Can a party represent its constituency?

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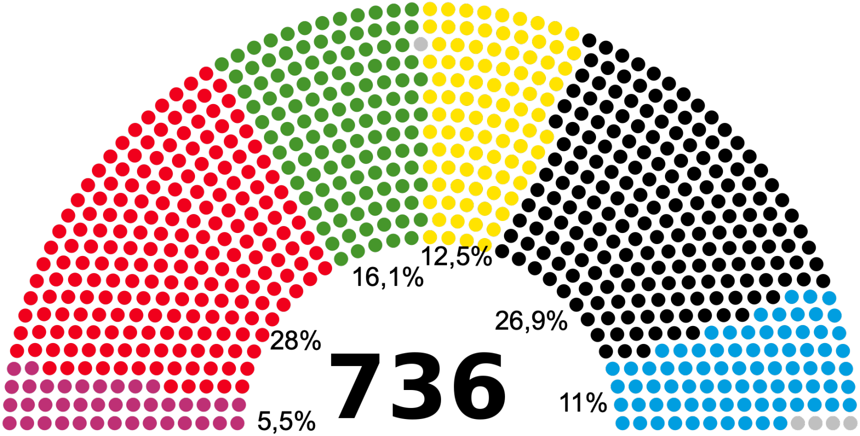
Introduction

We will consider elections to political bodies where x% for particular party gives them nx% seats and n is the number of seats.



Introduction

Politicians in one party also represent different political-ideological values.
Main problem: Is there a way to form and order the list so that different political values will be represented, no matter how many politicians are elected?

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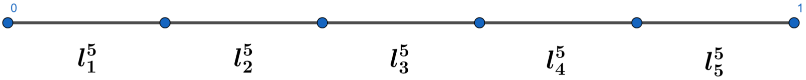
Model

$I = [0, 1]$ - set of different public political-ideological values for a given party. Each party member is represented as point with a value in I. We assume that points have uniform distribution over I.

Model

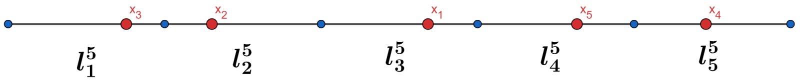
For a given k we define the sets $I_i^k, i = 1, 2, ..., k$ as:
$I_1^k = [0, 1/k)$
$I_2^k = [1/k, 2/k)$
.
.
.
$I_k^k = [(k-1)/k, 1]$



Model

Definition

$(x_1, ..., x_k)$ is a representative body if each point x_j is in different I_i^k.



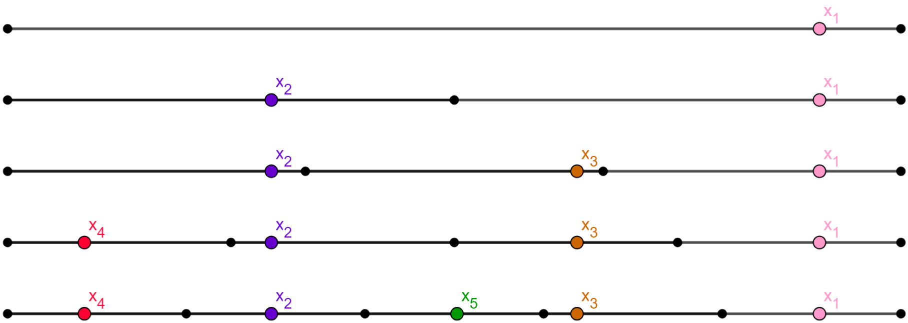
Model

Definition

An order list $(x_1, ..., x_n)$ is a representative list if $(x_1, ..., x_k)$ is a representative body for each point $k = 1, 2, ..., n$.

In the present model can a representative list be formed? What are its characteristics?

