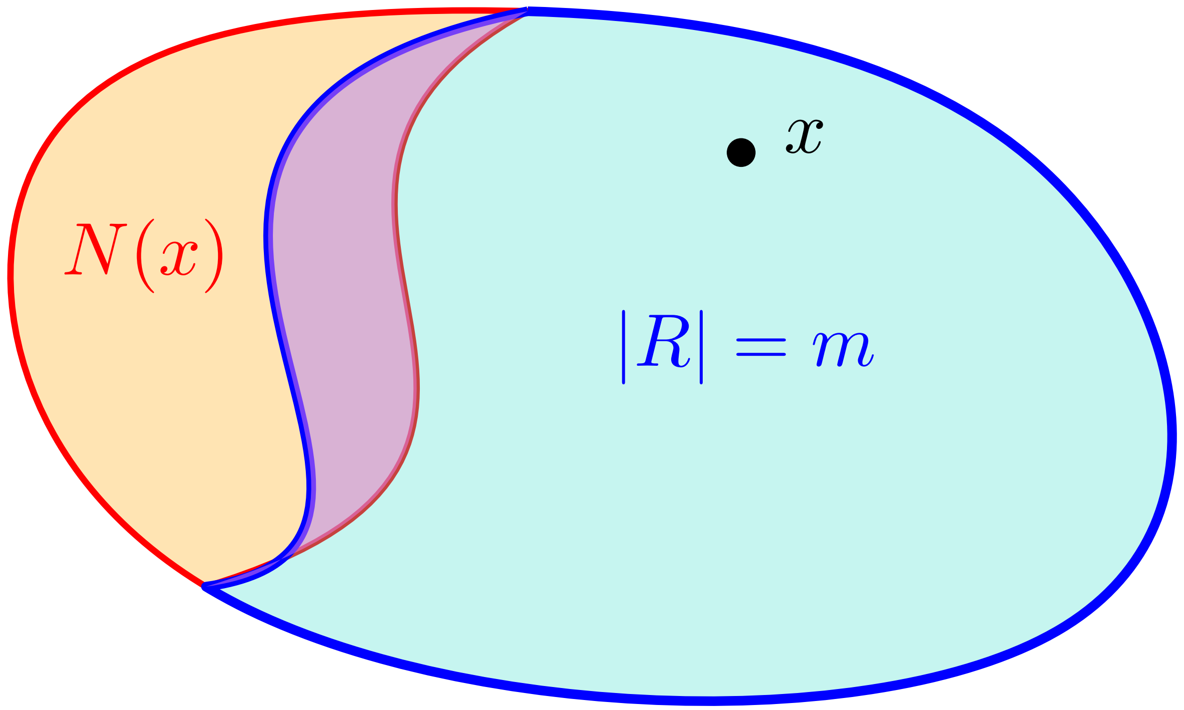
Every X has at least $q - d$ edges

$q - d > \frac{2(m-1)^2}{9}$

$(n-1, m-1, q-d, G')$

$e(G) = e(G') + d \geq \frac{(n-1)(q-d)}{2m-n-1} + d > \frac{nq}{2m-n}$

2. $d(x) > n - m$ for every x



$q \leq e(R) = e(V \setminus N(x)) + e(R \cap N(x), V \setminus N(x))$

$q \leq e(V \setminus N(x)) + \frac{d(x)-(n-m)}{d(x)} e(N(x), V \setminus N(x))$

$\frac{n-m}{d(x)} e(N(x), V \setminus N(x)) \leq e(G) - q$

$e(G)$ should be large, right?

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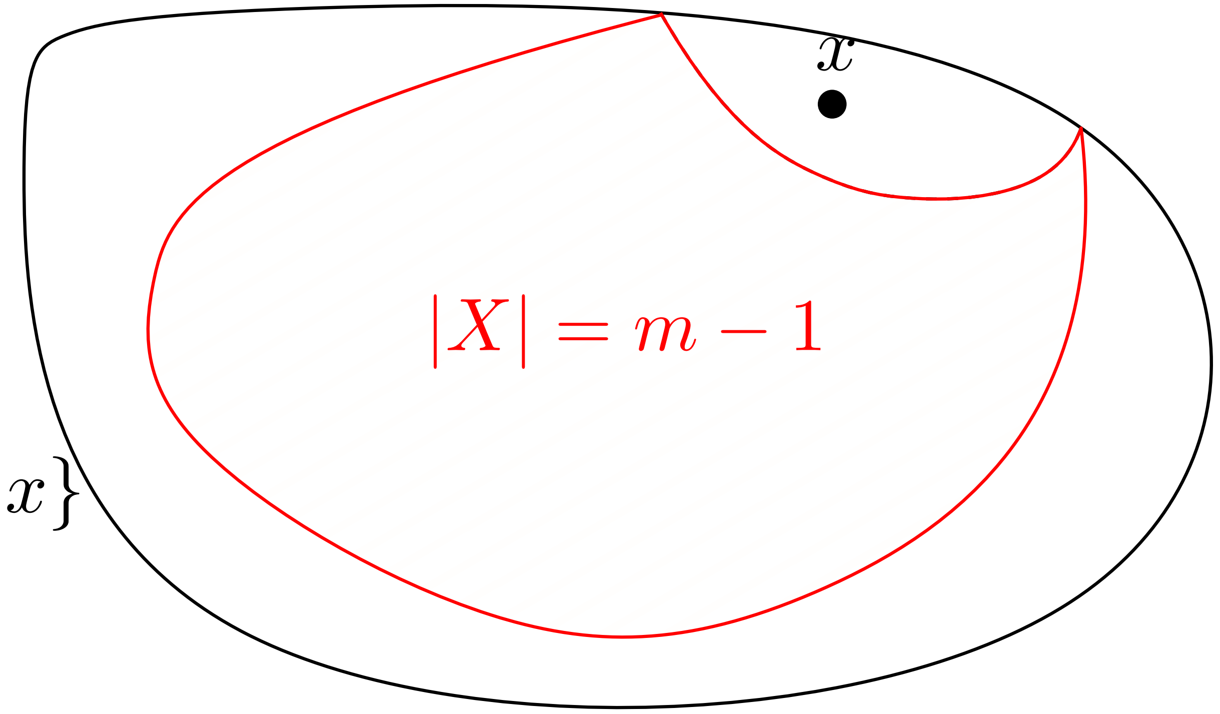

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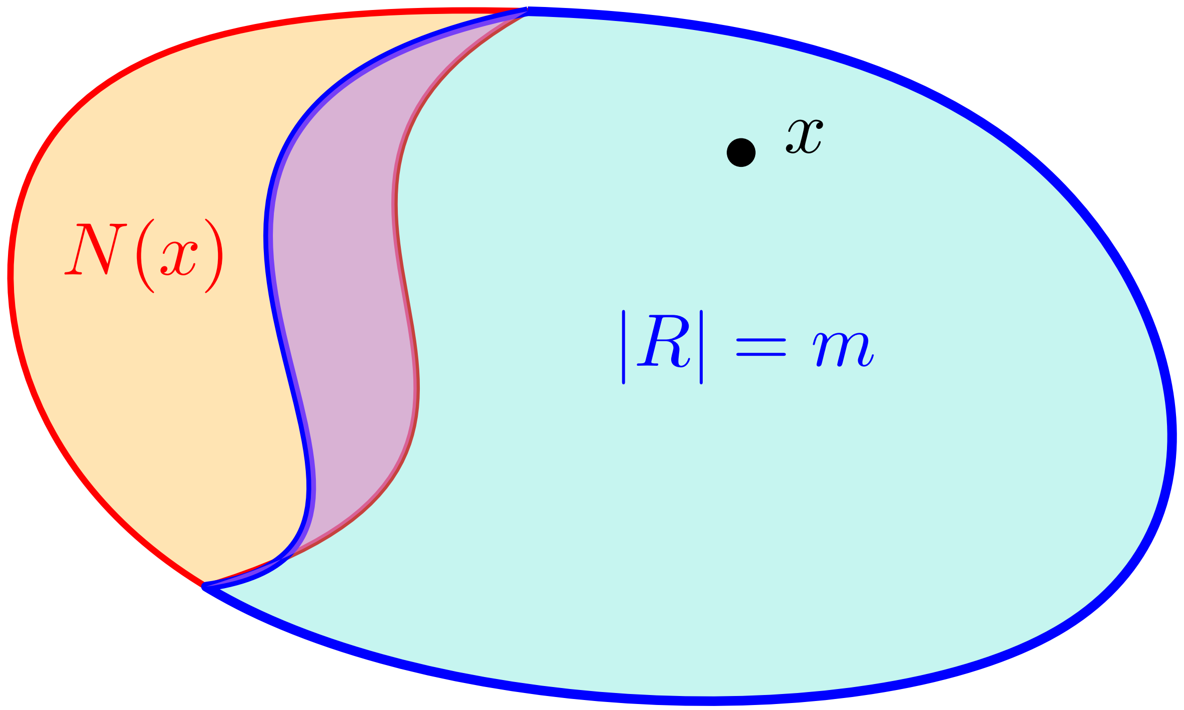

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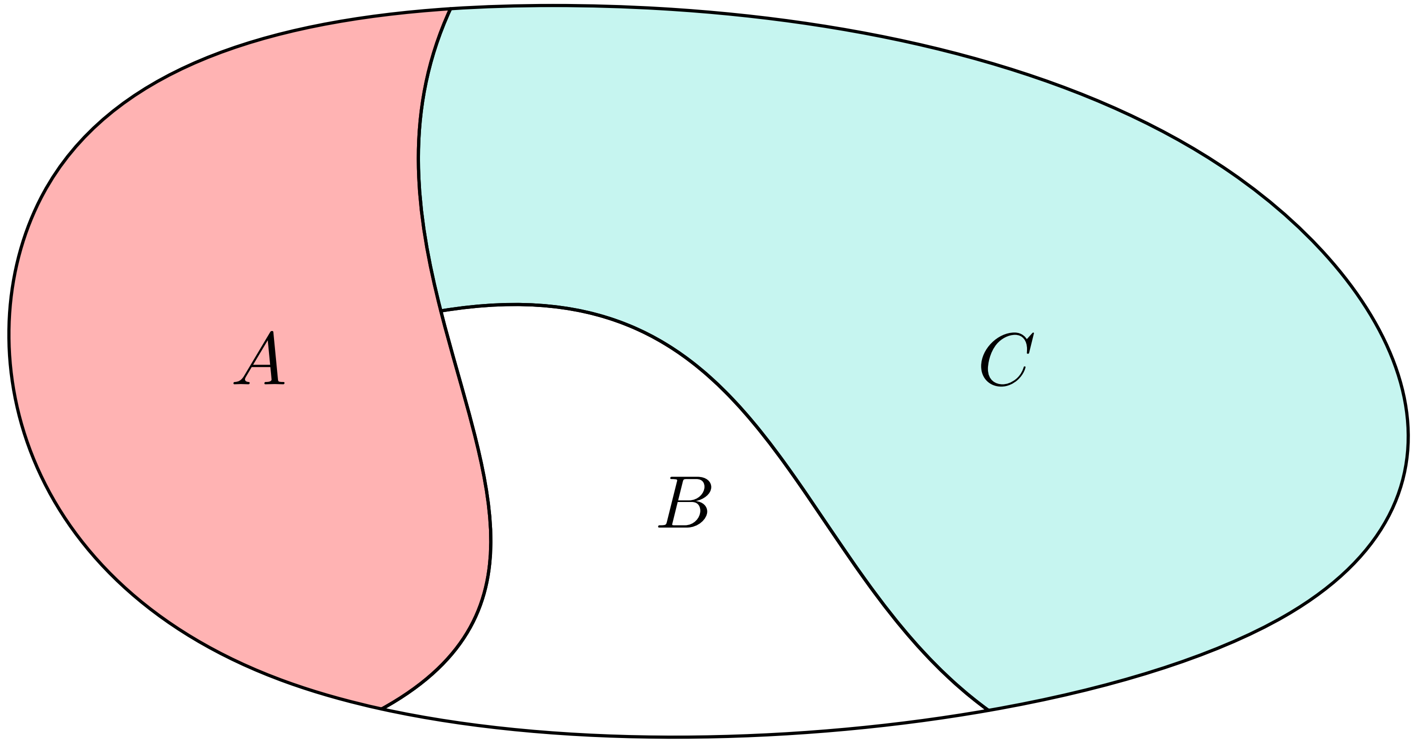
Sum over x

$2(n-m)e(G) \leq n(e(G) - q)$

Lower bounds on $e(G)$

G - *extremal*, $A, B \subseteq V(G)$ - disjoint, indpendent sets

Then $\frac{2}{9}n^2 + e(A, B) \leq |E|$



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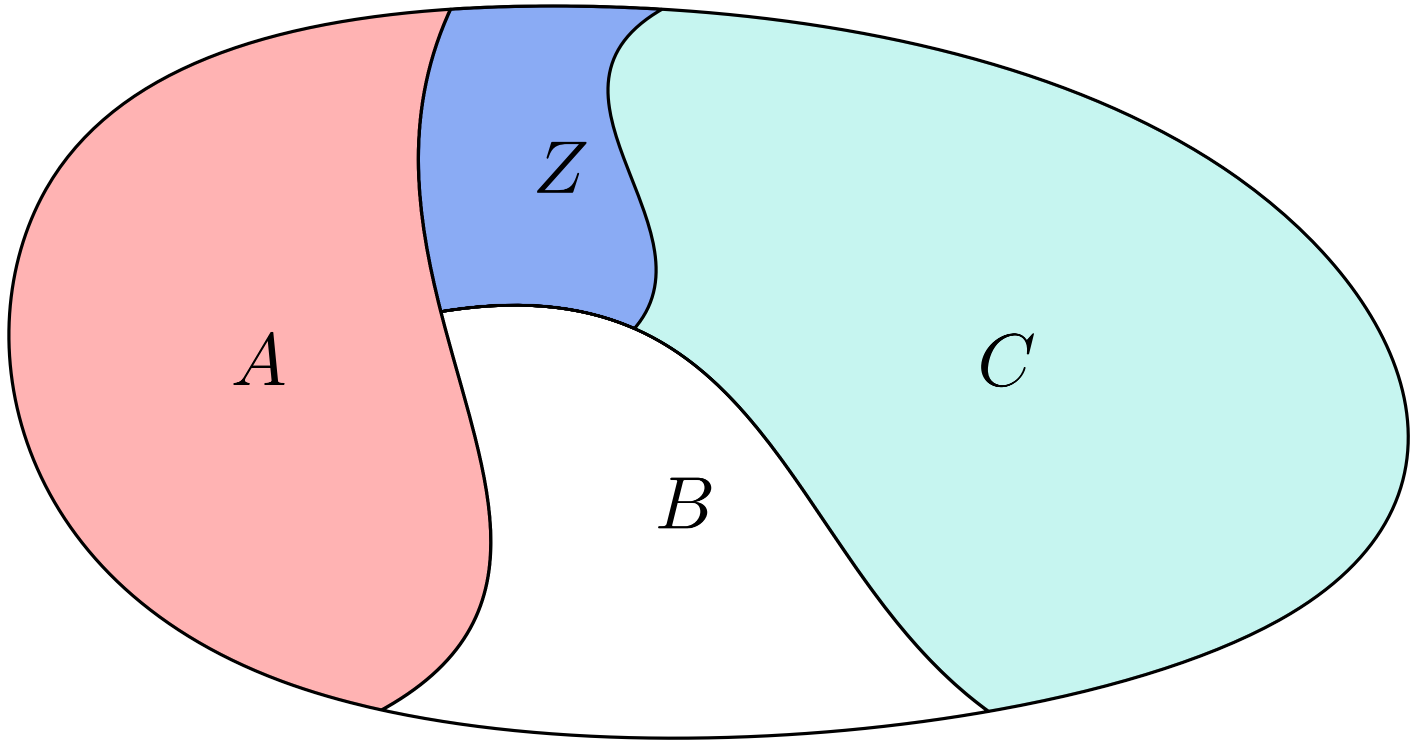
Then $\frac{2}{9}n^2 + e(A,B) \leq |E|$

