A Survey of the Game “Lights Out!”

based on an article by Rudolf Fleischer and Jiajin Yu

Bartłomiej Błoniarz

The game

The original game Lights Out! is played on a 5×5-grid where each cell has a button and a light.
Pressing the button will switch the related light and the lights of neighboring buttons.
Given an initial configuration of lights, the goal is to switch all lights off.

Problem statement

We are given an undirected graph $G = (V, E)$ with n nodes, where each node $v \in V$ has a state (light) $C_v \in \{0, 1\}$. We say v is off if $C_v = 0$, and on if $C_v = 1$.

The objective of the game is to reach the all-off configuration from the given initial configuration by a finite sequence of activations.
This initial configuration is considered solvable if this can be achieved.

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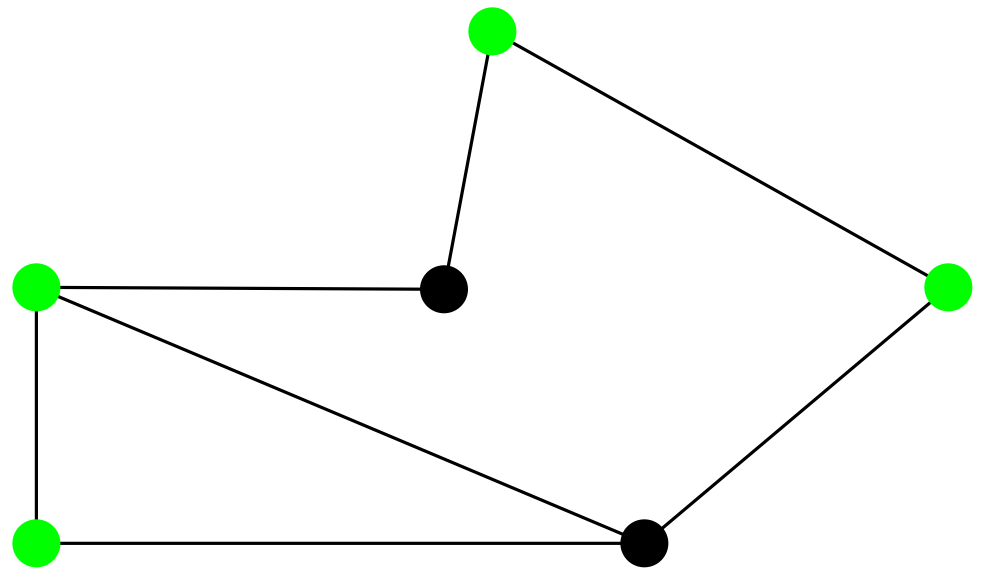

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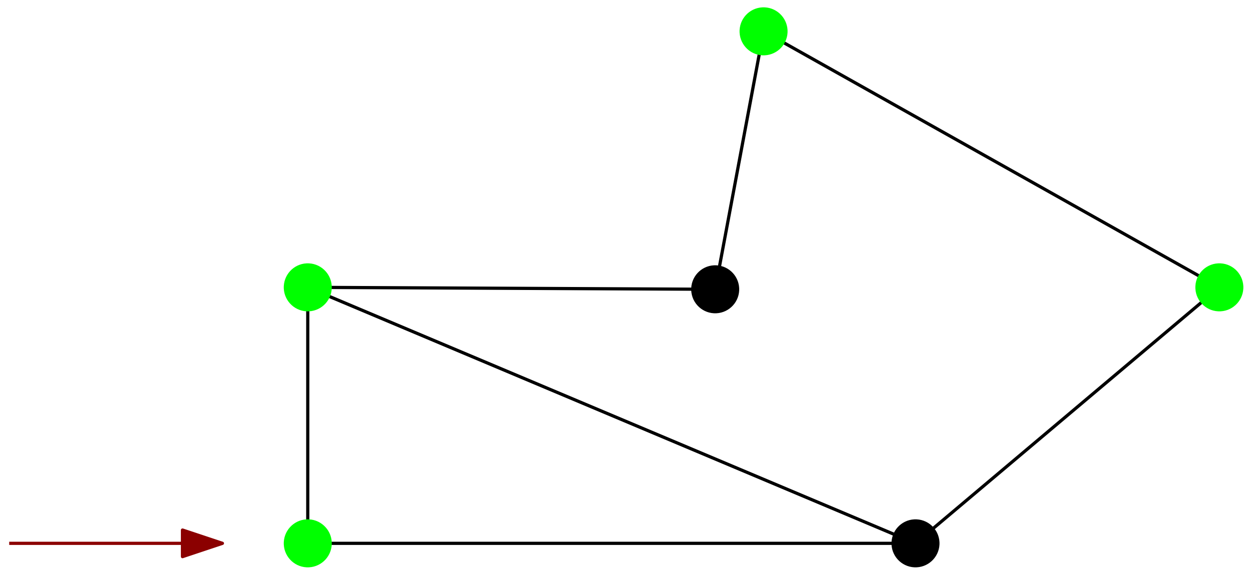

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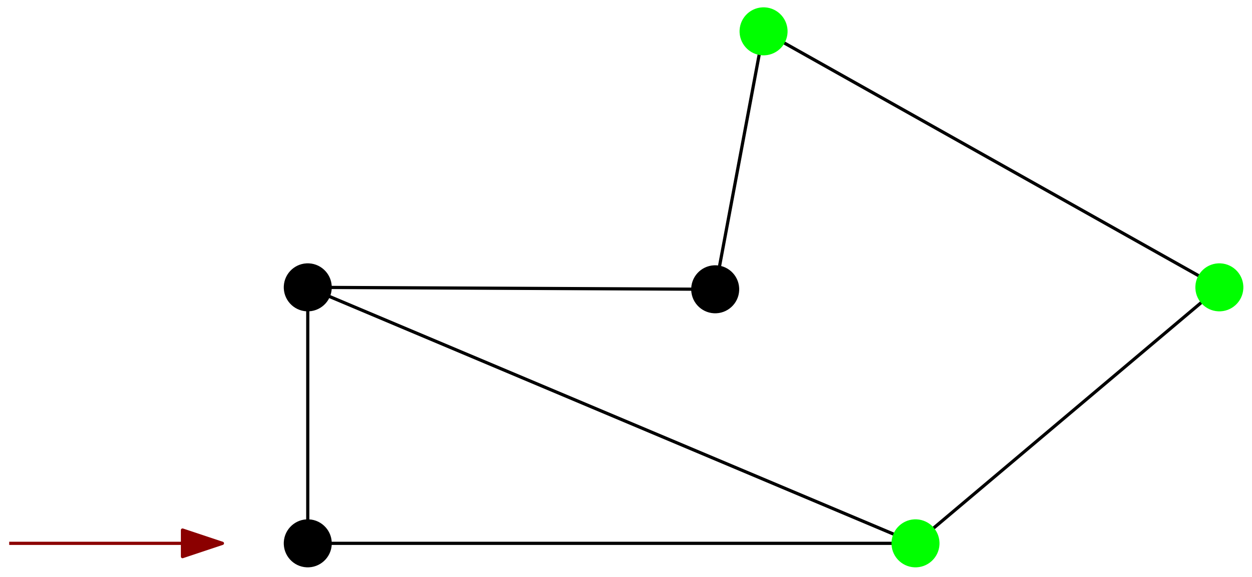

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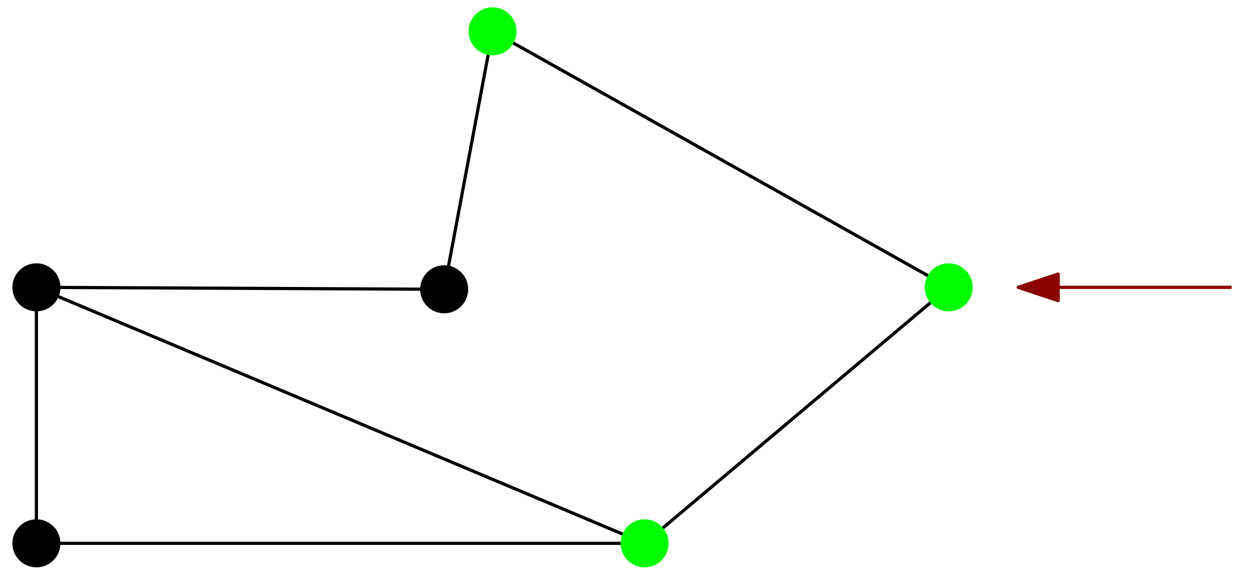