Model



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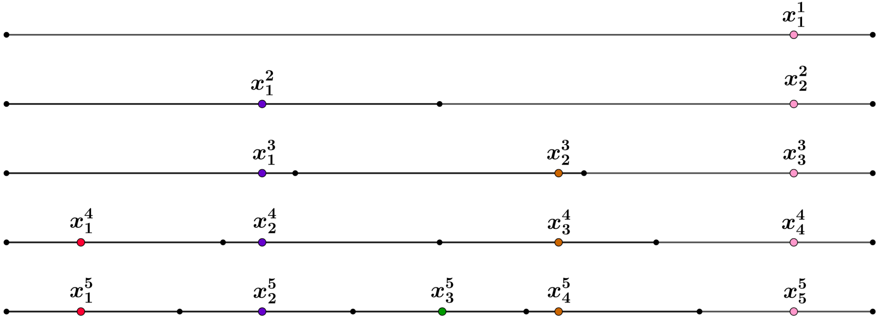
Problem

Theorem

For a given $n > 0, n \in \mathbb{Z}$ there exists a vector $(x_1, ..., x_n)$ such that for each $k \leqslant n, k \in \mathbb{Z}$ each of the components of the vector $(x_1, ..., x_k)$ is a point in a different I_i^k if and only if $n \leqslant 17$.

Proof

Let x_k^n denote that number of the sequence $x_1, \ldots, x_n$ which lies in the interval $[\frac{k-1}{n}, \frac{k}{n})$, $n = 1, \ldots, N$ $k = 1, \ldots, n$ $N = 18$.



Proof

Let's consider all possible values for x_5^9. By symmetry, it is sufficient to examine below cases:

1° – $\frac{4}{9} \leqslant x_5^9 \leqslant \frac{5}{11}$

2° – $\frac{5}{11} \leqslant x_5^9 \leqslant \frac{6}{13}$

3° – $\frac{6}{13} \leqslant x_5^9 \leqslant \frac{7}{15}$

4° – $\frac{7}{15} \leqslant x_5^9 \leqslant \frac{8}{17}$

5° – $\frac{8}{17} \leqslant x_5^9 \leqslant \frac{1}{2}$

6° – $x_5^9 = \frac{1}{2}$

Proof

But why we want to consider those particular cases?

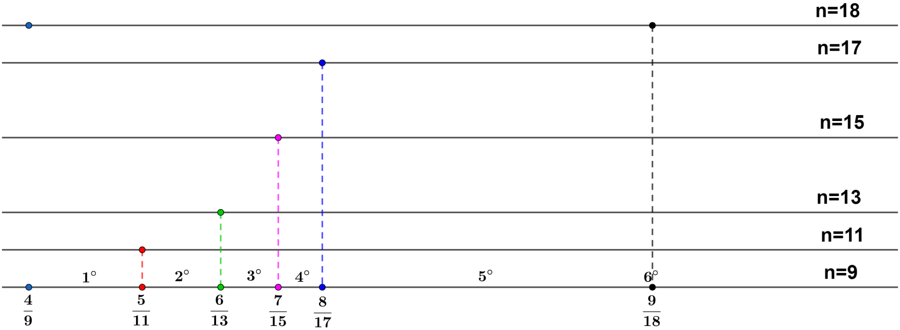
We know that $\frac{4}{9} \leqslant x_5^9 < \frac{5}{9}$, but we assume $\frac{4}{9} \leqslant x_5^9 \leqslant \frac{1}{2}$, to not consider cases symmetrical over $\frac{1}{2}$.

Fractions which are borders of our cases: $\frac{5}{11}$, $\frac{6}{13}$, $\frac{7}{15}$, $\frac{8}{17}$, $\frac{1}{2}$ are accordingly ends of sections $I_5^{11}, I_6^{13}, I_7^{15}, I_8^{17}, I_9^{18}$.

We don't consider sections like this: $I_{k_1}^{12}$, $I_{k_2}^{14}$, $I_{k_3}^{16}$, because $\left[\frac{4}{9}, \frac{1}{2}\right]$ is included in segments I_6^{12}, I_7^{14}, I_8^{16}, so we know right away that $x_5^9 = x_6^{12} = x_7^{14} = x_8^{16}$.

Proof

But why we want to consider those particular cases?