$|A| + |Z| = \frac{n}{2}$

$\frac{n^2}{18} \leq e(A, Z) + e(Z)$



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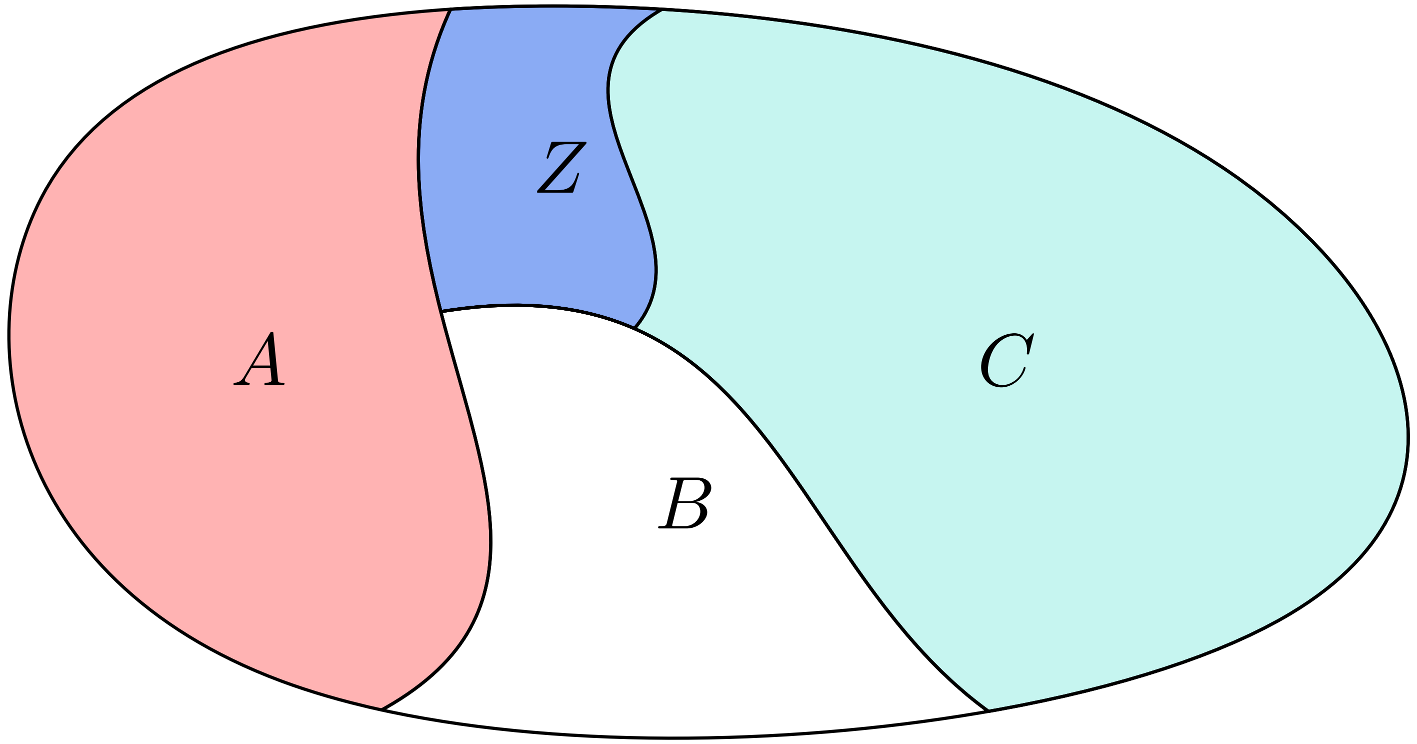

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$$\frac{n^2}{18} \leq \frac{\frac{1}{2}n - |A|}{|C|} e(A,C) + \left(\frac{\frac{1}{2}n-|A|}{|C|}\right)^2 e(C)$$

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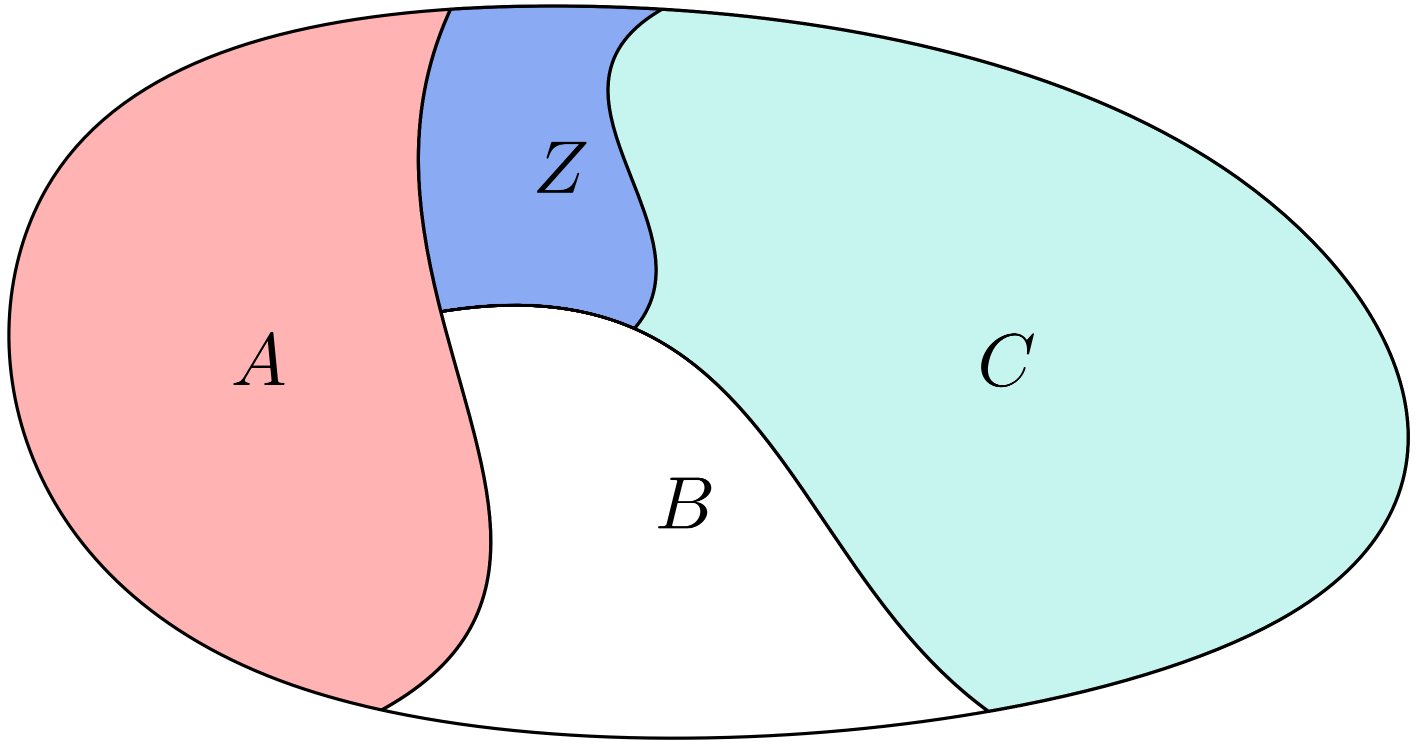

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$$\frac{n^2|C|}{9(n-2|A|)} \leq e(A,C) + \frac{\frac{1}{2}n - |A|}{|C|} e(C)$$

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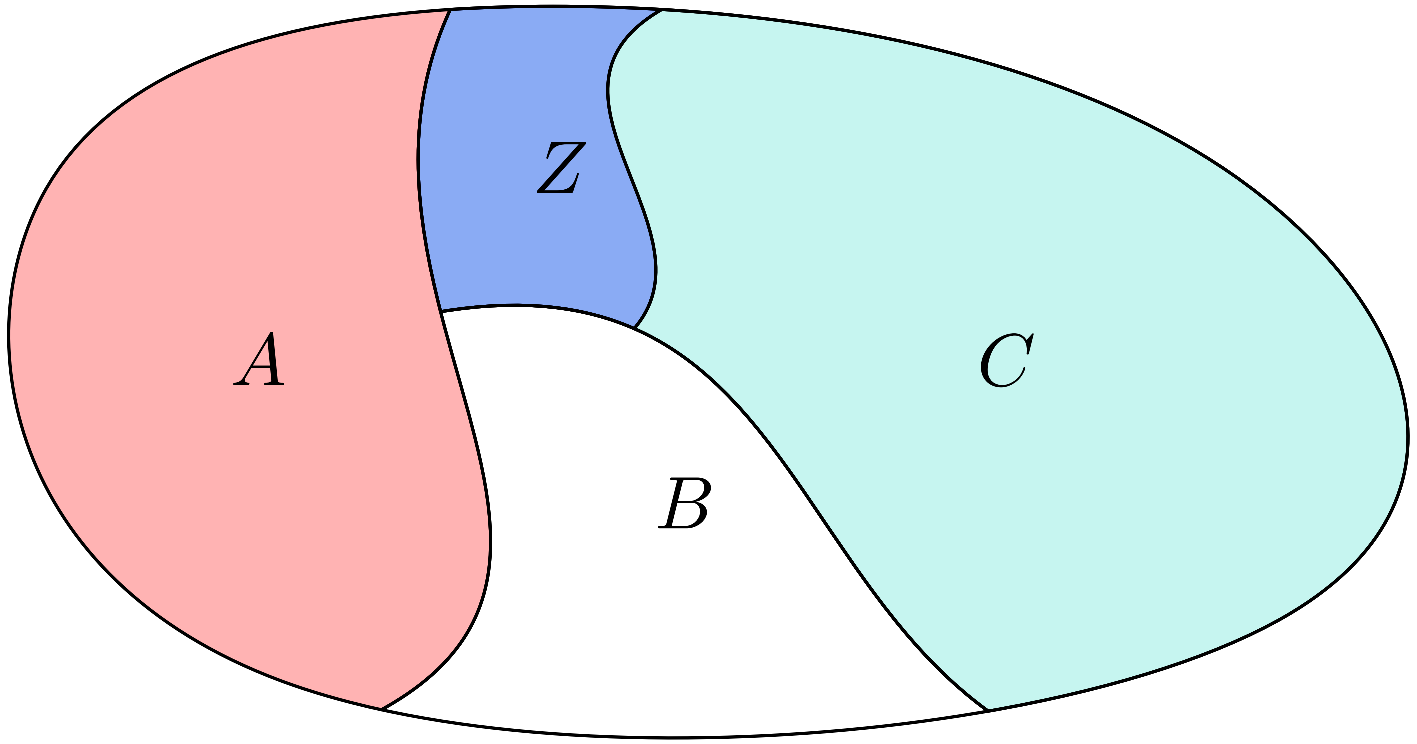

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Lower bounds on $e(G)$

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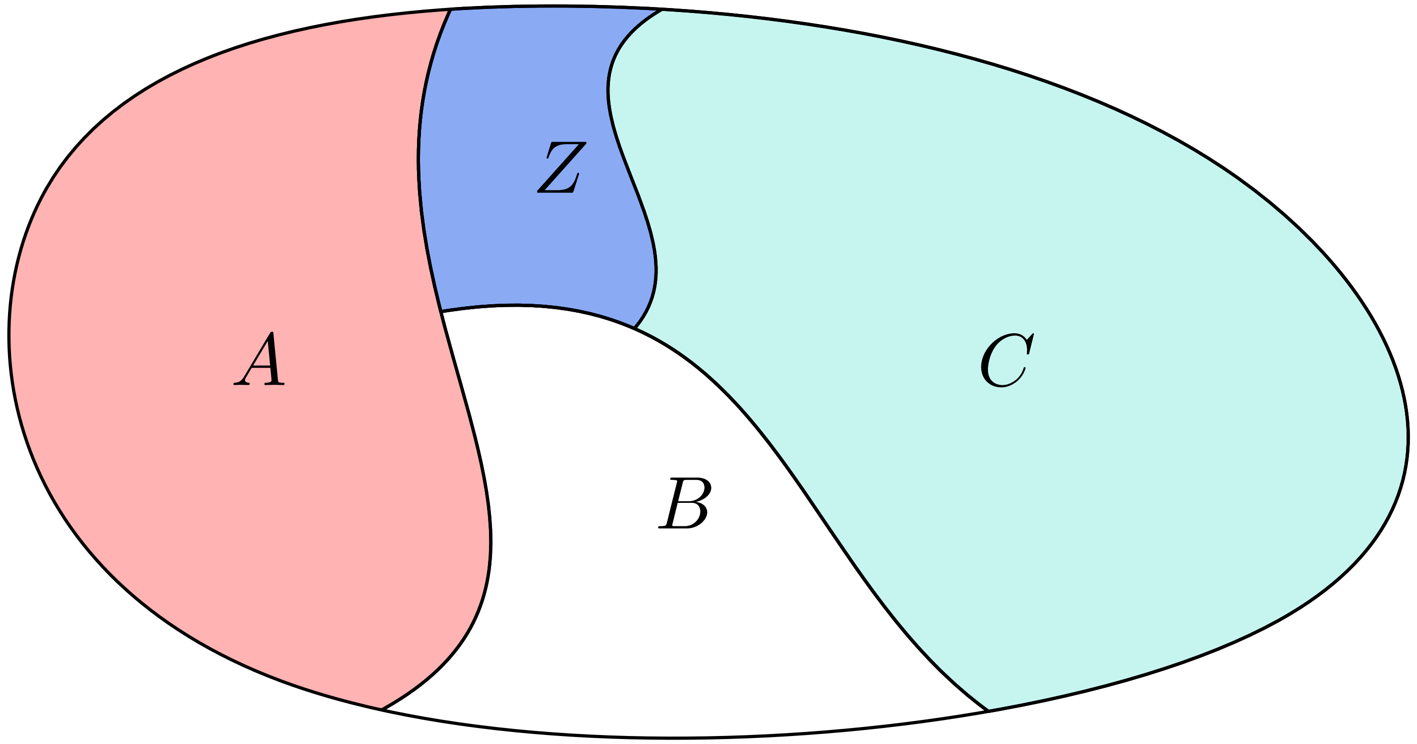

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$$\frac{n^2|C|}{9}\frac{2}{|C|} \leq |E| - e(A,B)$$

$e(G) \geq \frac{7}{24}n^2$ - preparations

G - $\triangle$-free, m - natural
Every $X \subseteq G$ of size m spans at least $\frac{2}{9}m^2$ edges
There exist $A, B \subseteq G$, A, B independent so that
$t = |A| + |B|$, $n \leq \frac{m}{3} + \frac{3t}{4}$