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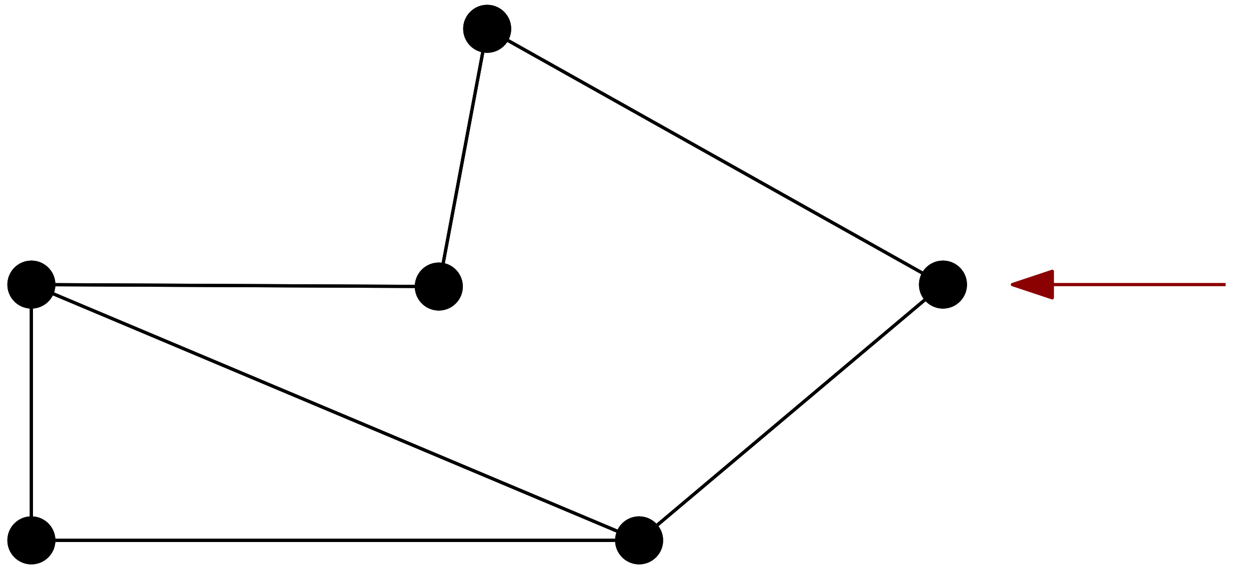

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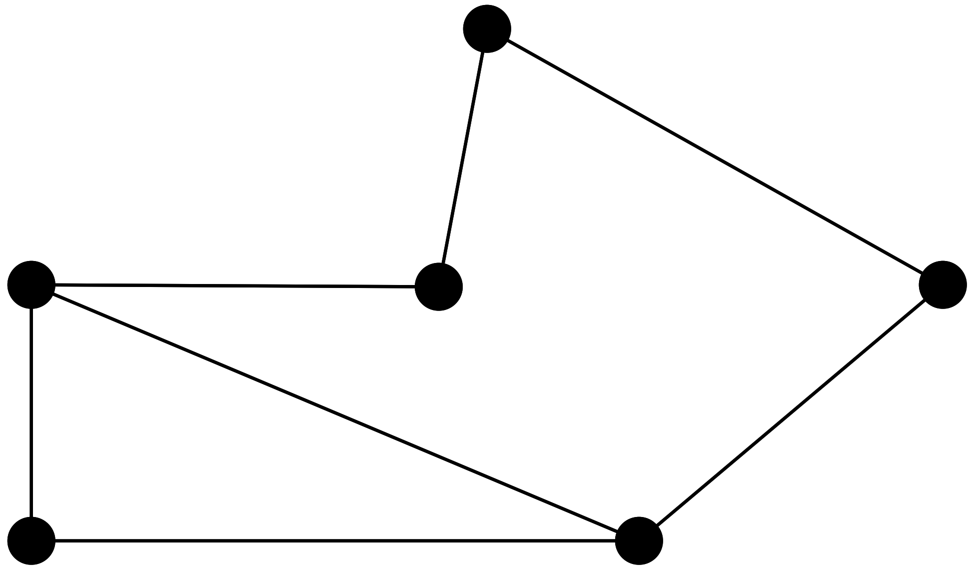

Observations:

- The sequence in which the activations take place does not affect the outcome

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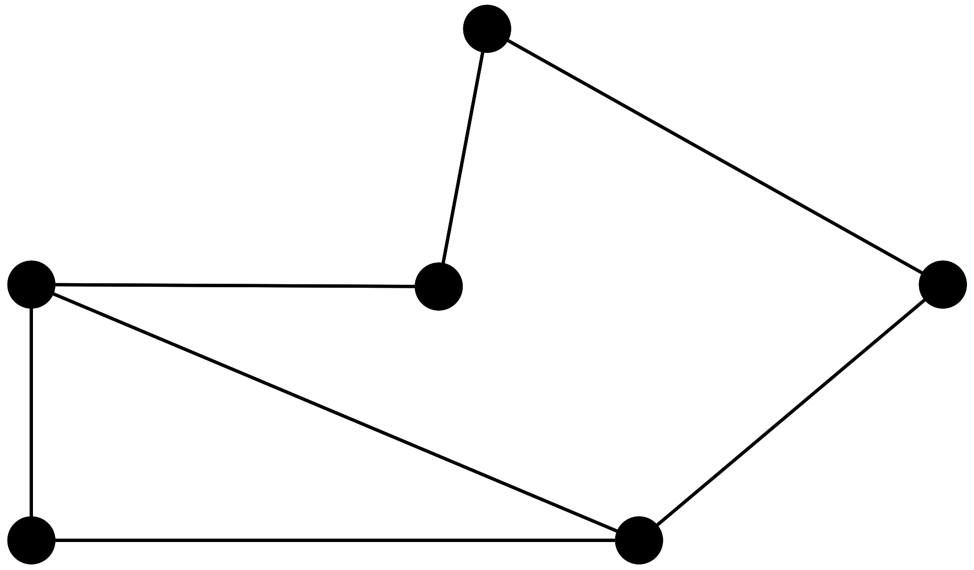

Observations:

- The sequence in which the activations take place does not affect the outcome
- It is unnecessary to activate any light more than once

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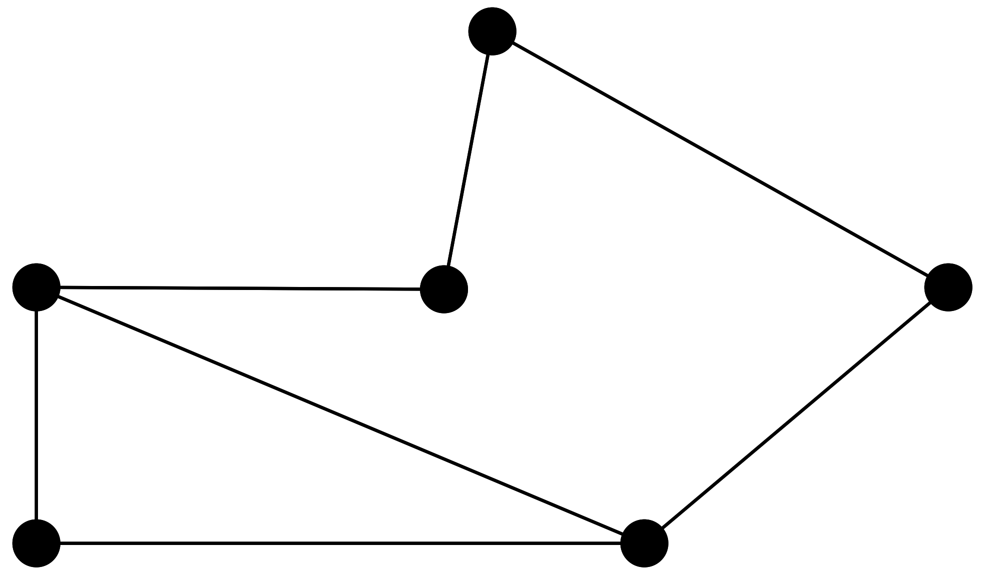

Observations:

- The sequence in which the activations take place does not affect the outcome
- It is unnecessary to activate any light more than once

This implies that we can simplify the sequence of activations by reducing it to a set.