Proof

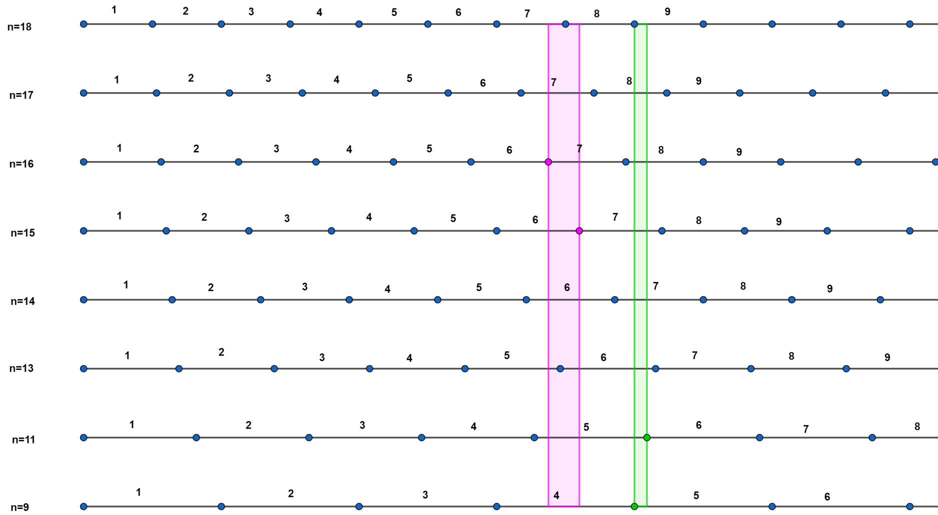
Consider first case: $\frac{4}{9} \leqslant x_5^9 \leqslant \frac{5}{11}$.
We have:

$x_5^9 = x_5^{11} = x_7^{14} = x_7^{15} = x_8^{16} = x_8^{17} \Rightarrow$

$x_4^9 = x_6^{15} \wedge x_6^{15} = x_6^{14} = x_7^{17} = x_7^{16} \wedge \frac{3}{8} \leqslant x_7^{16} = x_6^{15} = x_4^9 < \frac{2}{5} \Rightarrow$

$x_4^9 = x_5^{11} \Rightarrow x_4^9 = x_5^9$, which is contradictory.

Proof


n=18
1 2 3 4 5 6 7 8 9
n=17
1 2 3 4 5 6 7 8 9
n=16
1 2 3 4 5 6 7 8 9
n=15
1 2 3 4 5 6 7 8 9
n=14
1 2 3 4 5 6 7 8 9
n=13
1 2 3 4 5 6 7 8 9
n=11
1 2 3 4 5 6 7 8
n=9
1 2 3 4 5 6

Proof

Other cases can be solved similarly.

Construction for $N = 17$

For $N = 17$ we can construct such a sequence that satisfies wanted properties:

$$\frac{4}{7} \leqslant x_1 < \frac{7}{12}, \quad \frac{2}{7} \leqslant x_2 < \frac{5}{17}, \quad \frac{16}{17} \leqslant x_3 < 1, \quad \frac{1}{14} \leqslant x_4 < \frac{1}{13},$$

$$\frac{8}{11} \leqslant x_5 < \frac{11}{15}, \quad \frac{5}{11} \leqslant x_6 < \frac{6}{13}, \quad \frac{1}{7} \leqslant x_7 < \frac{2}{13}, \quad \frac{14}{17} \leqslant x_8 < \frac{5}{6},$$

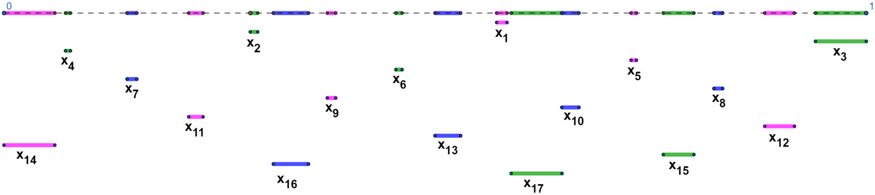
$$\frac{3}{8} \leqslant x_9 < \frac{5}{13}, \quad \frac{11}{17} \leqslant x_{10} < \frac{2}{3}, \quad \frac{3}{14} \leqslant x_{11} < \frac{3}{13}, \quad \frac{15}{17} \leqslant x_{12} < \frac{11}{12},$$

$$\frac{1}{2} \leqslant x_{13} < \frac{9}{17}, \quad 0 \leqslant x_{14} < \frac{1}{17}, \quad \frac{13}{17} \leqslant x_{15} < \frac{4}{5}, \quad \frac{5}{16} \leqslant x_{16} < \frac{6}{17},$$

$$\frac{10}{17} \leqslant x_{17} < \frac{11}{17}.$$

Construction for $N = 17$

How it looks visually:



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Corollary

Above solution is one of the 768 different solutions (2 times more if we count symmetrical solutions separately). All of those solutions have to satisfy common restrictions, for example:

$$\frac{2}{7} \leqslant x_2^5 < \frac{5}{17}, \quad \frac{8}{11} \leqslant x_4^5 < \frac{11}{15}, \quad \frac{5}{11} \leqslant x_3^6 < \frac{6}{13}, \quad \frac{4}{7} \leqslant x_4^6 < \frac{7}{12},$$

$\frac{3}{8} \leqslant x_9 < \frac{5}{13}$

Corollary

A party can construct an ordered representative list $(x_1^*, ..., x_n^*)$ if and only if it contains not more than seventeen names. Strangly, x_9 - middle name on the list - is not located at the center of the political spectrum.

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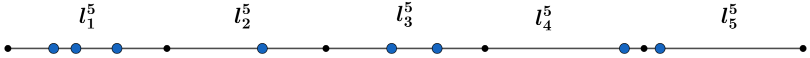
Variations

Consider situation in which we want different fractions of party are represented, but we can let them to have different number of politicians inside the subparties.



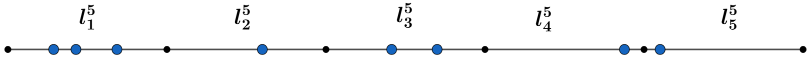
In the example above l_1^5 is represented by three chosen politicians, l_2^5 by one etc.

Variations

Definition

$(x_1, ..., x_{k+d})$ is a representative body with at most d irregularities if each I_i^k contains at least one x_j.

Example for $k = 5,\ d = 3$:



Variations

Definition

An order list $(x_1, ..., x_{N+d})$ is a representative list with at most d irregularities if $(x_1, ..., x_{k+d})$ is a representative body with at most d irregularities for each point $k = 1, 2, ..., N$.

Variations

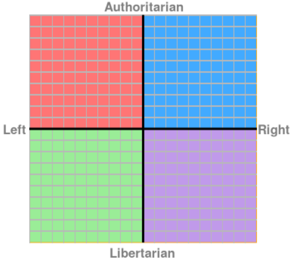
N=O(d^3) (L.Graham)

N $\leqslant 24801d^3 + 942d^2 + 3$ for $d \geqslant 1$ (K. Levy)

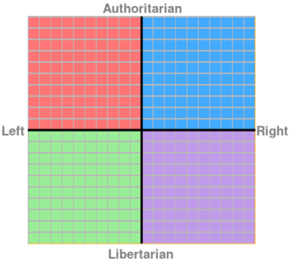
N $\geqslant 2d$ for any positive integer d and that $N < 200d$ for all sufficiently large d (S. V. Konyagin)

Variations

Another interesting variations would be representing political-ideological values on some m-dimentional structure, instead of our $I = [0, 1]$. However, this creates a problem how to define representative list in this case. In my opinion two-dimentional case is a great example - because of well known political square compass.



Variations



Political views of each politician could be points, then I could be minimal square or rectangle containing all of the points.

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References

K. Levy - Lower and upper bounds on irregularities of distribution
https://www.scopus.com/record/display.uri?eid=2-s2.0-85097550717origin=inwardtxGid=0e9d4065fe81ad6b49aaf32747c731e0
S. V. Konyagin - On irregularity of finite sequences
https://link.springer.com/article/10.1134/S0081543821040052)
L. Graham - A note on irregularities of distribution
https://www.degruyter.com/document/doi/10.1515/9783110298161.760/htr
M. Warmus - A supplementary note on the irregularities of distributions:
https://www.sciencedirect.com/science/article/pii/0022314X76900020?ref=p
$RR - 2rr = 7885c162a907bfc3$
A. Kats - Can a party represent its contituency?
http://www.jstor.org/stable/30023969.

End

Thank you!