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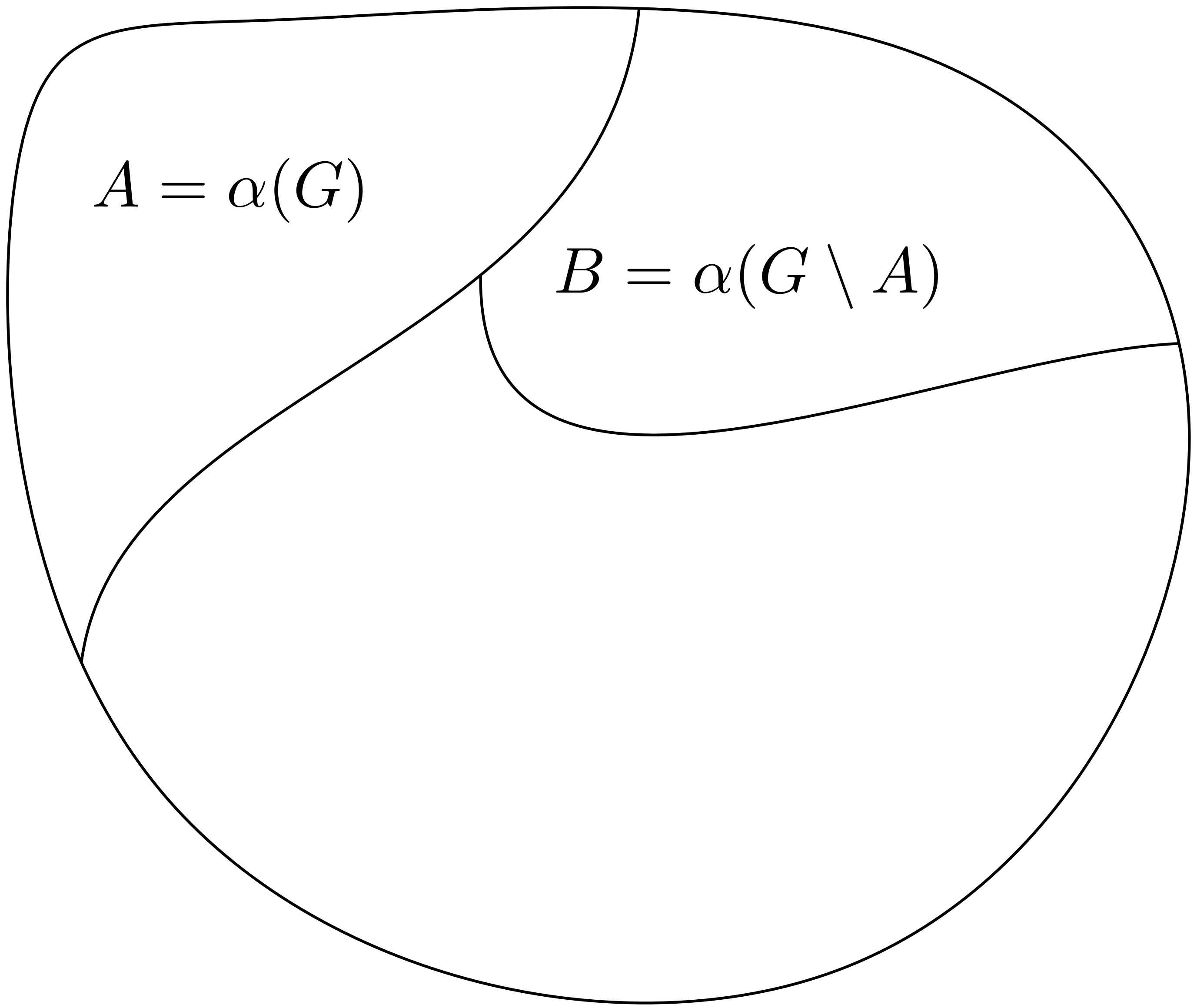

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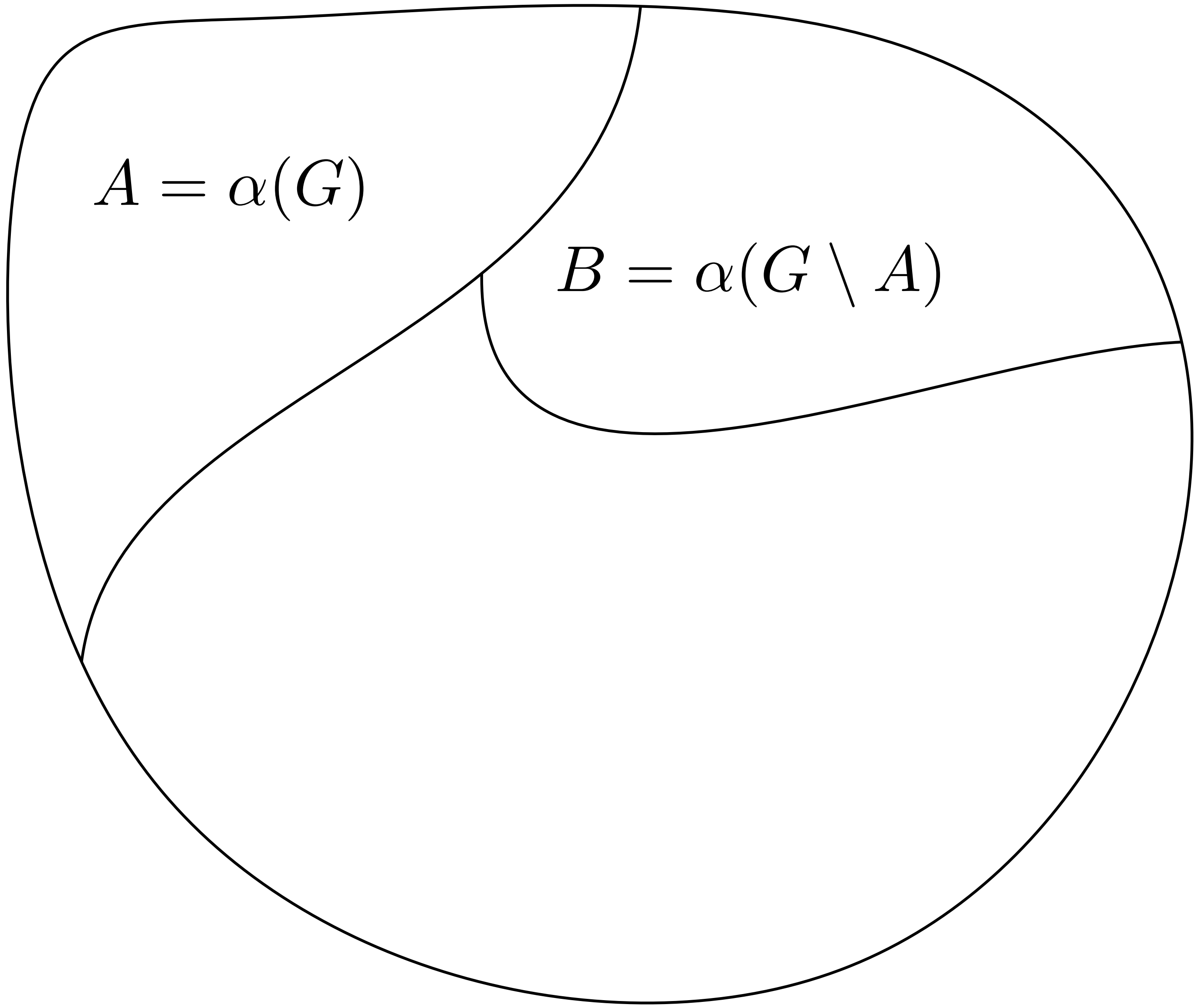
$m \geq \frac{3}{4}n$



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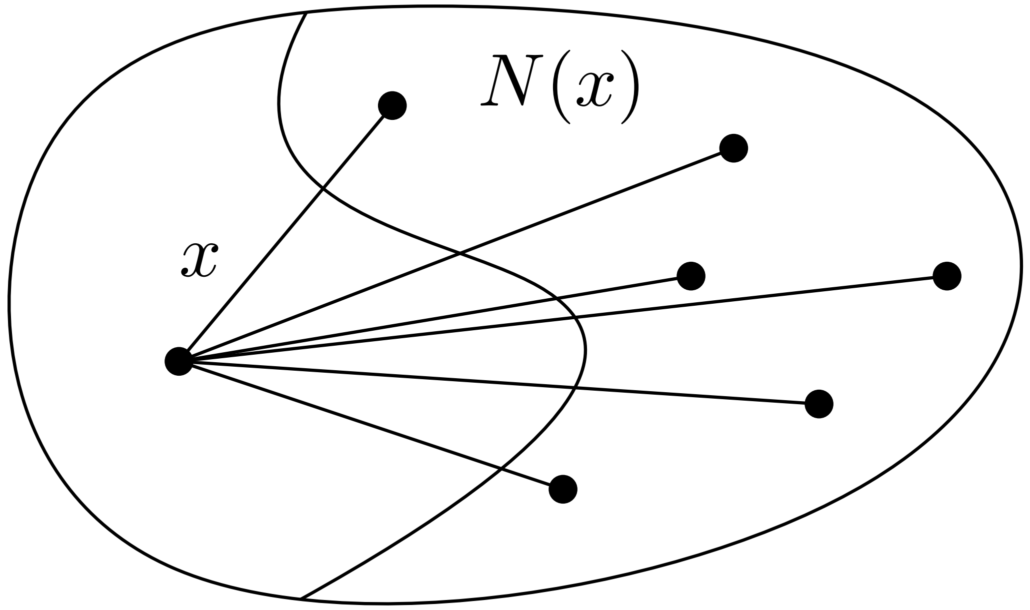
G - *extremal*
$d(x) \geq \frac{n}{2}$



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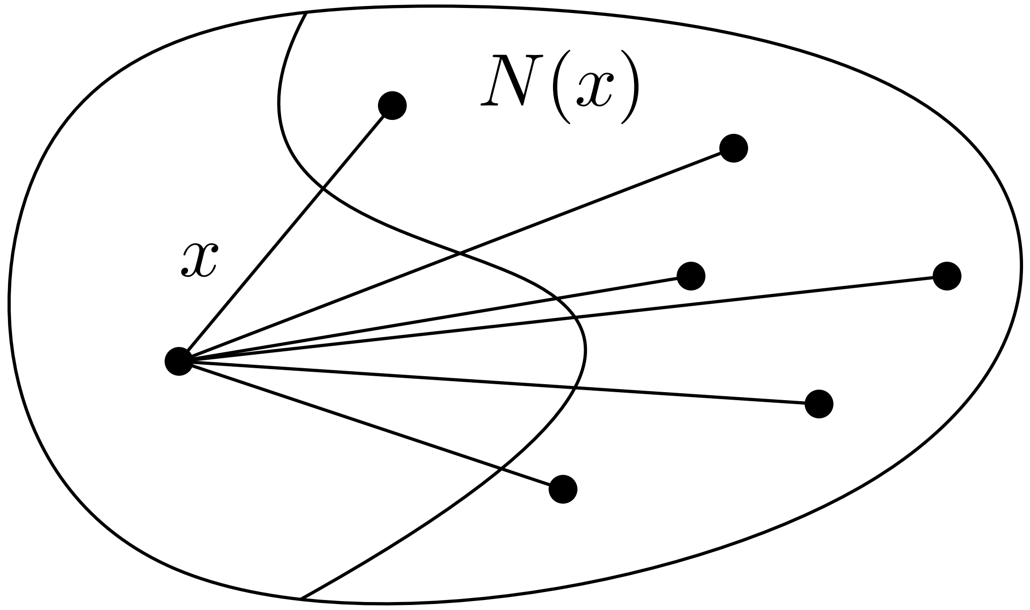

$G := N(x), m := \frac{n}{2}$

There exist $A, B \subseteq N(x)$, so that A, B - independent and $|A| + |B| \geq \frac{4}{3}d(x) - \frac{2}{9}n \geq \frac{4}{9}n$

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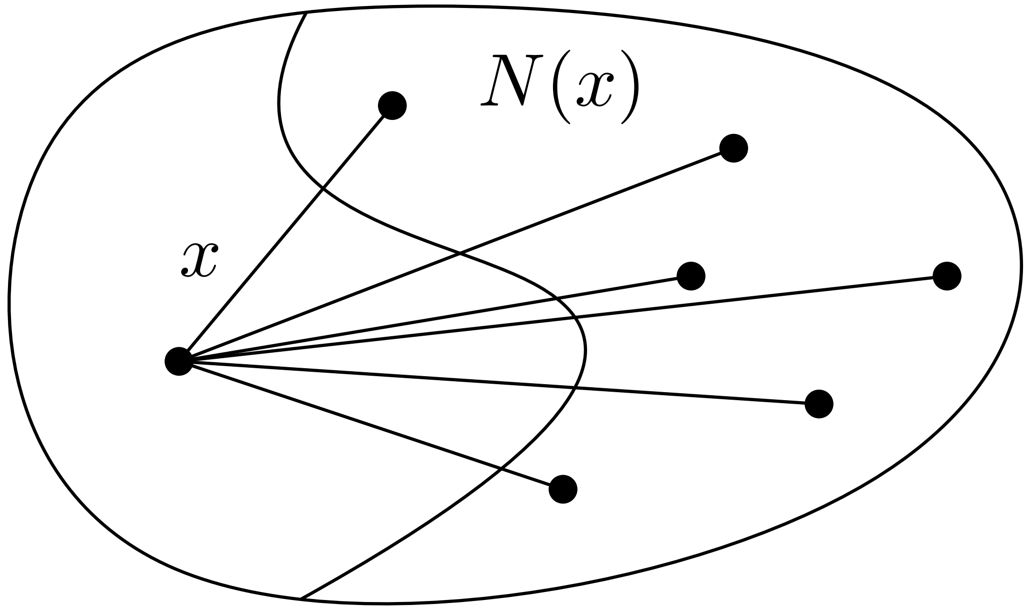

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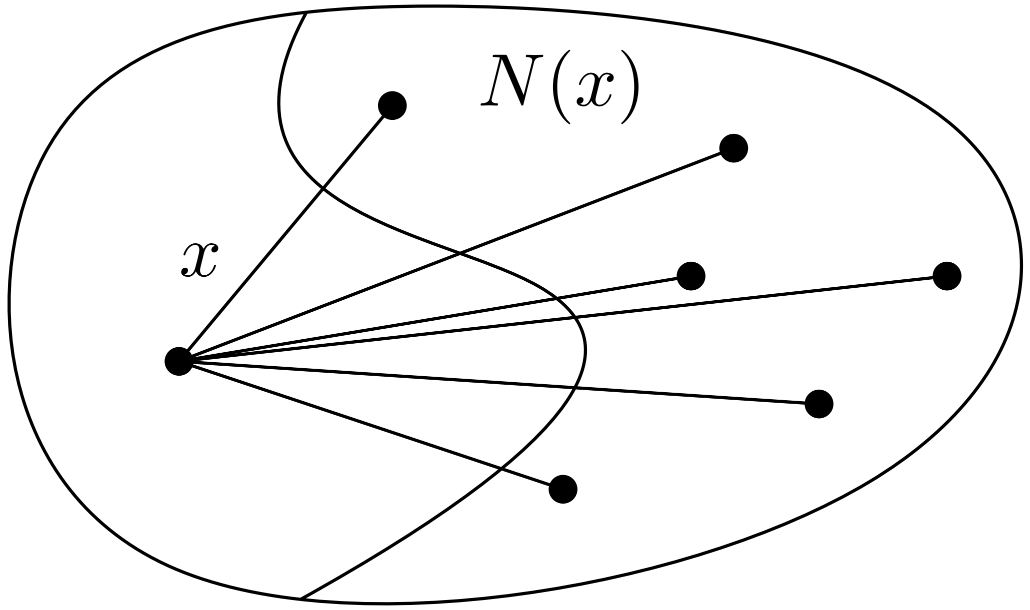