Generalizations

We can generalize the game by introducing a neighborhood vector F of dimension n.
For each node v:

- If $F_v = 1$, activating node v will also switch its state. This is the model that was previously introduced.
- If $F_v = 0$, activating node v will not change its state, only the states of its neighboring nodes will change

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We denote the generalized version of the problem with neighborhood vector F as $F-$Lights Out!

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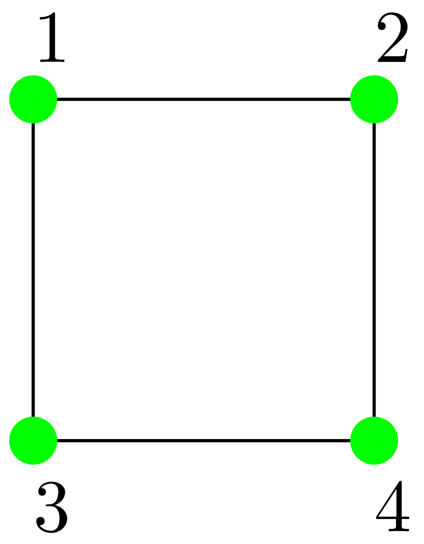

$X_1 = \{4\}$

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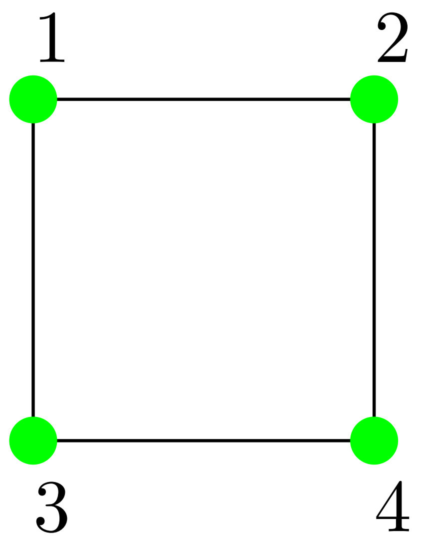

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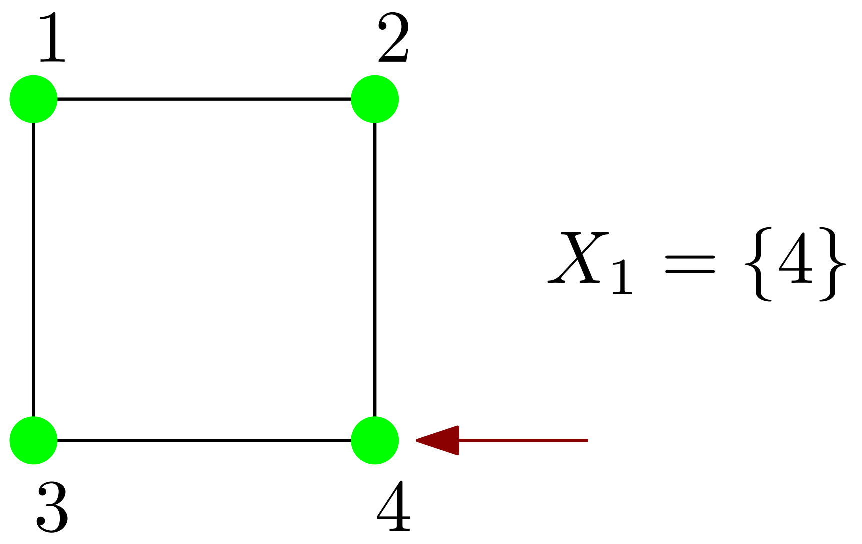

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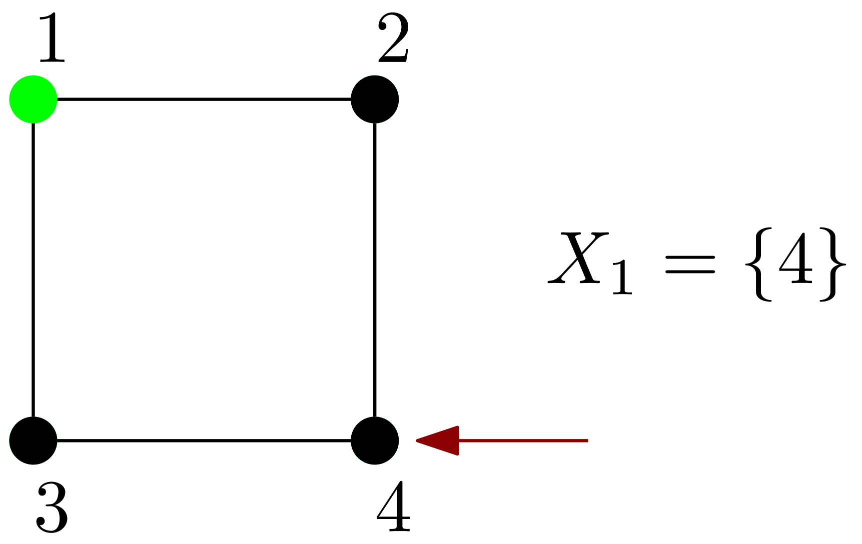

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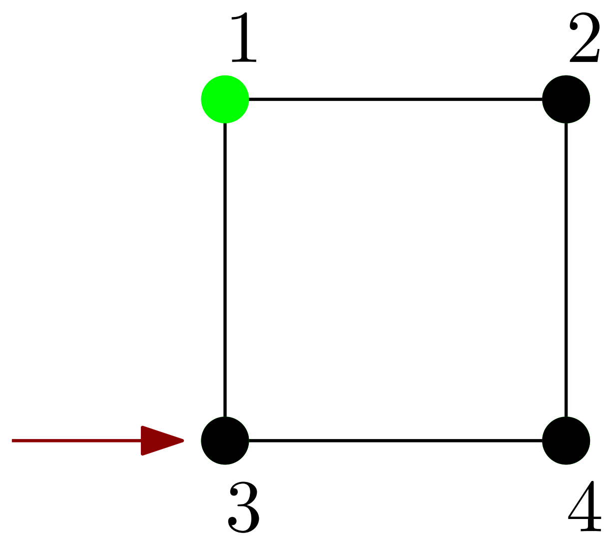

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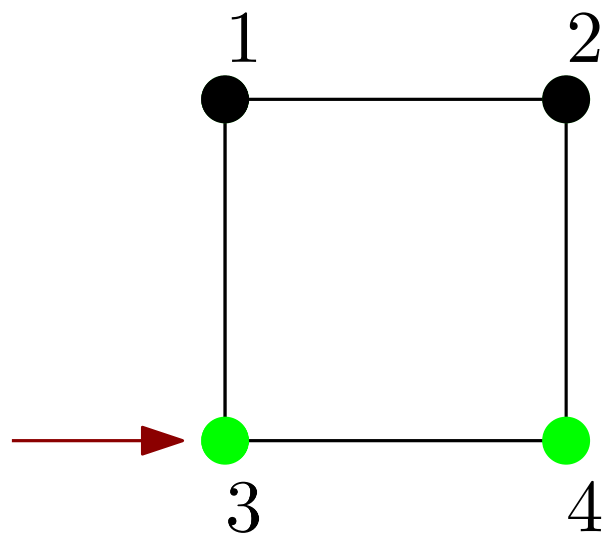