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In *extremal* graph G, if $d(x) \geq \frac{n}{2}$, then:

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$|E(G)| = \gamma n^2, \gamma > \frac{1}{4}$

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$$\alpha(G) \leq \frac{n}{3}$$

$$e(G) \geq \frac{7}{24}n^2$$

Three independent sets

G - *extremal* with γn^2 edges
Then there exist three disjoint independent subsets V_1, V_2, V_3, so that $|V_1| + |V_2| + |V_3| \geq \frac{n}{3(1-2\gamma)}$

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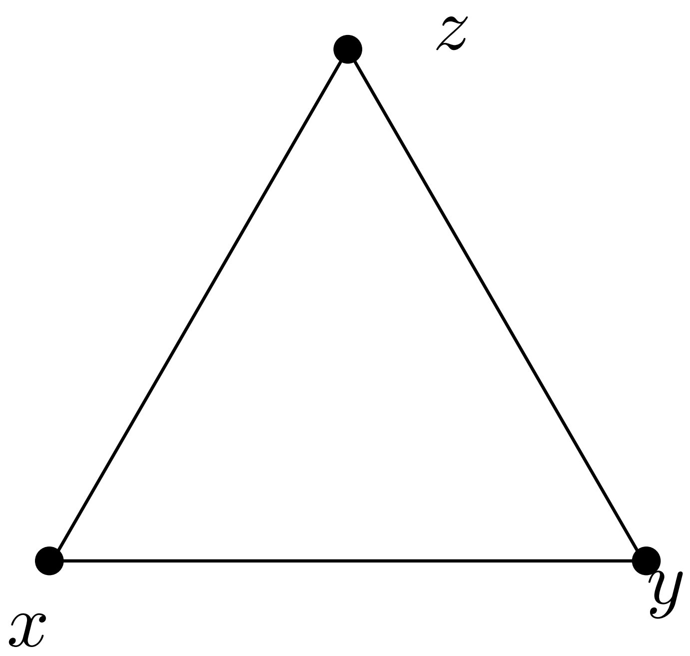

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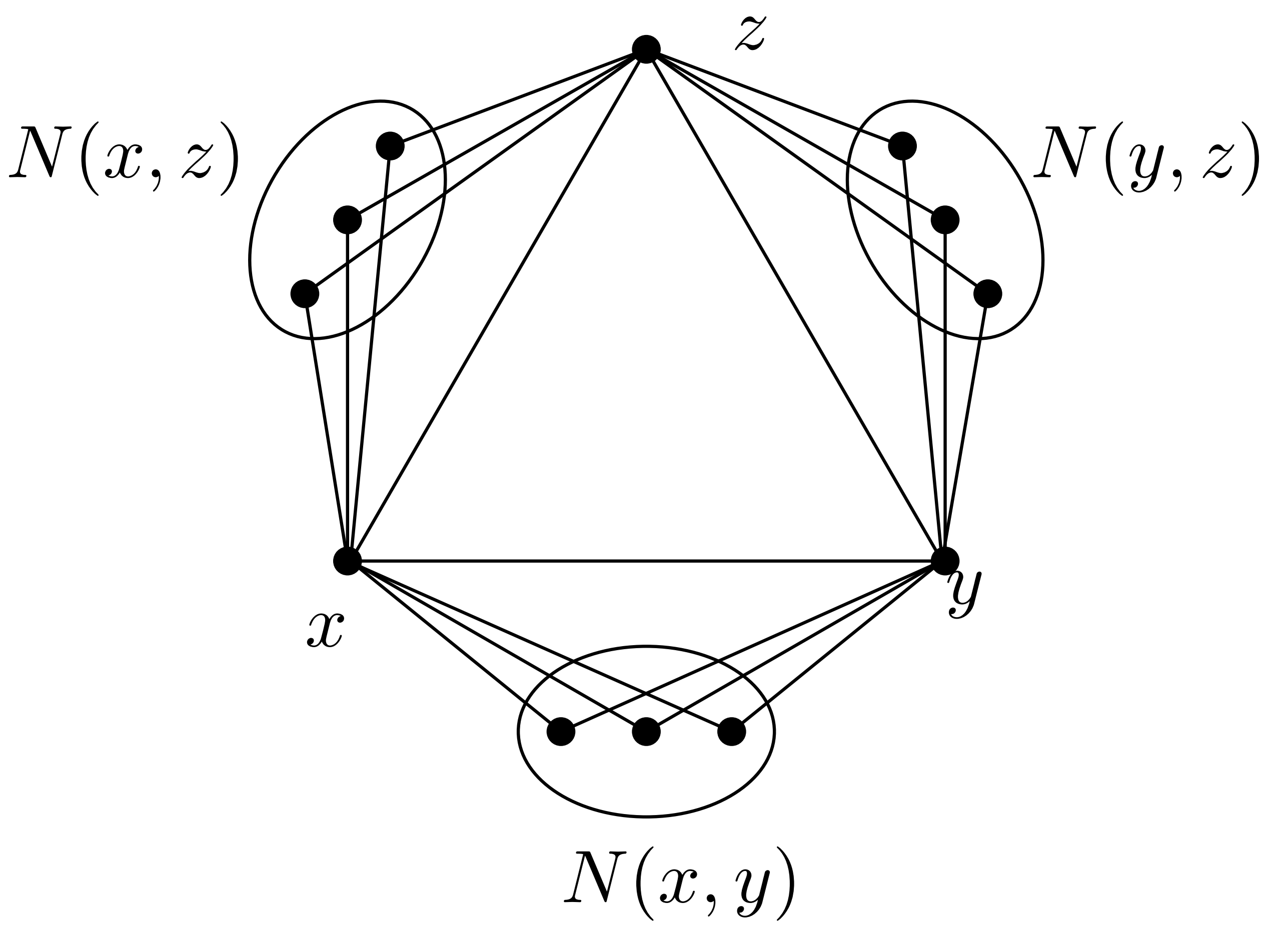

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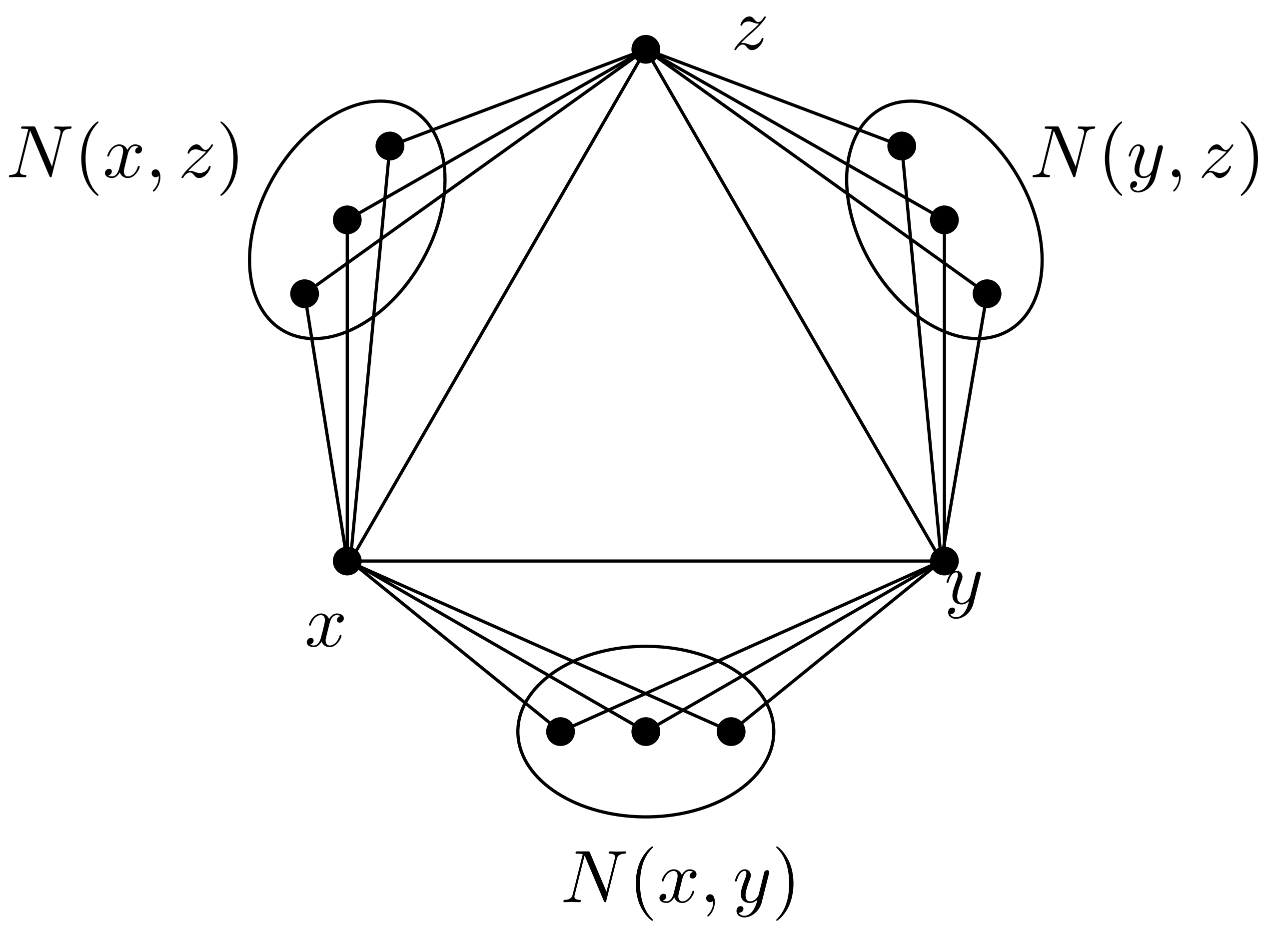

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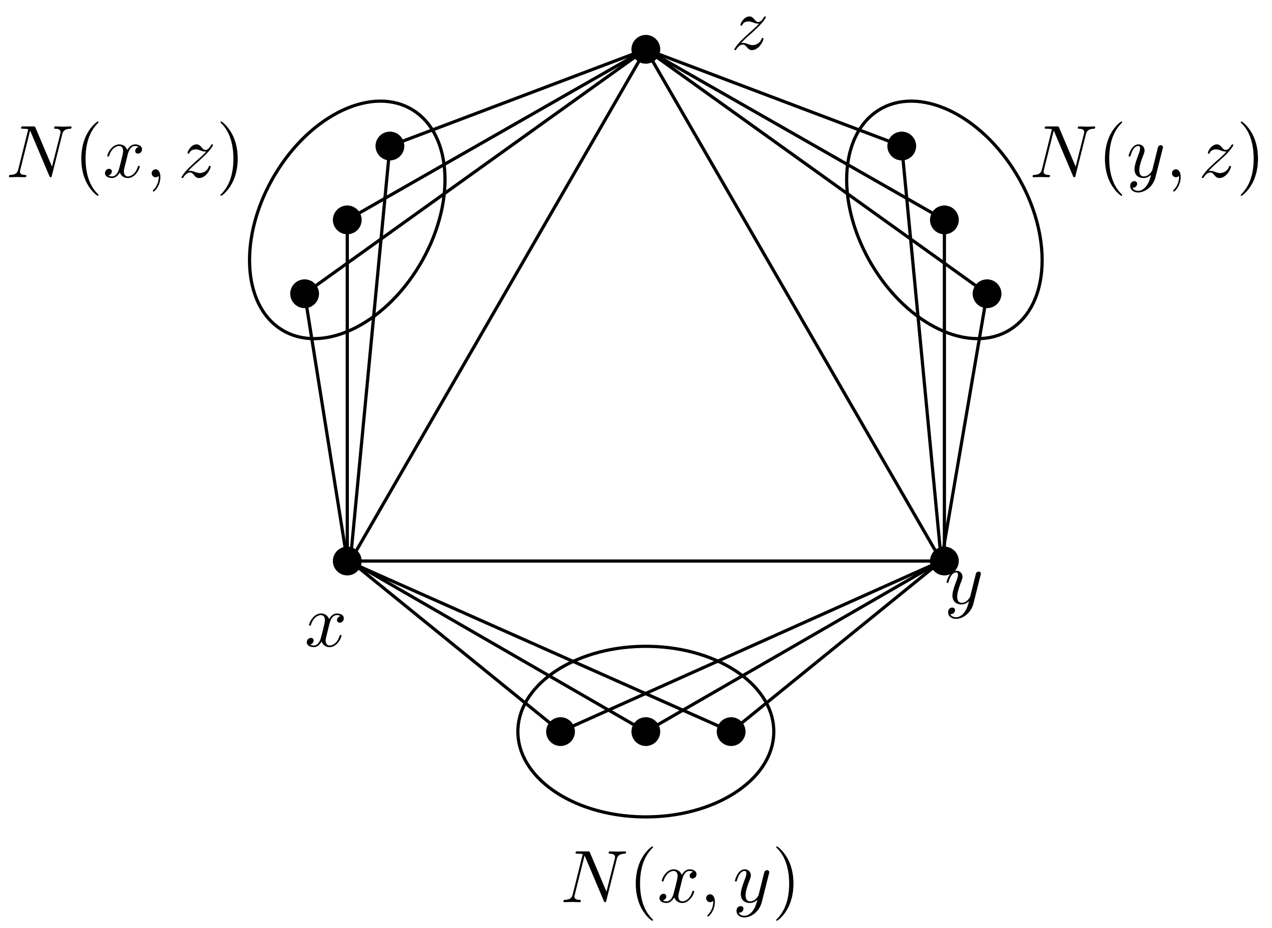

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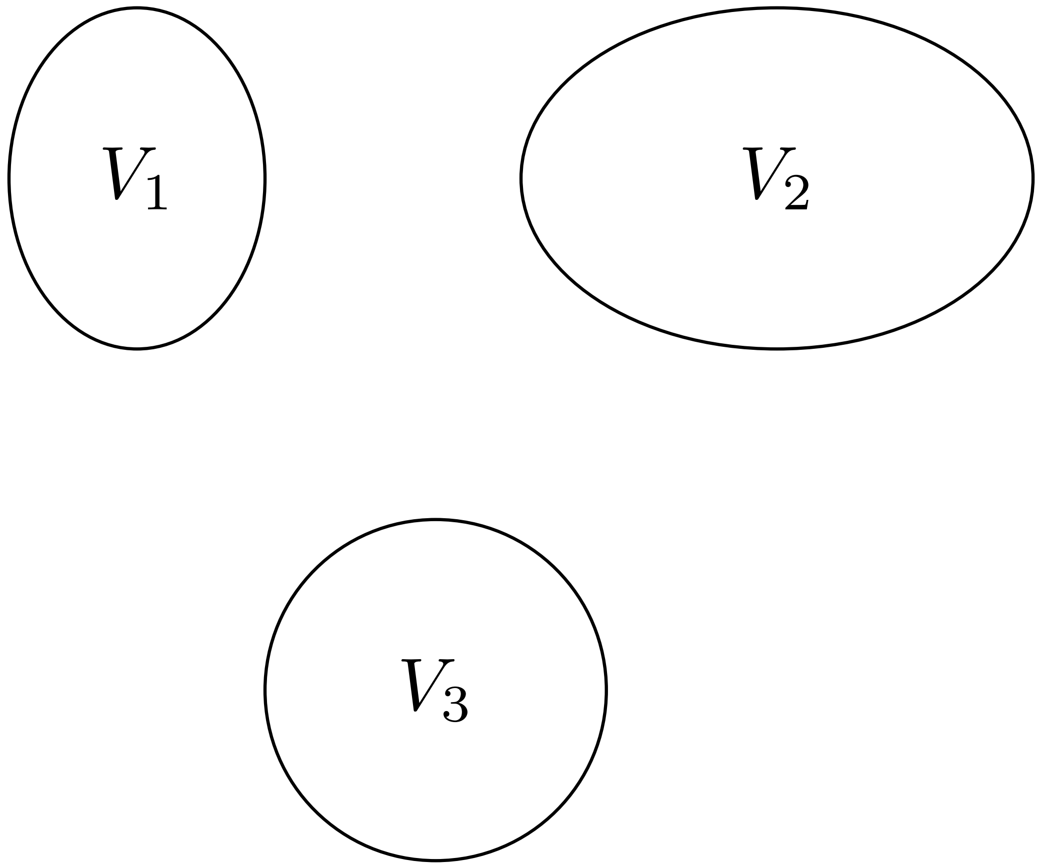
$$\sum_{y \in N(x)} t_{xy}^2 \geq \frac{4t(x)^2}{d(x)}$$

Leftovers



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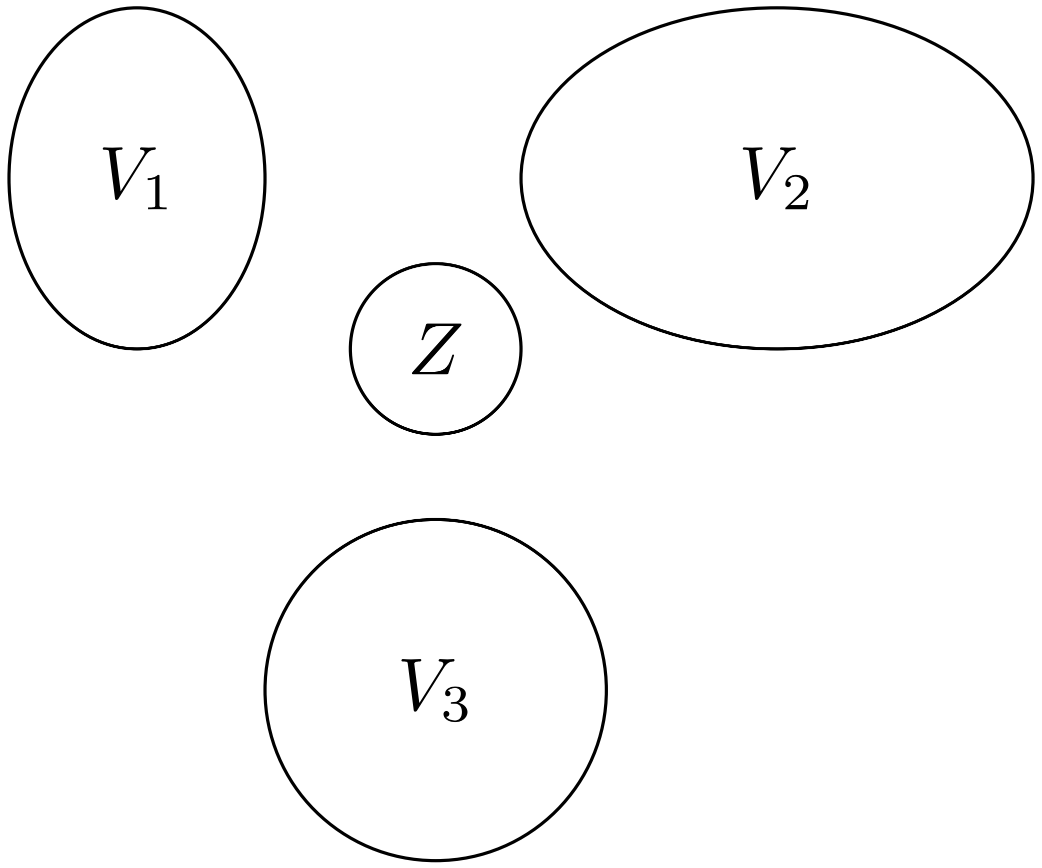

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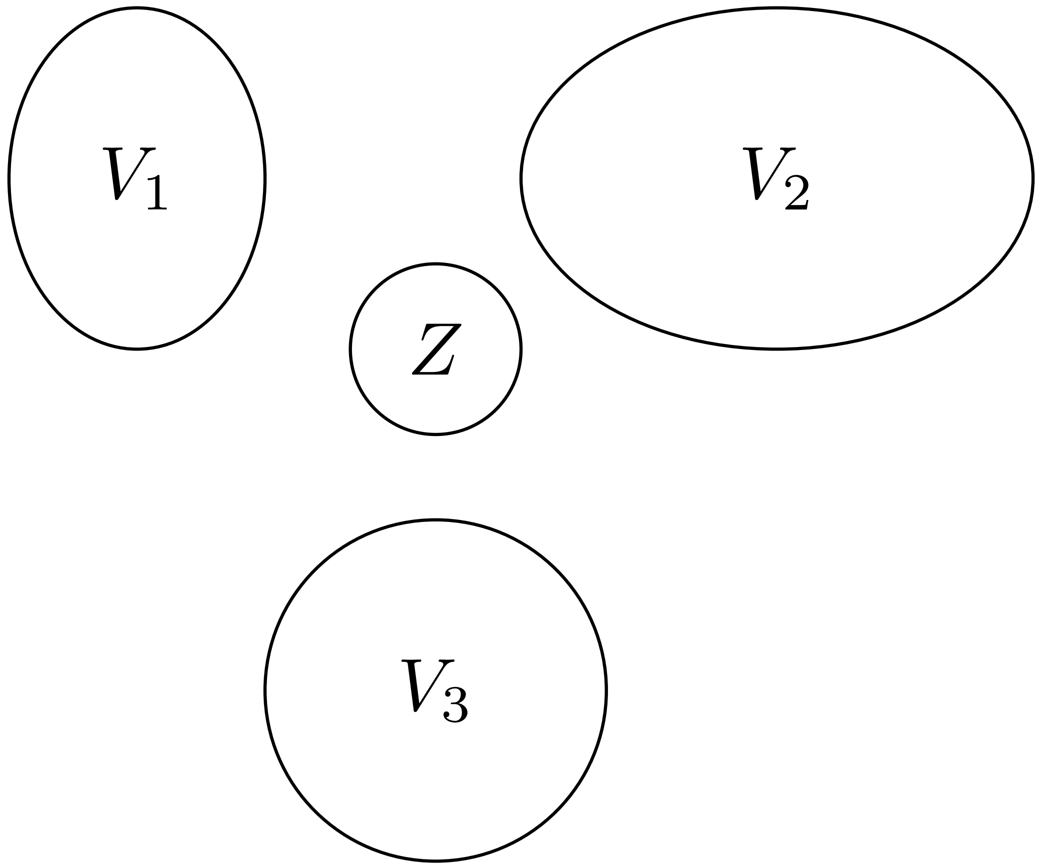

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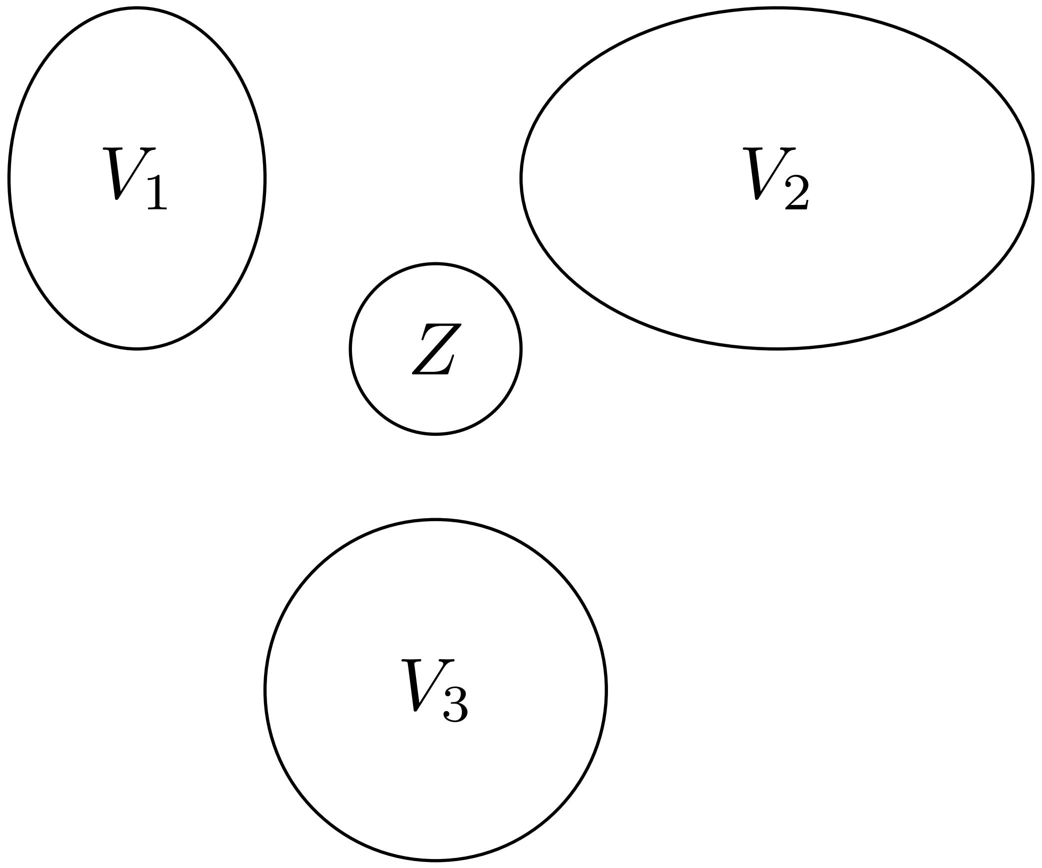

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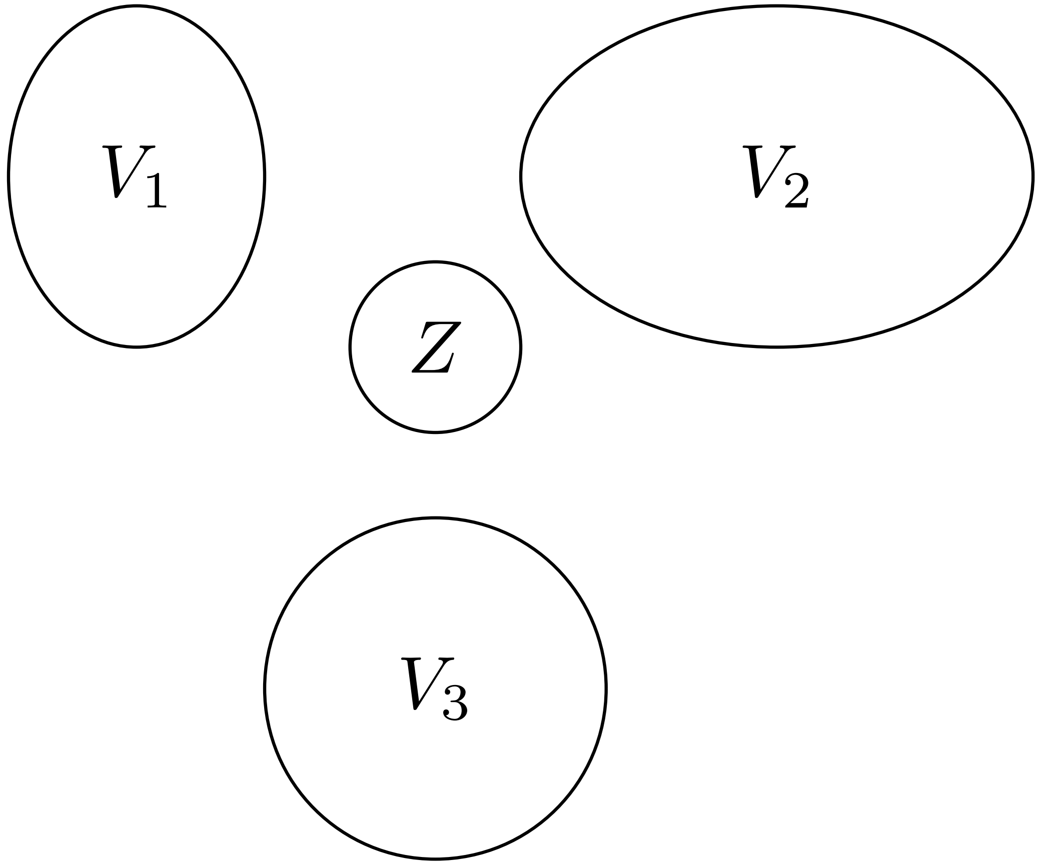

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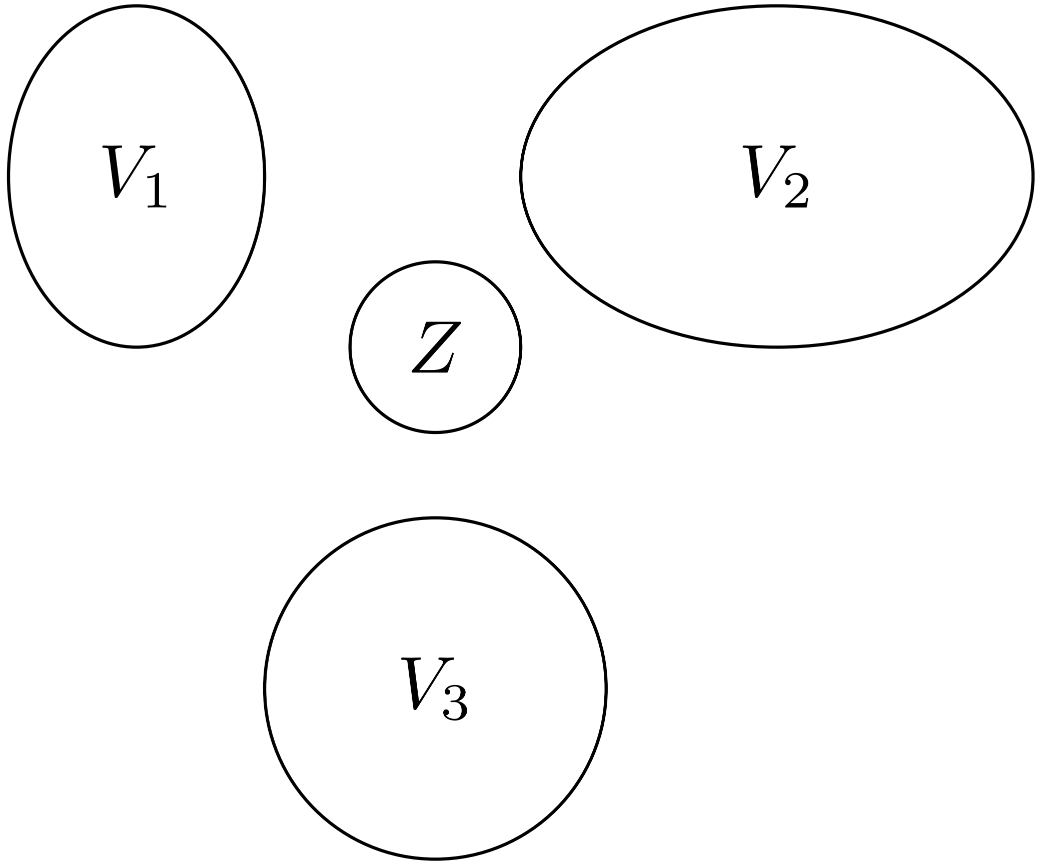