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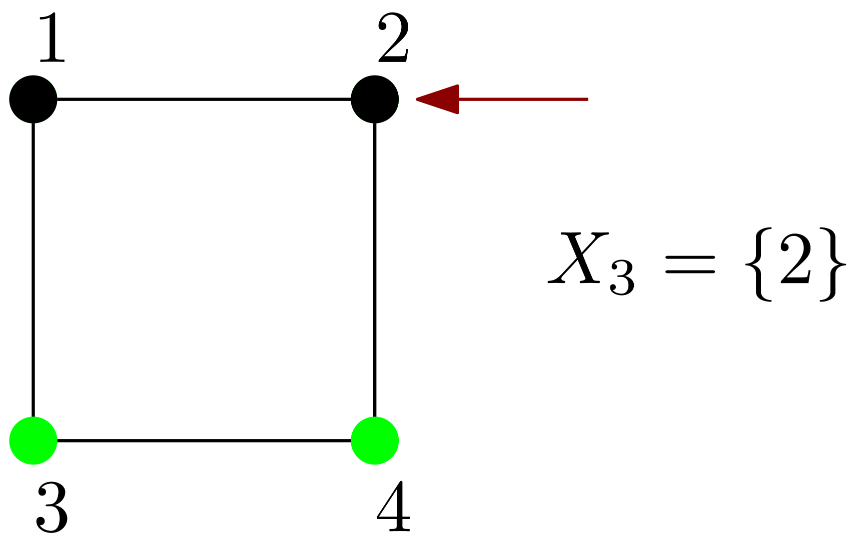

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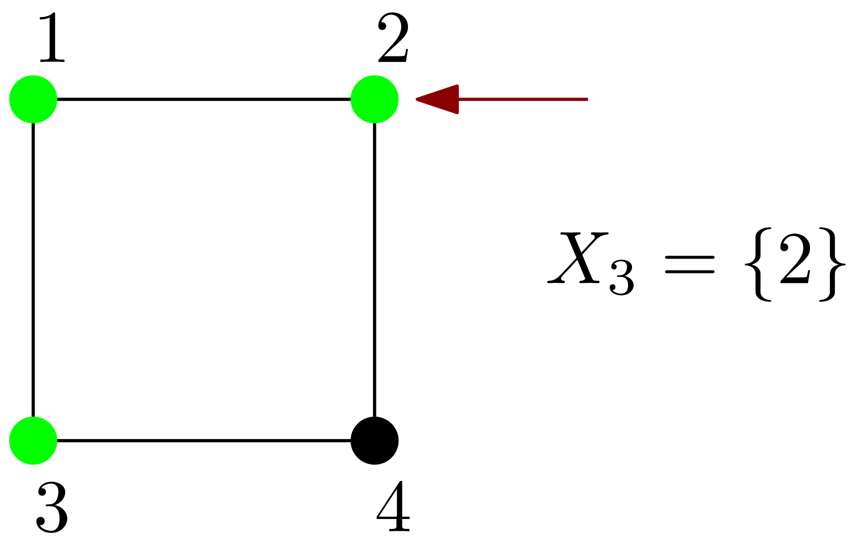

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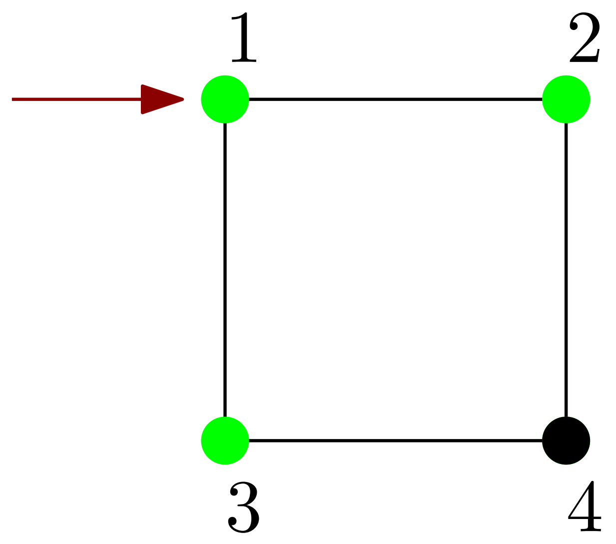

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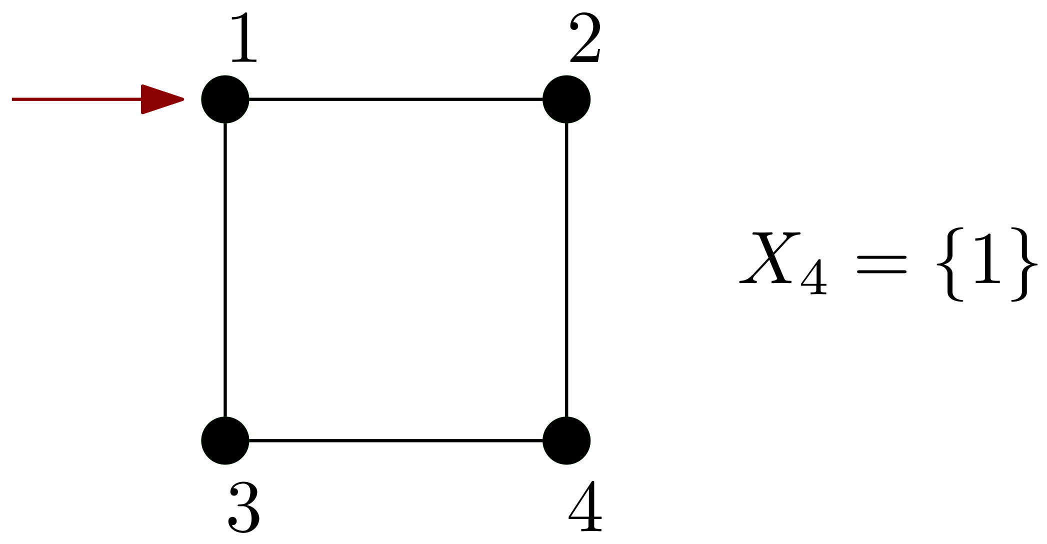

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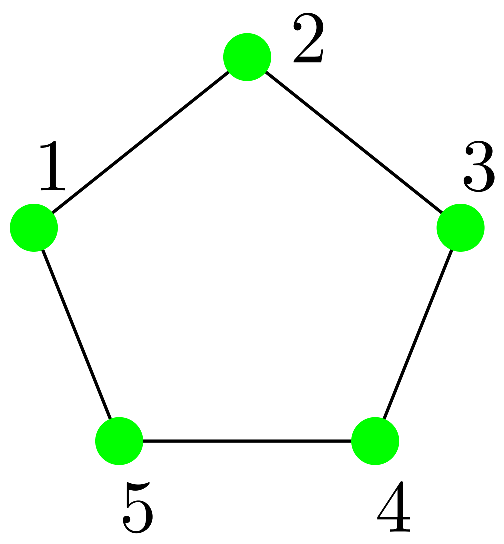

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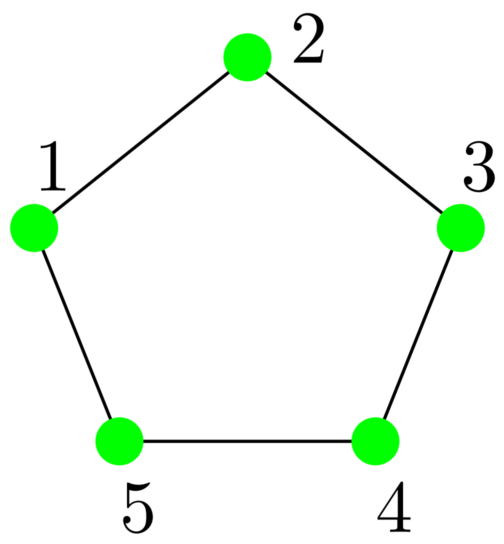

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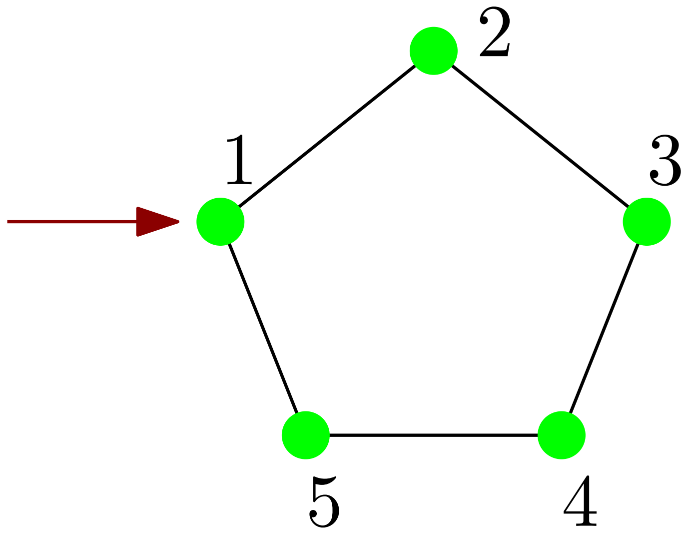