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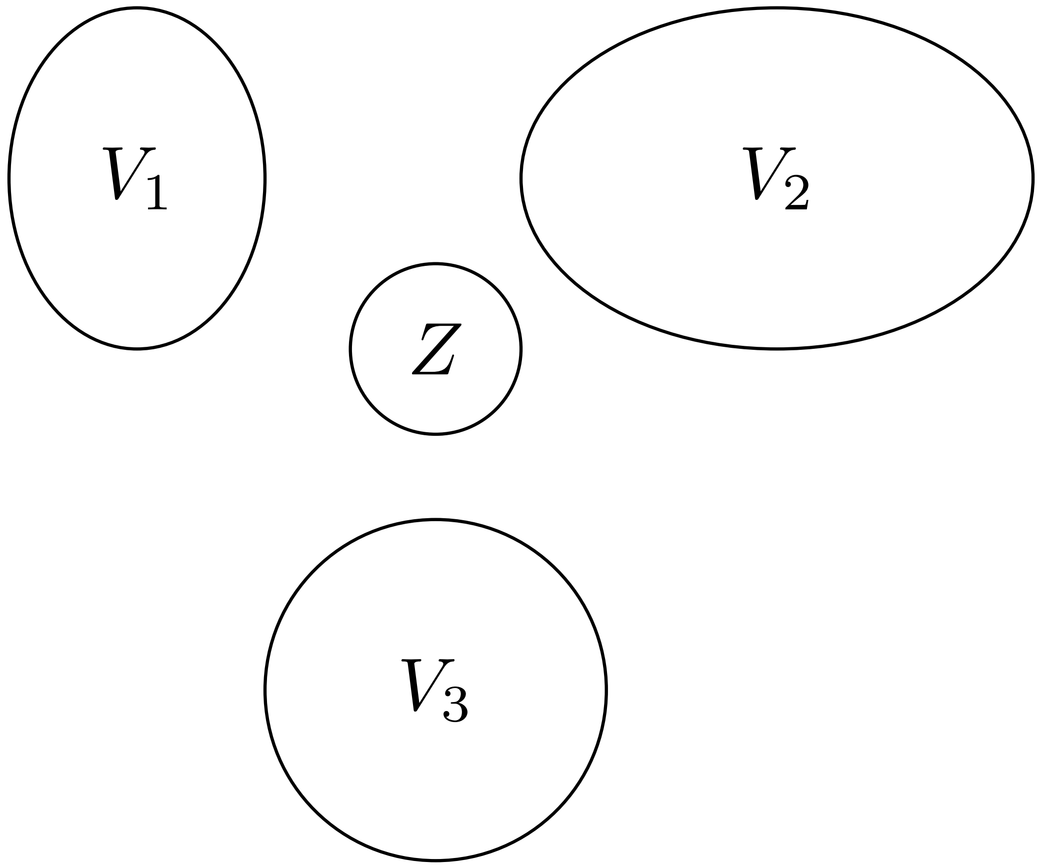

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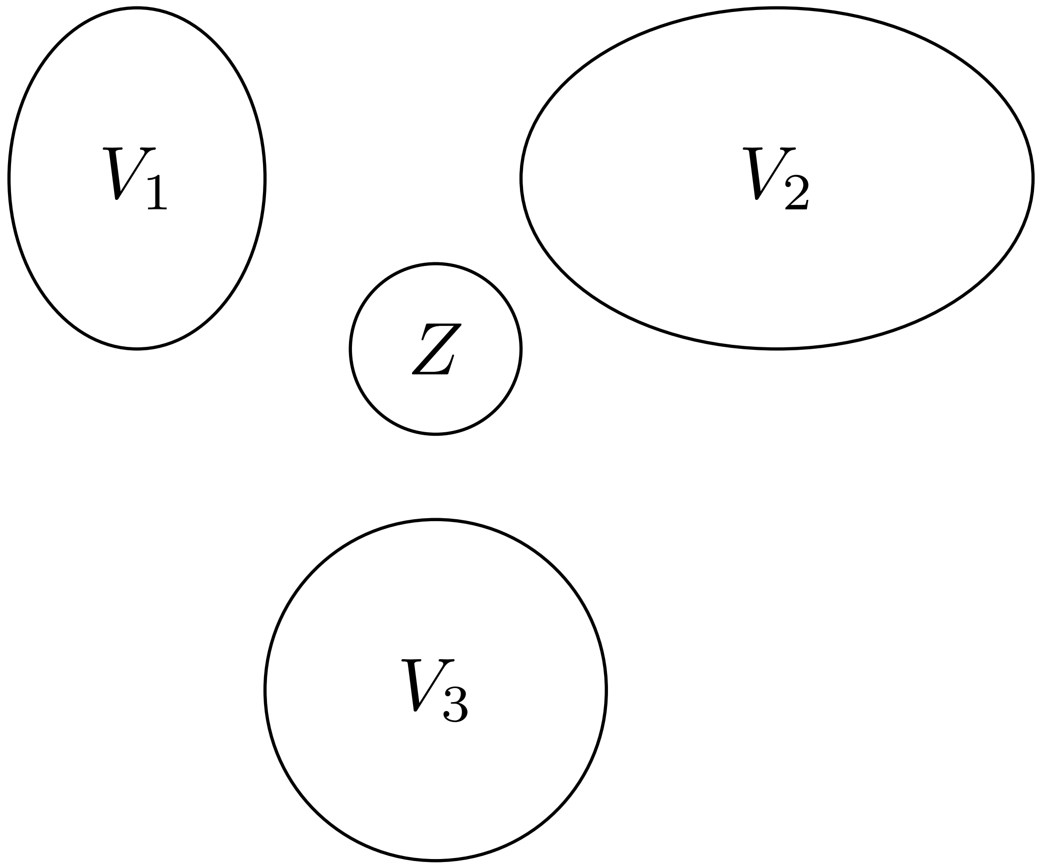

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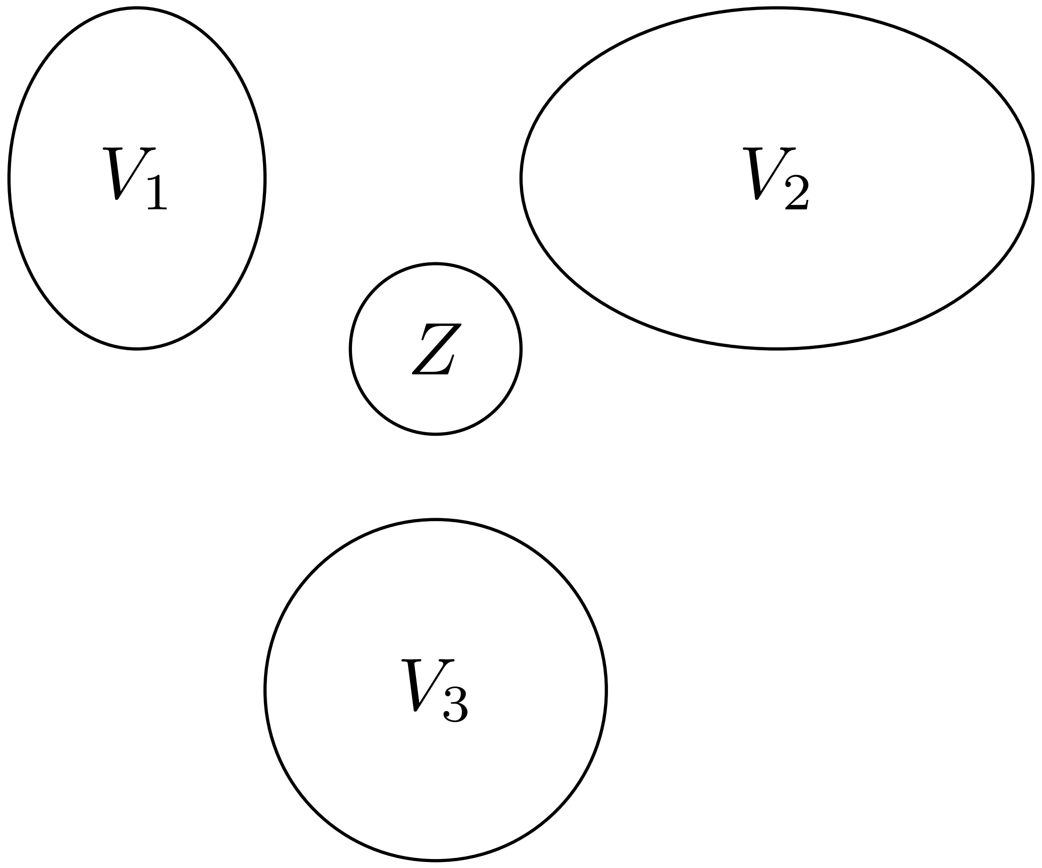
Almost there...



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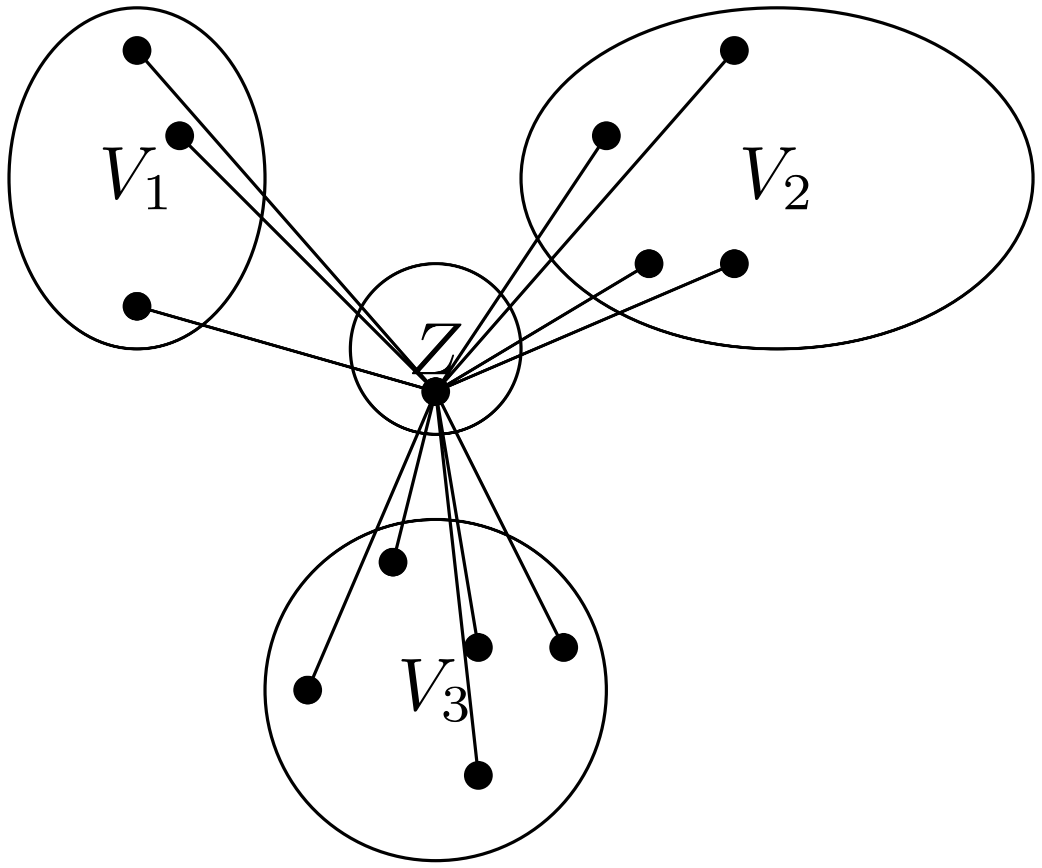

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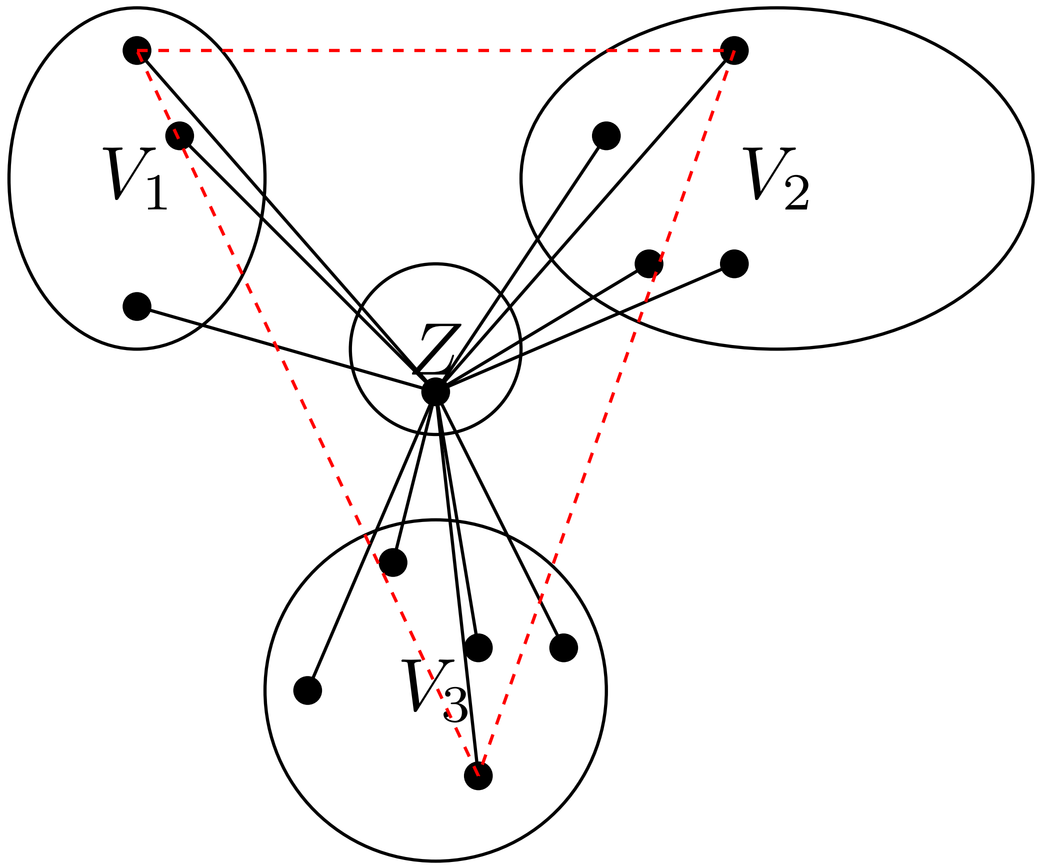

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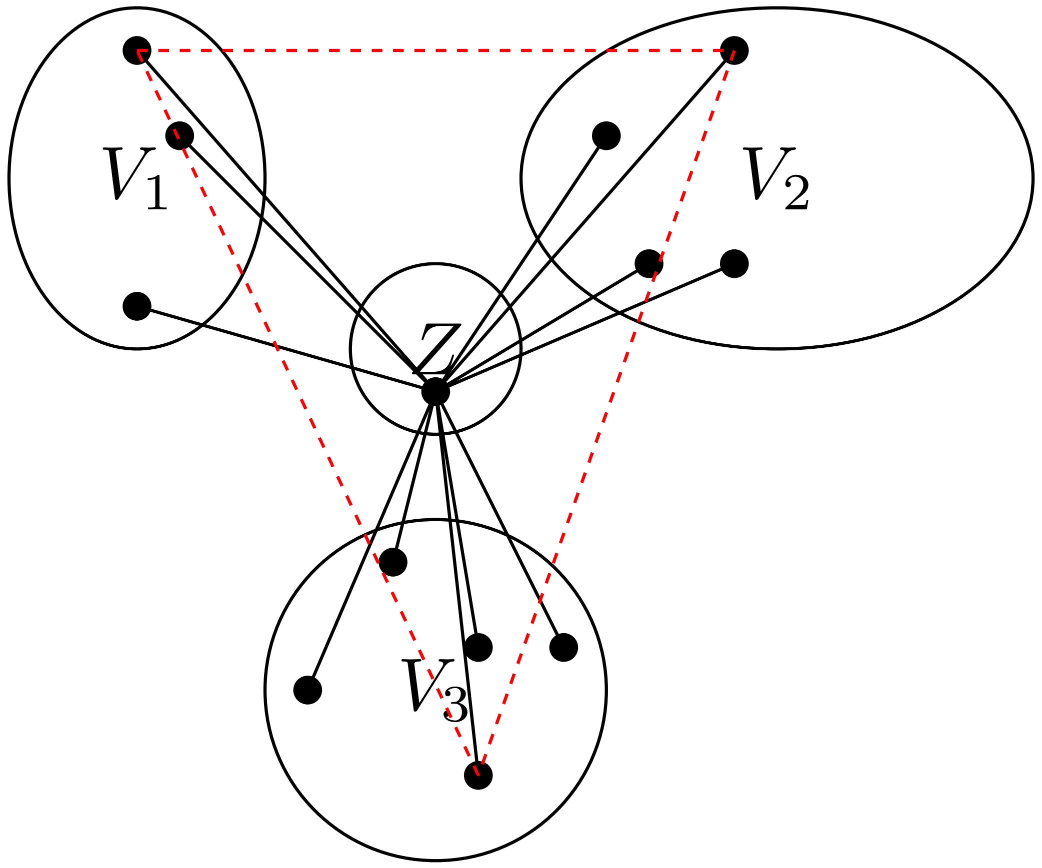

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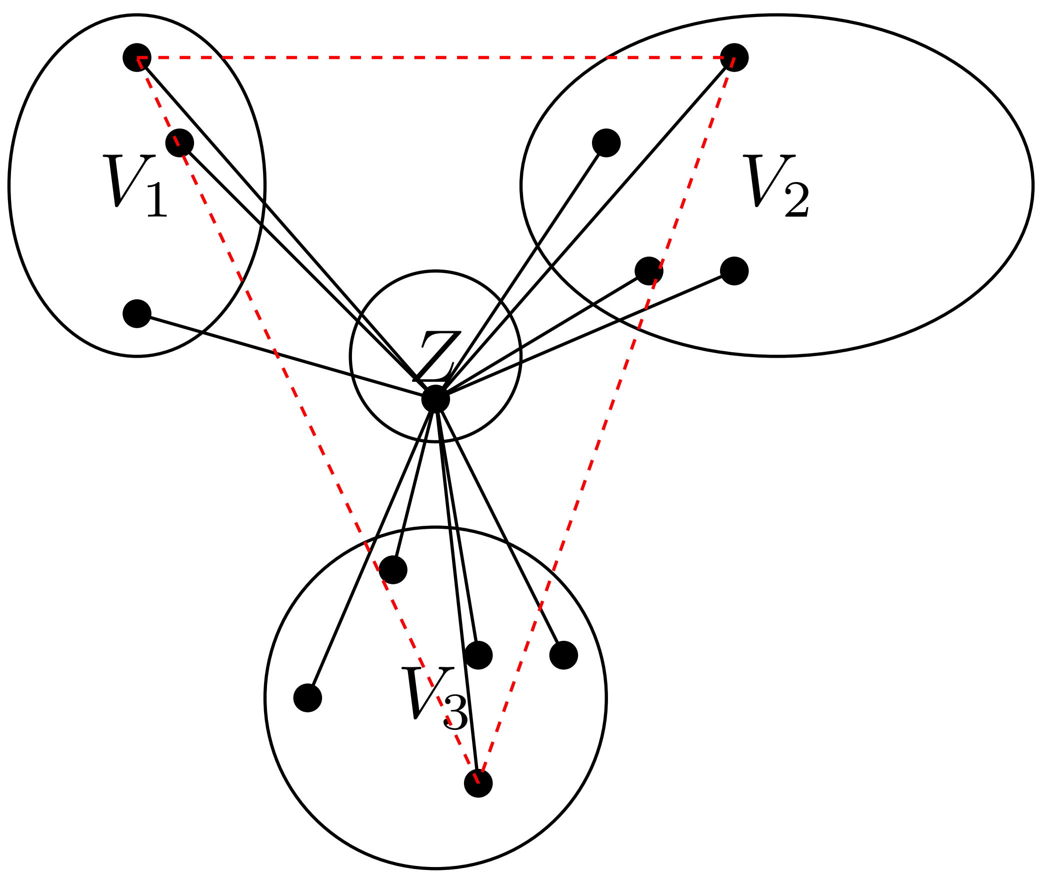

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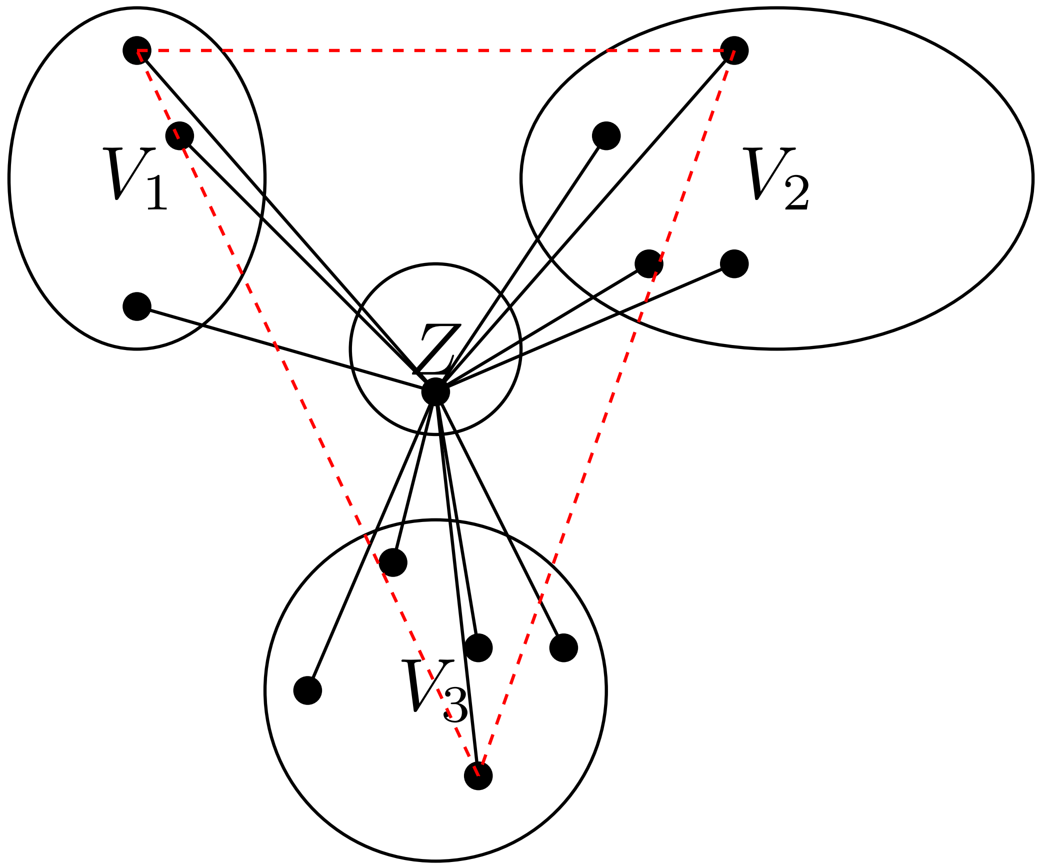

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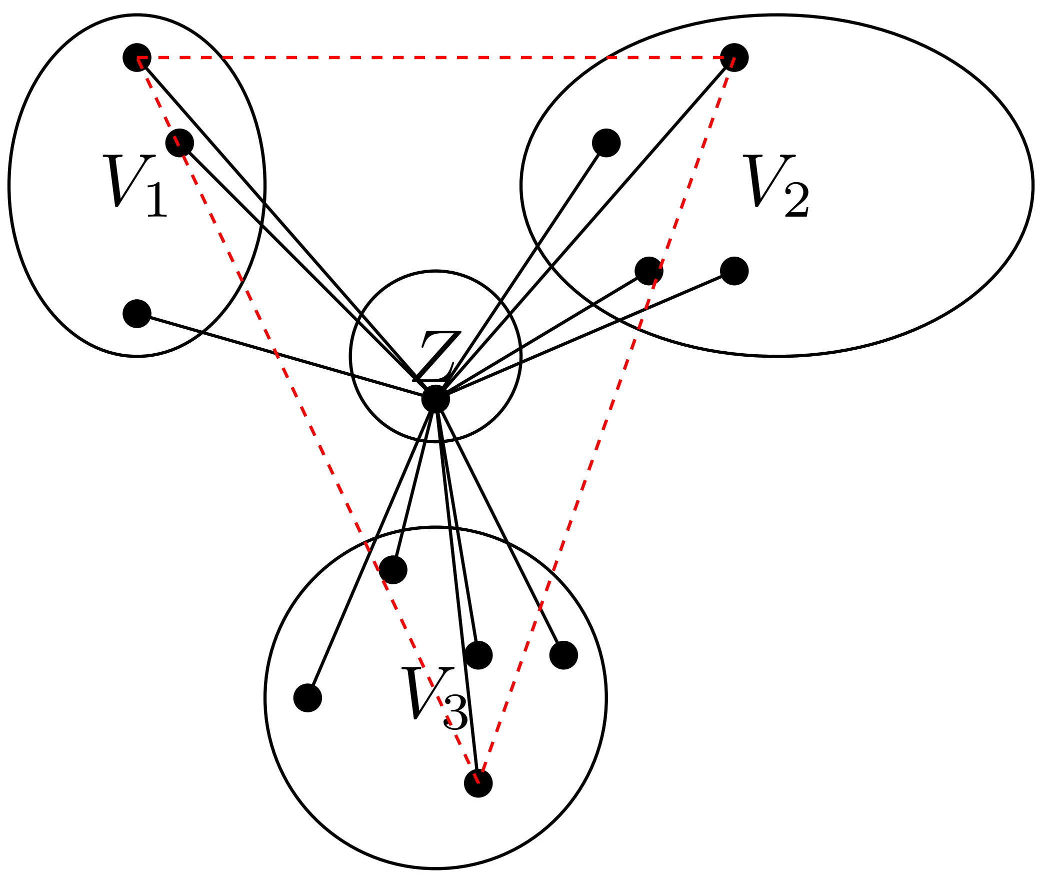

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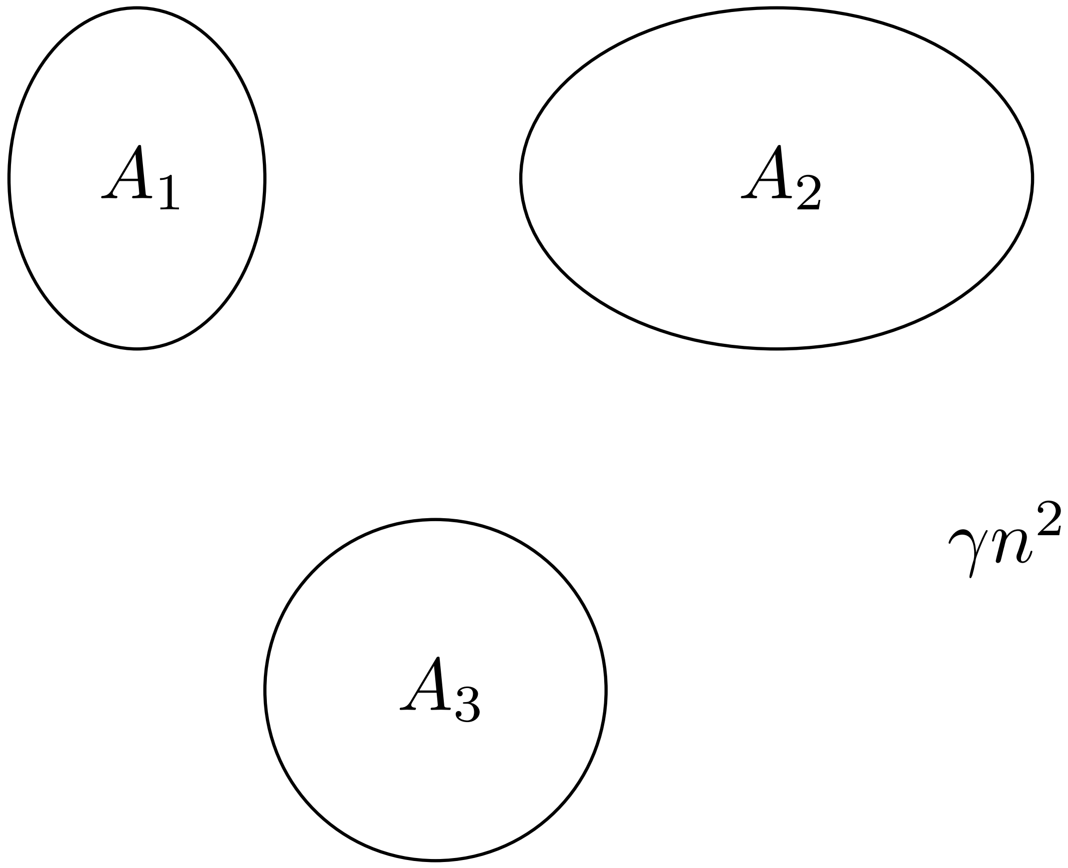
Finish