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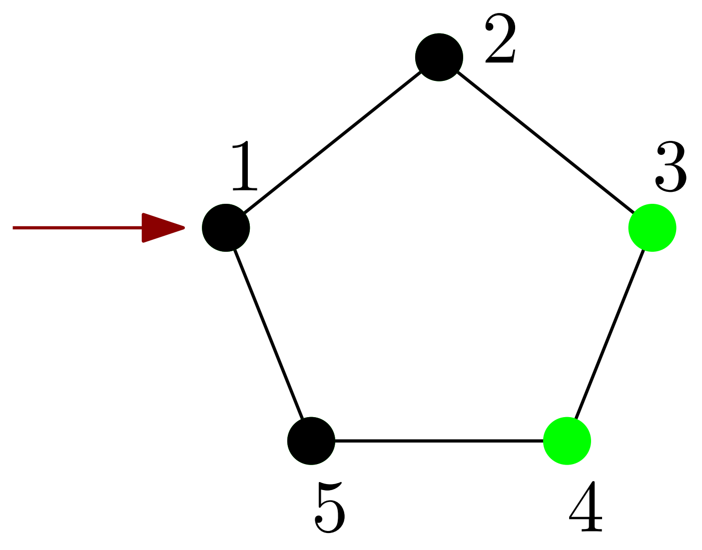

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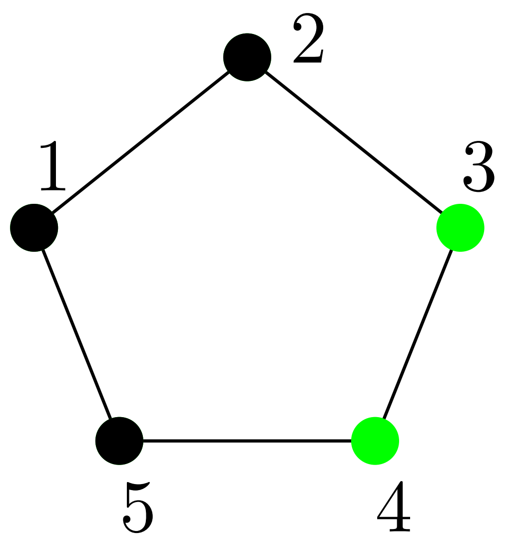

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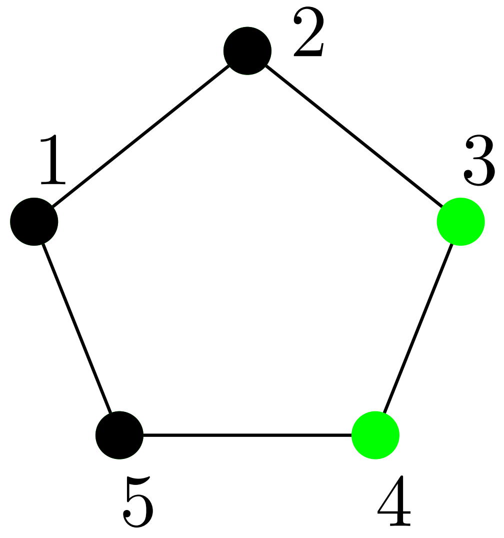
So activating these sets will change the states of nodes in $N[v]$ an even number of times (so they will stay off) and the remaining nodes will change their states an odd number of times (so they will switch off).



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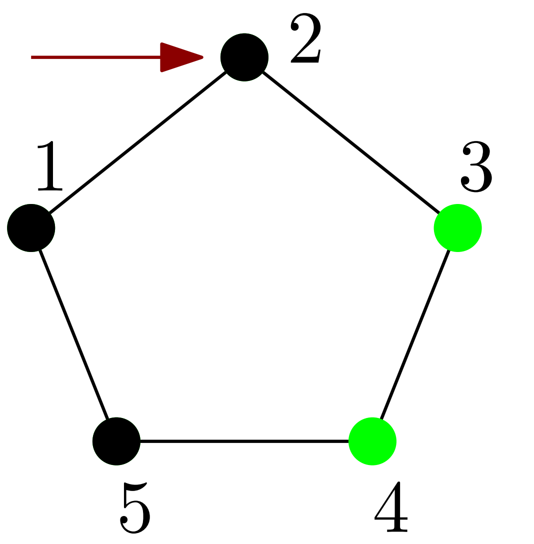

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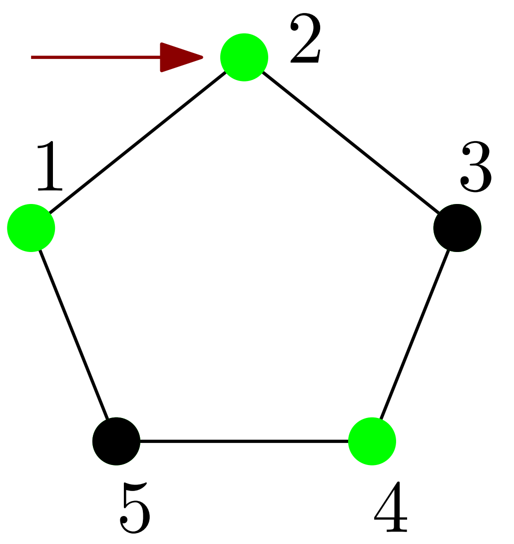

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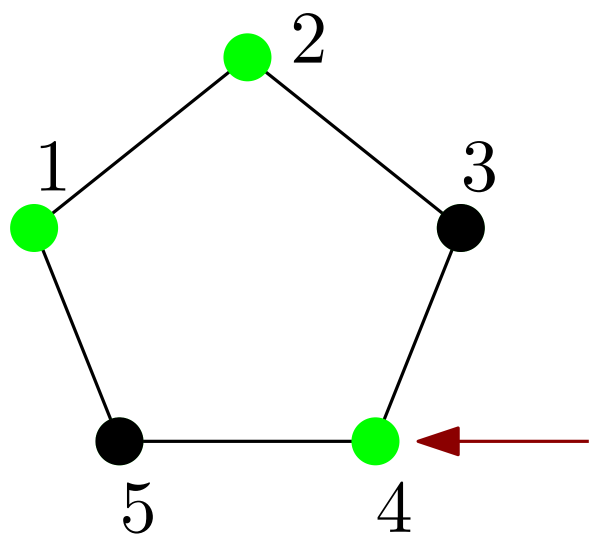

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All-on configuration

Theorem The all-on configuration is solvable for any undirected graph

Proof by induction

Assume we have a graph G with $n+1$ nodes and for every graph with n nodes the statement is true. For every node v, by induction we know that $G - \{v\}$ has a solution X_v. If X_v turns v off, it is also a solution for G. Otherwise we can assume that every X_v keeps v on.

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   Then G has an odd number of nodes.
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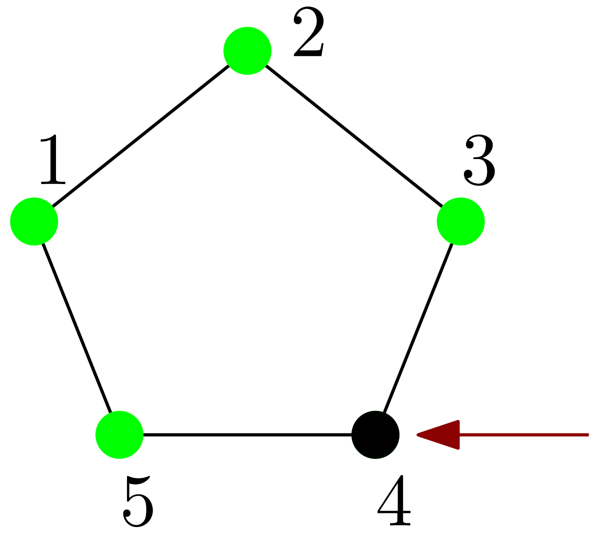

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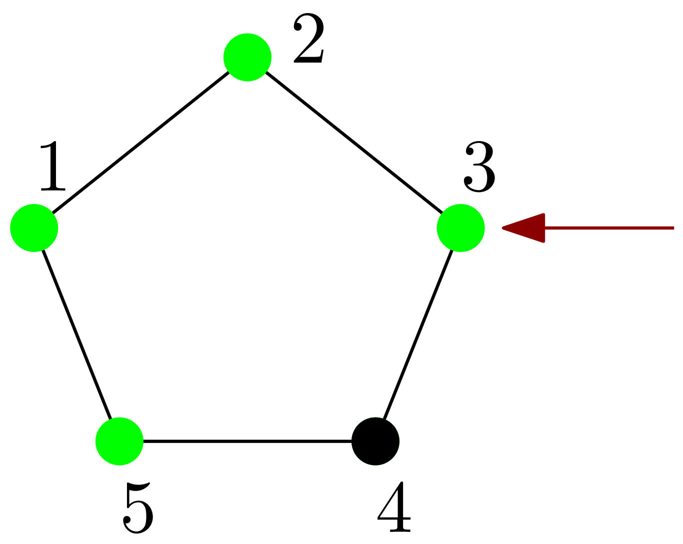