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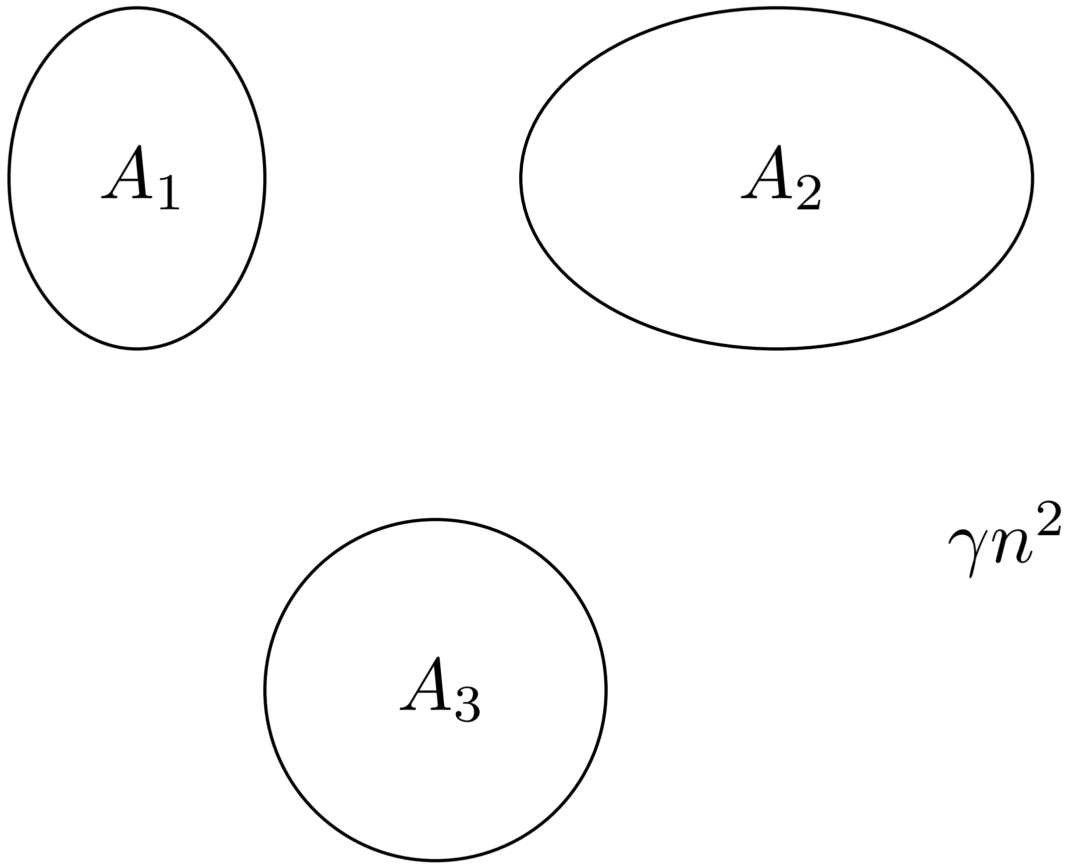

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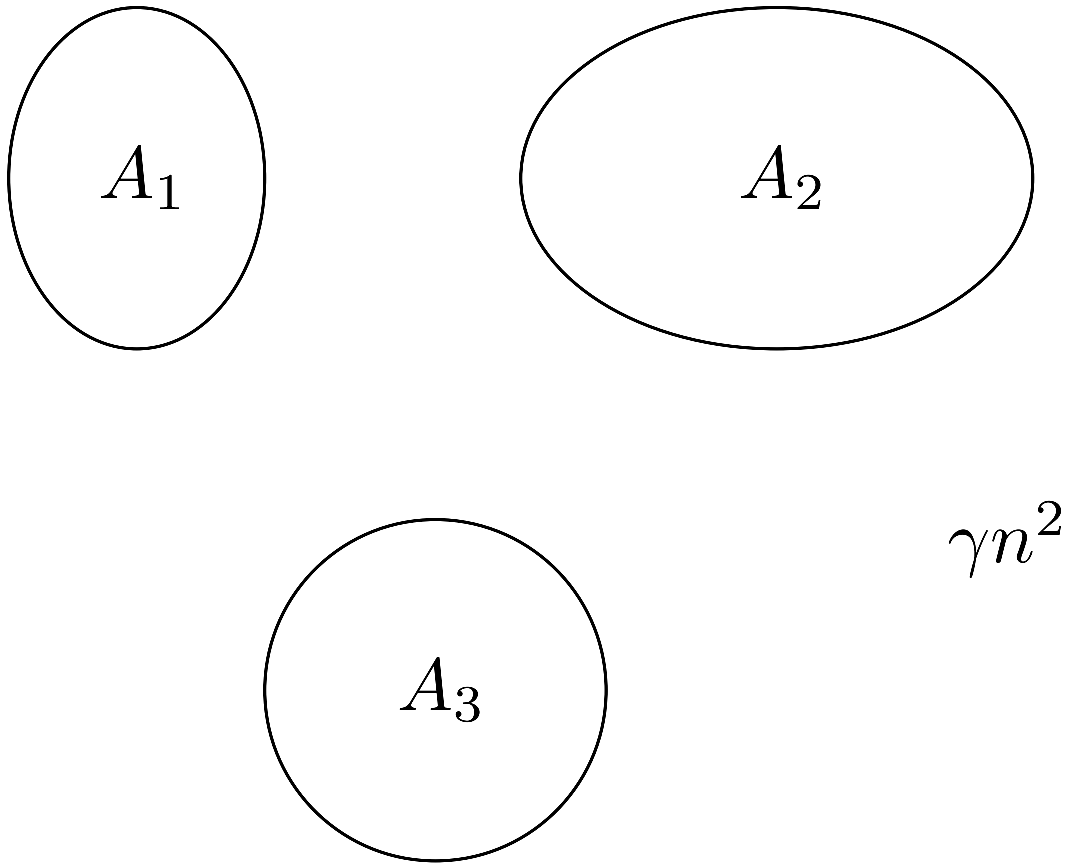

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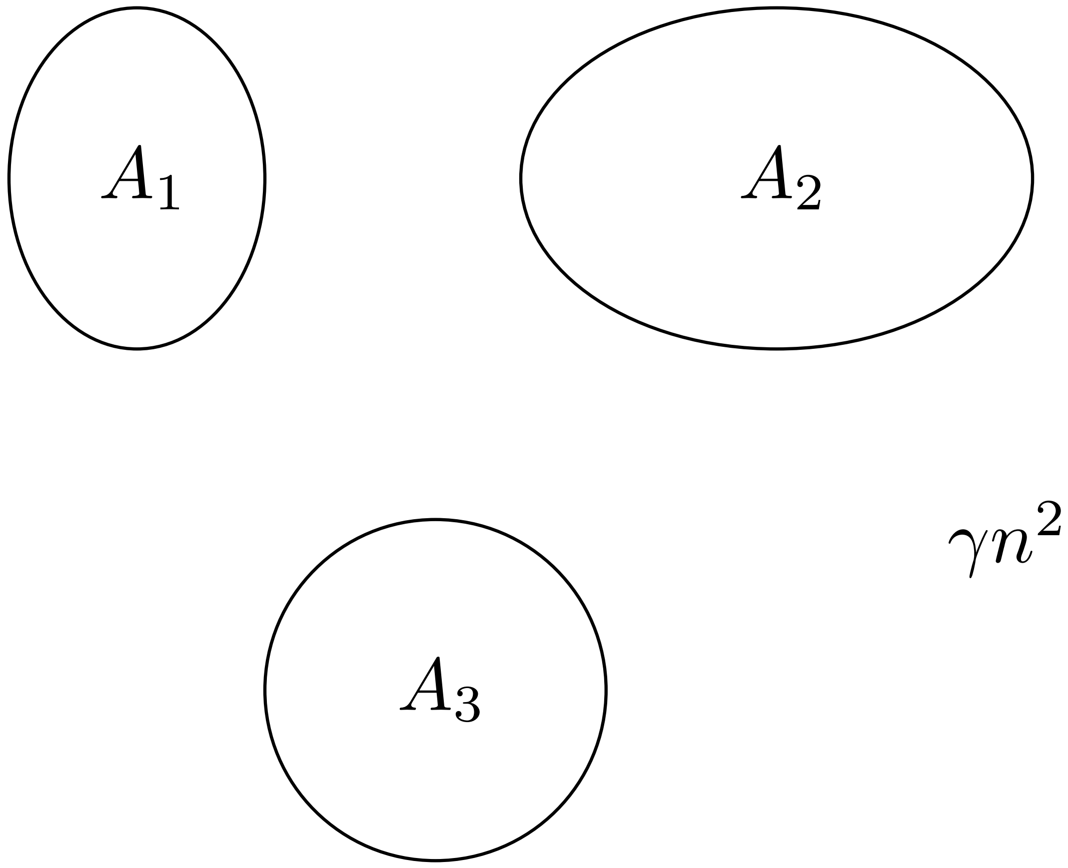
Finish



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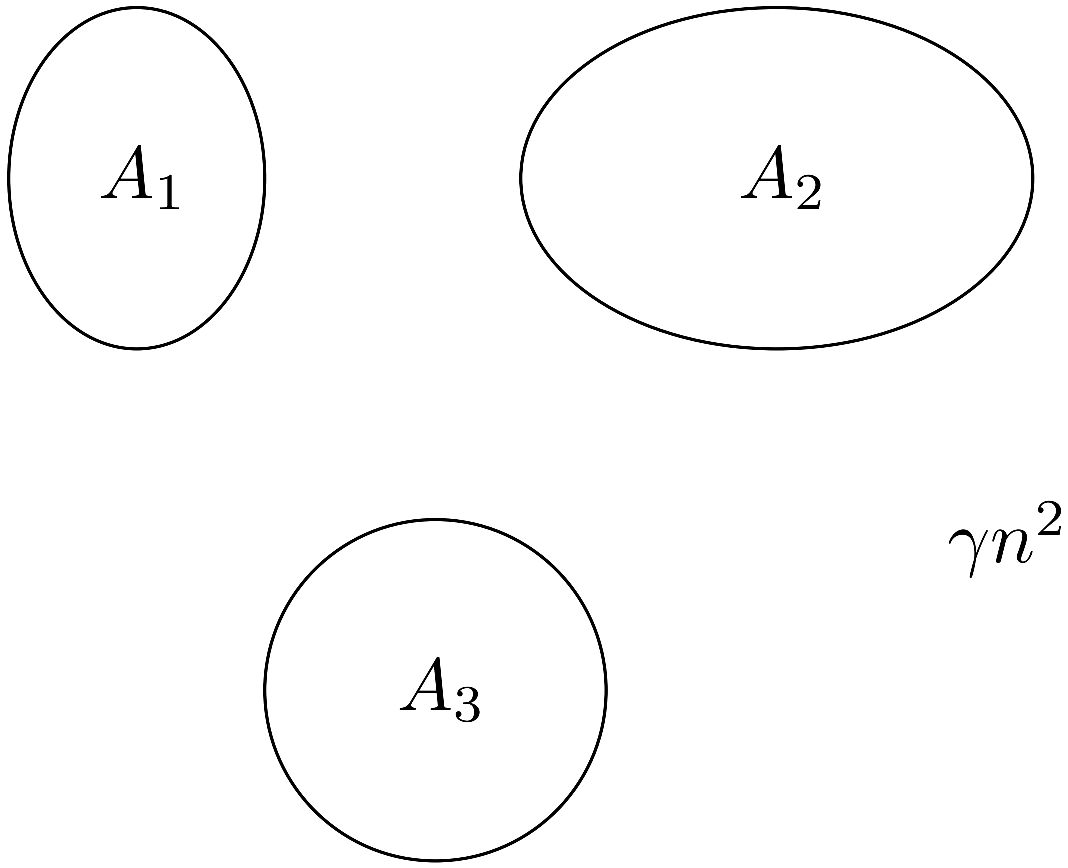
$36\gamma \geq 12 - 58\omega$

Finish



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$e(A_1) + e(A_2) + e(A_3) \leq \omega n^2$ for some $\omega \leq \frac{1}{60}$
Then $e(G) \geq (\frac{1}{3} - \frac{29}{18}\omega) n^2$

3. $\gamma \geq \frac{1+z+z^2}{3(1+2z)}$

4. $\gamma \leq \frac{1}{3} - \frac{z}{4} + \frac{z^2}{36}$

5. $z \leq \frac{3}{29}$

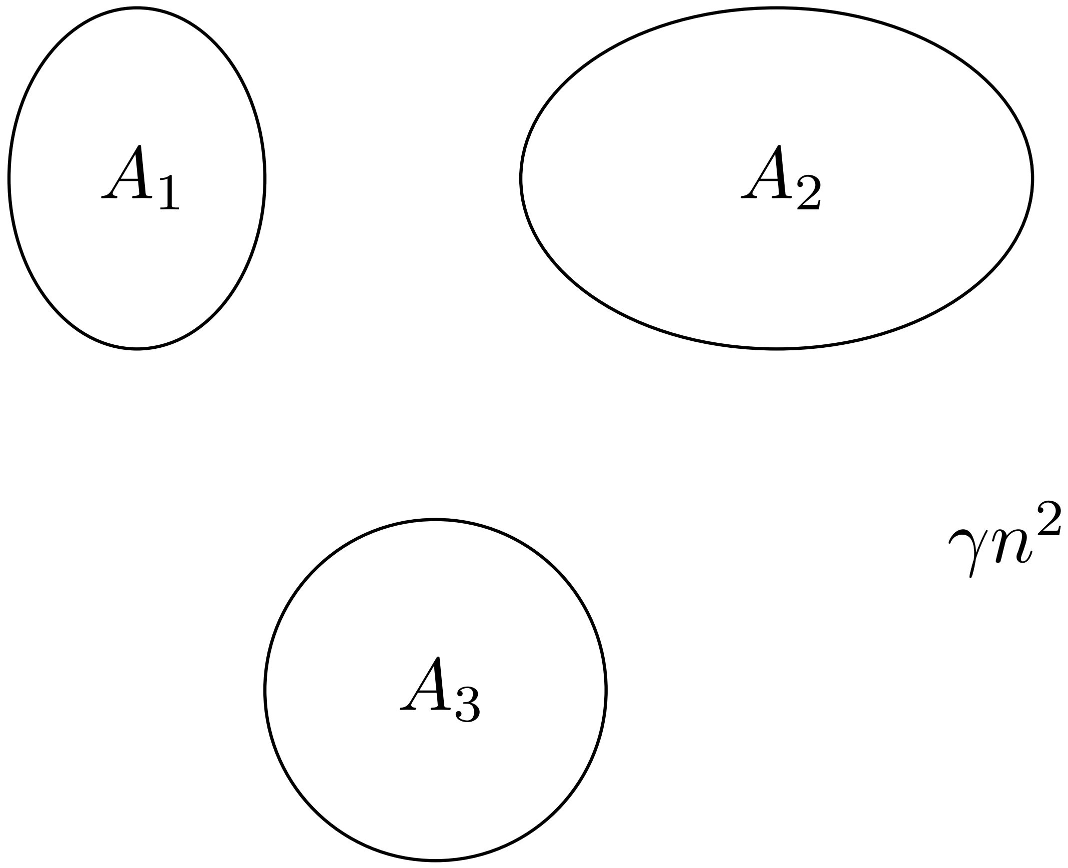
$$\omega n^2 = e(A_1) + e(A_2) + e(A_3)$$

$$\omega n^2 \leq \frac{2}{17} n |Z| + e(Z) \leq (\frac{2}{17} z + \frac{1}{3} z^2) n^2 < \frac{7}{46} z n^2 < \frac{1}{60} n^2$$

$$36\gamma \geq 12 - 58\omega$$

$$12 - (9 - \frac{4}{23}) z \leq 12 - 58\omega \leq 12 - 9z + z^2$$

Finish



$\gamma n^2 = e(G)$

If $x \in Z$, then there exists i so that $|N(x) \cap V_i| \leq \frac{2}{17}n$

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$$36\gamma \geq 12 - 58\omega$$

$$12 - (9 - \frac{4}{23})z \leq 12 - 58\omega \leq 12 - 9z + z^2$$

$$z(\frac{4}{23} - z) \leq 0$$

Thank you!