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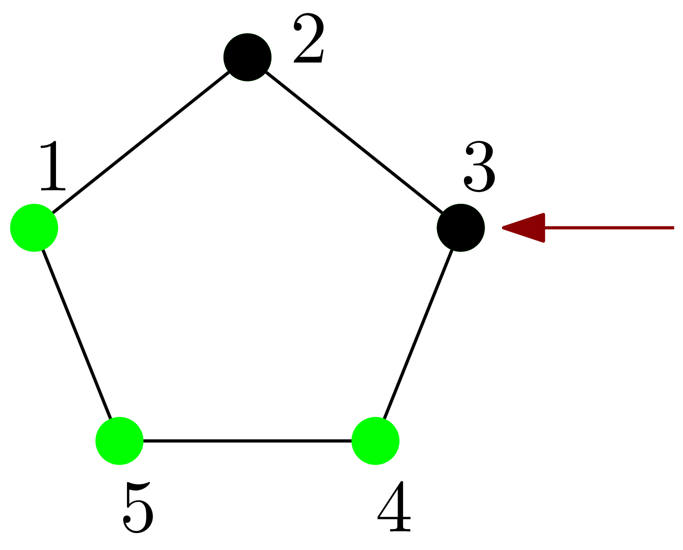

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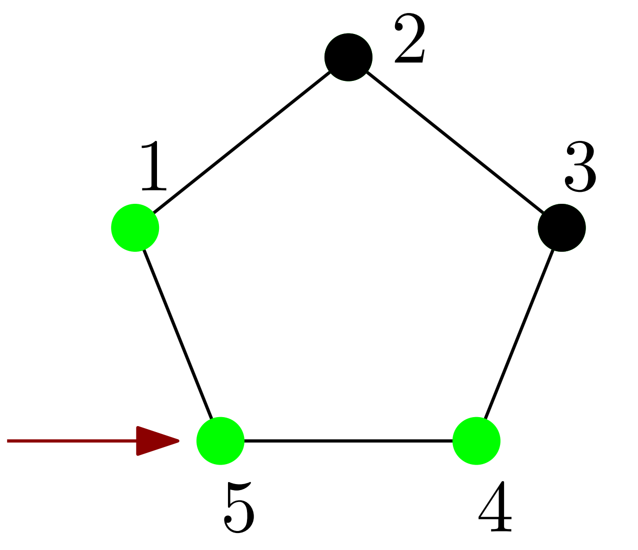

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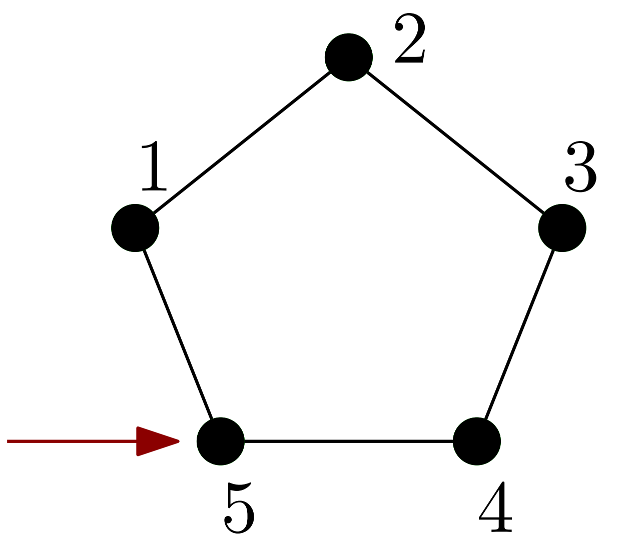

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Proof (sketch)

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3. By 2. equation $A_G^F \cdot x = F$, or equivalently $A_G^F \cdot x + F = 0$ has a solution.

Optimization Problems

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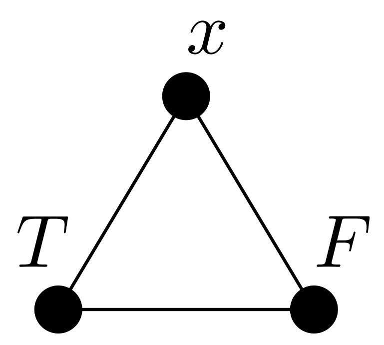
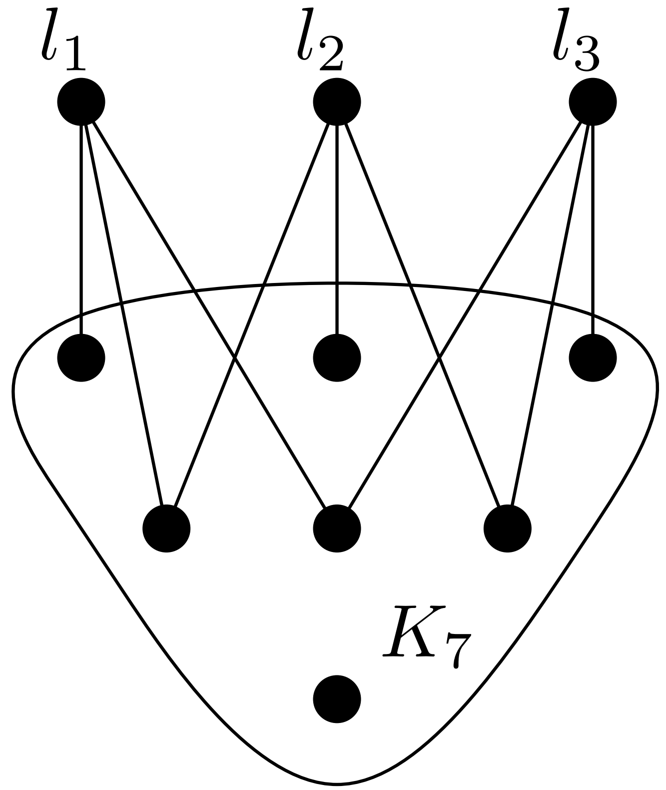
We will use gadgets:



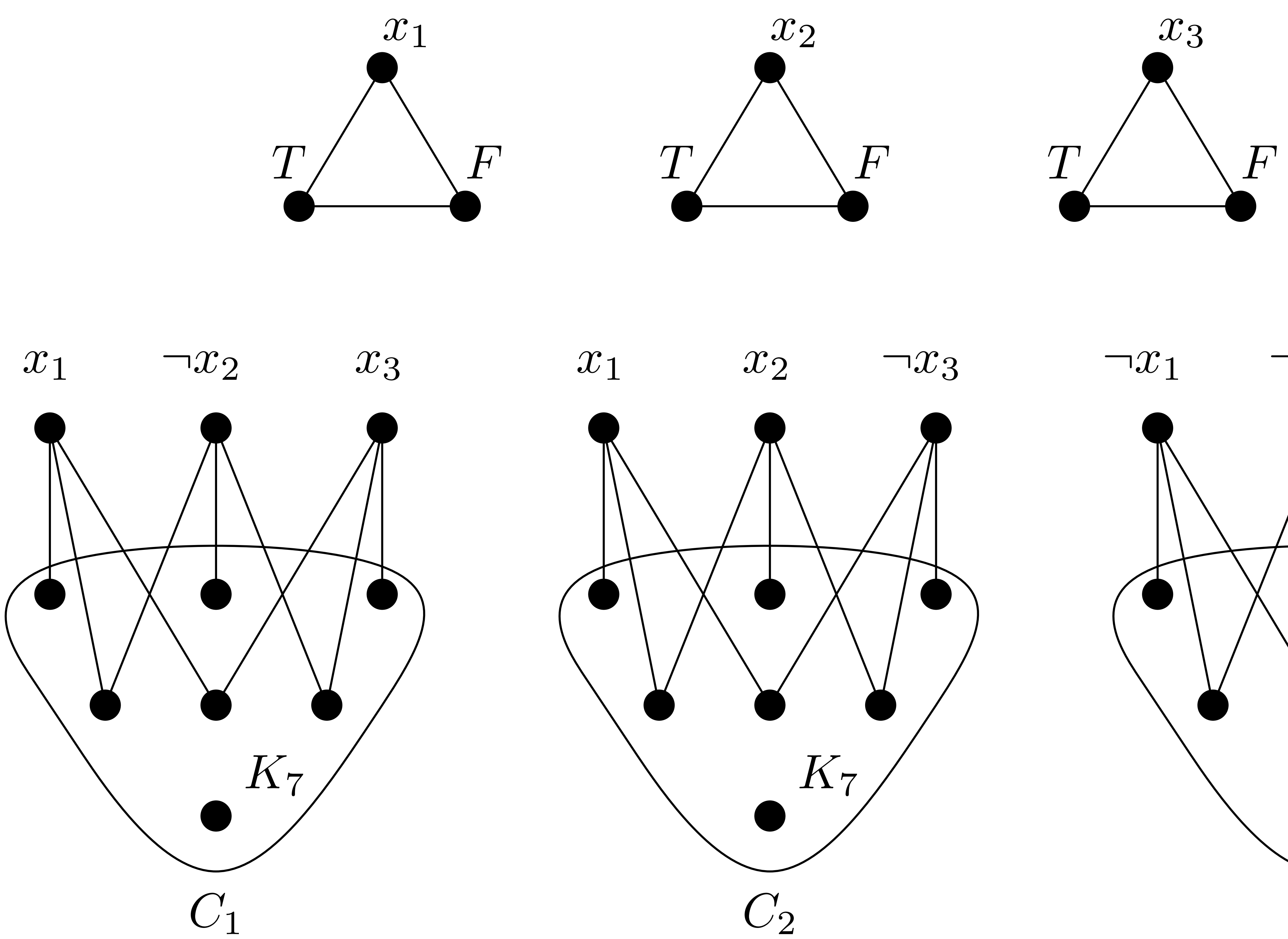
Variable gadget



Clause gadget

Example

$\phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$



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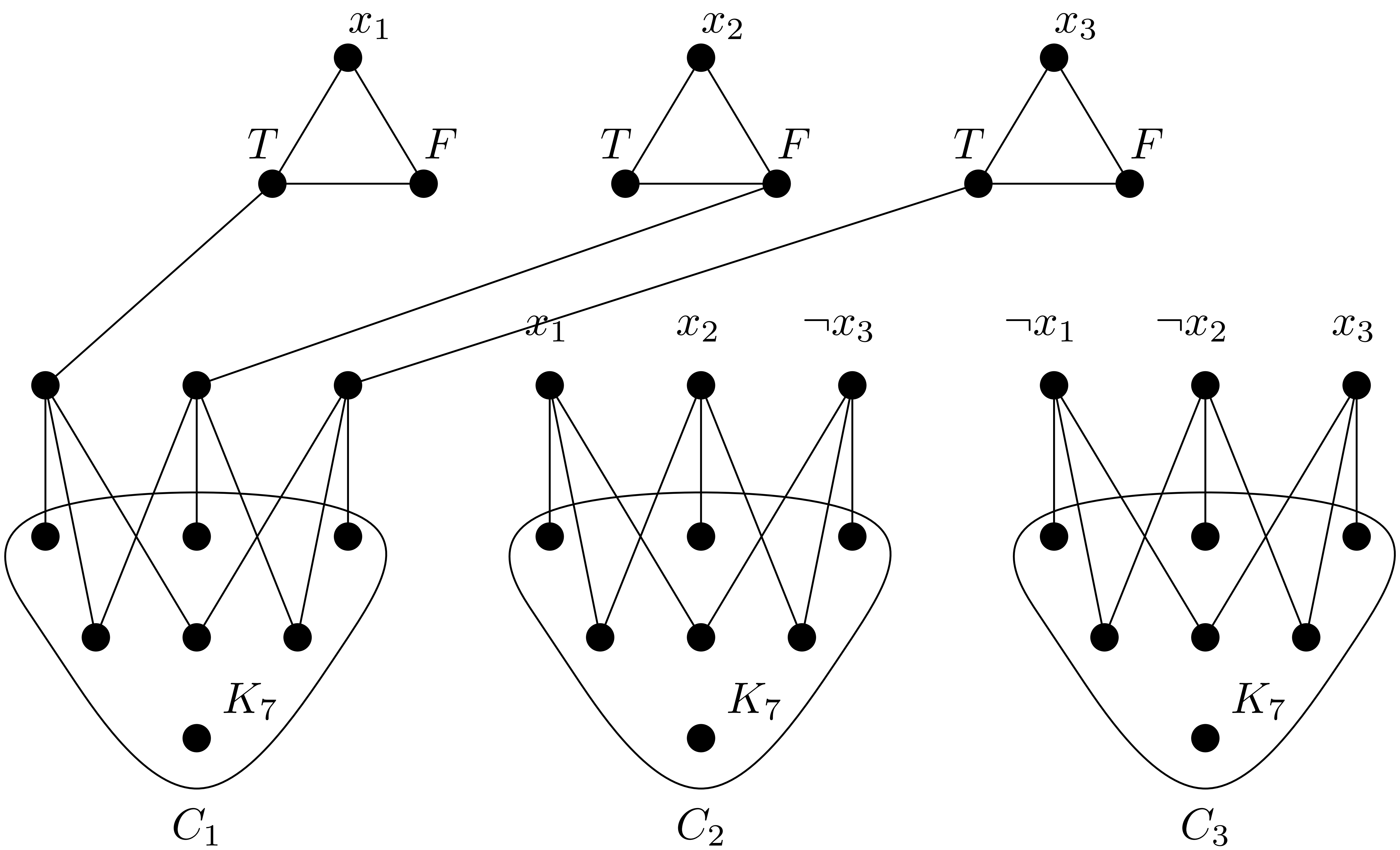

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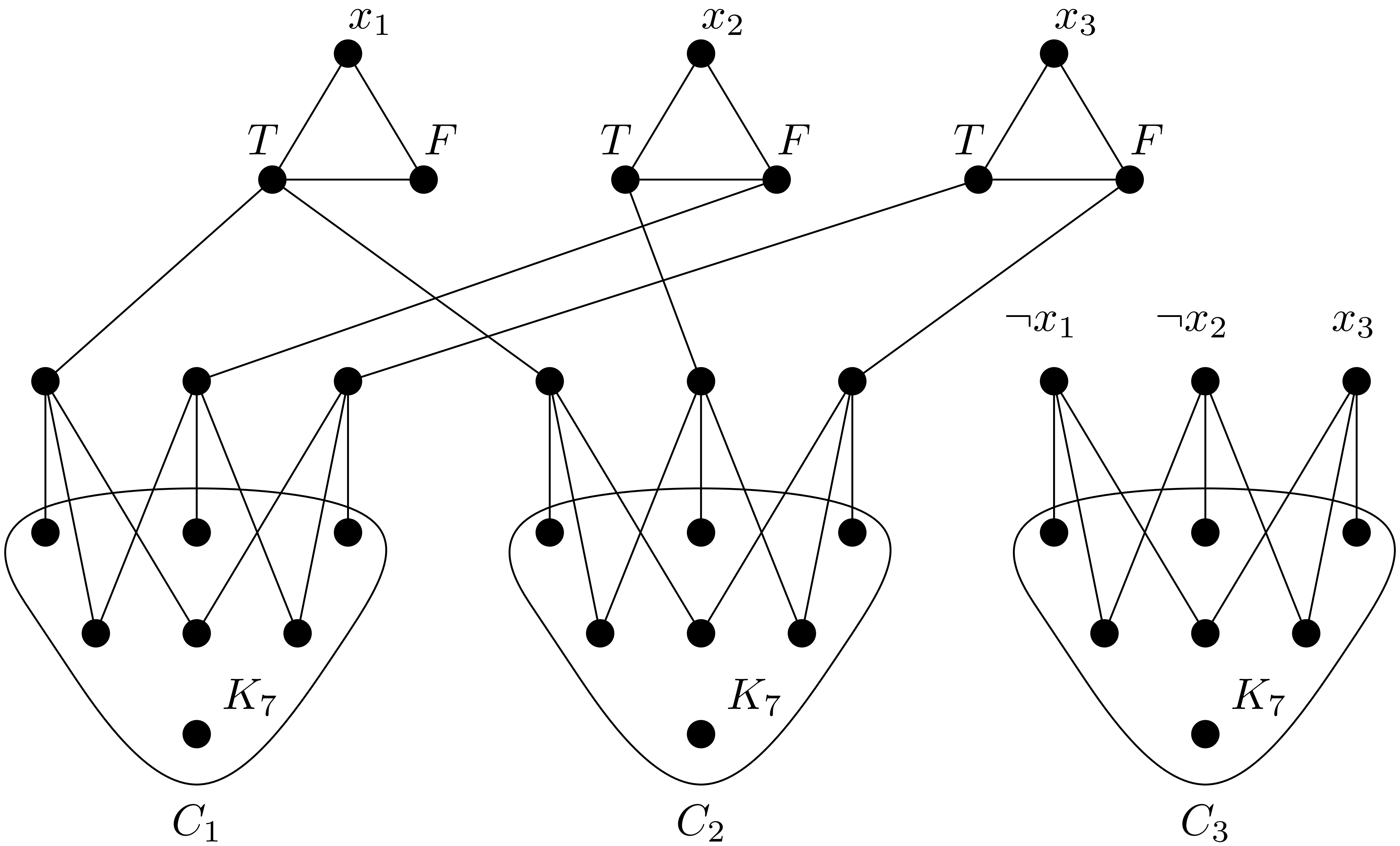

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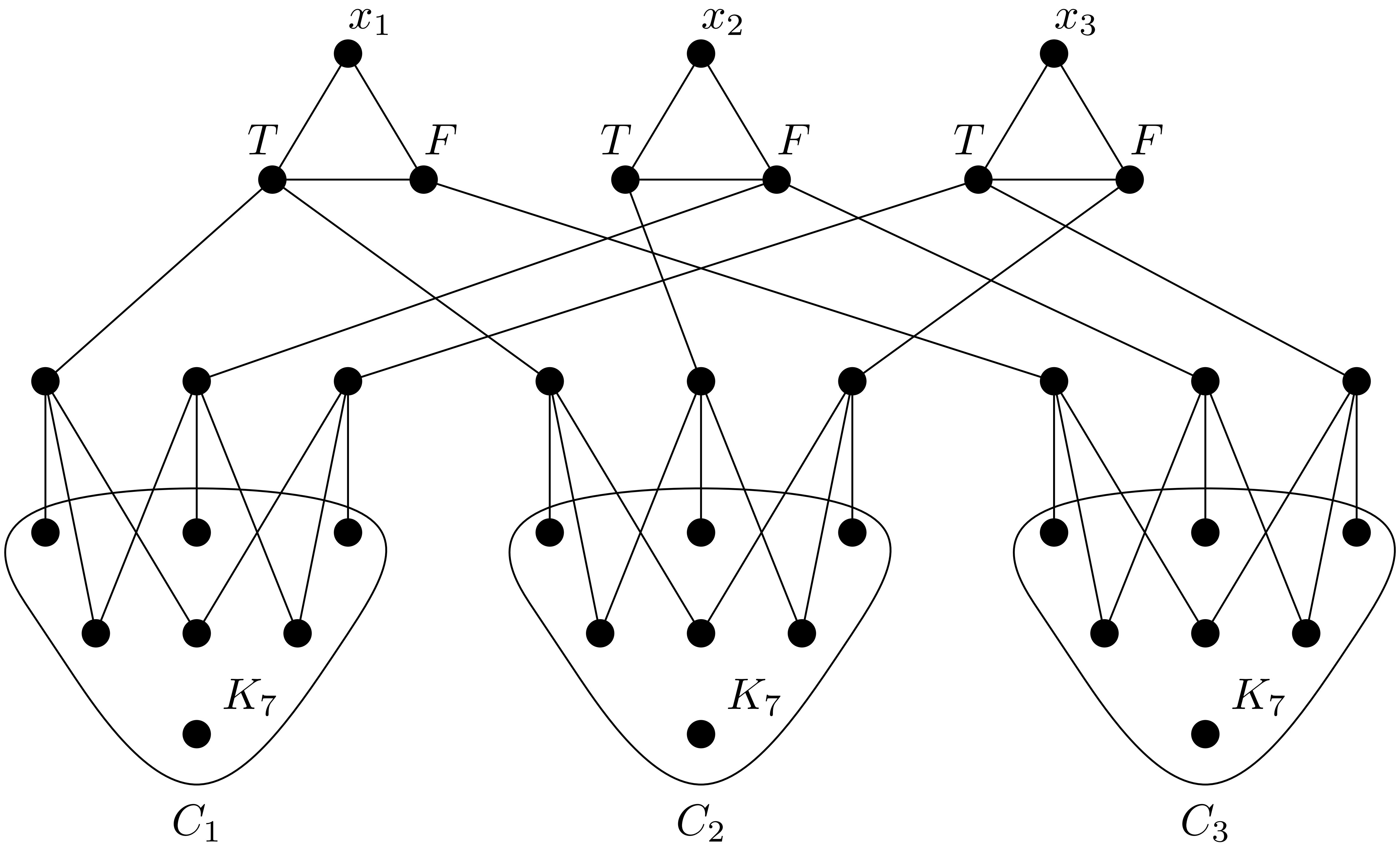

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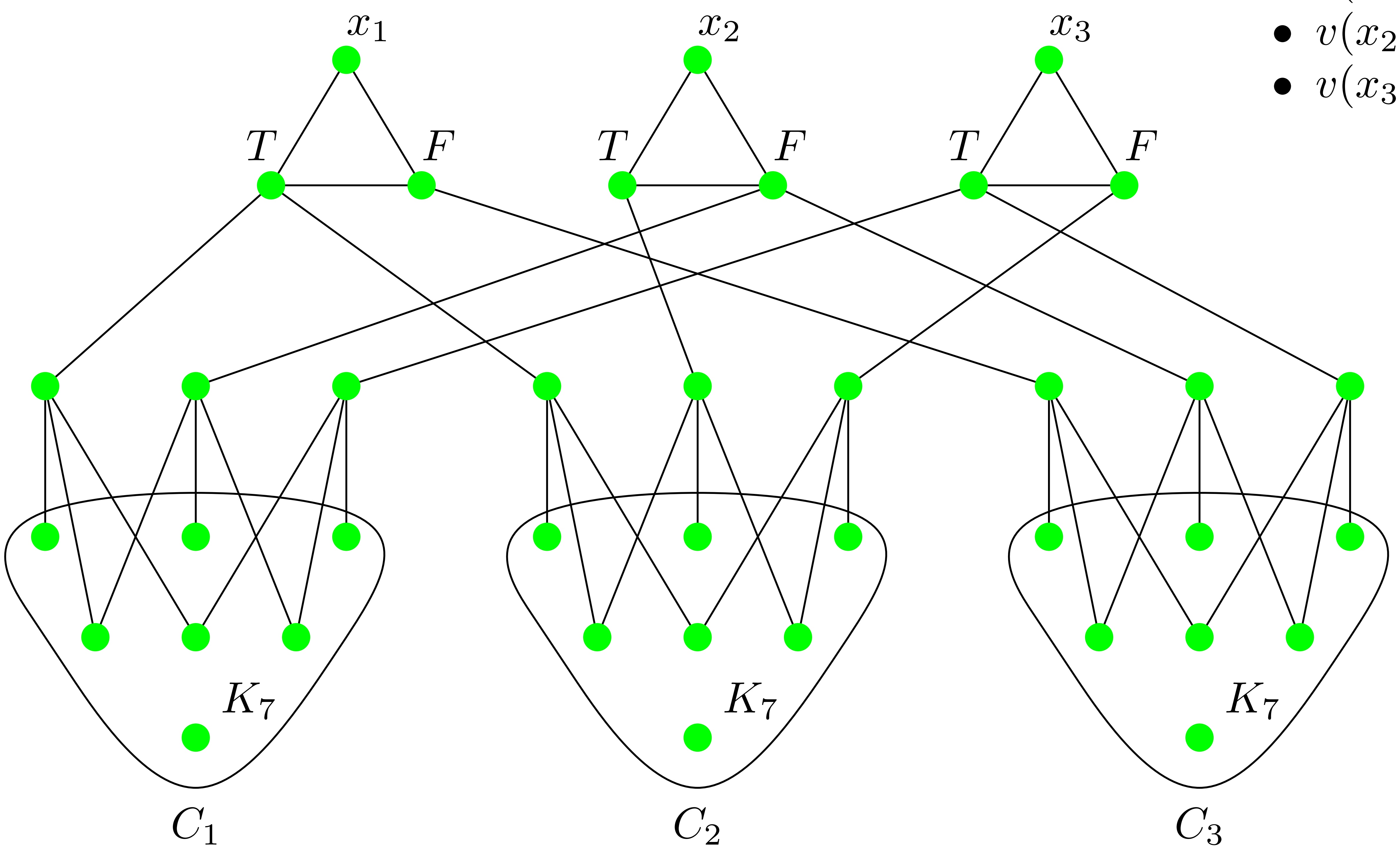

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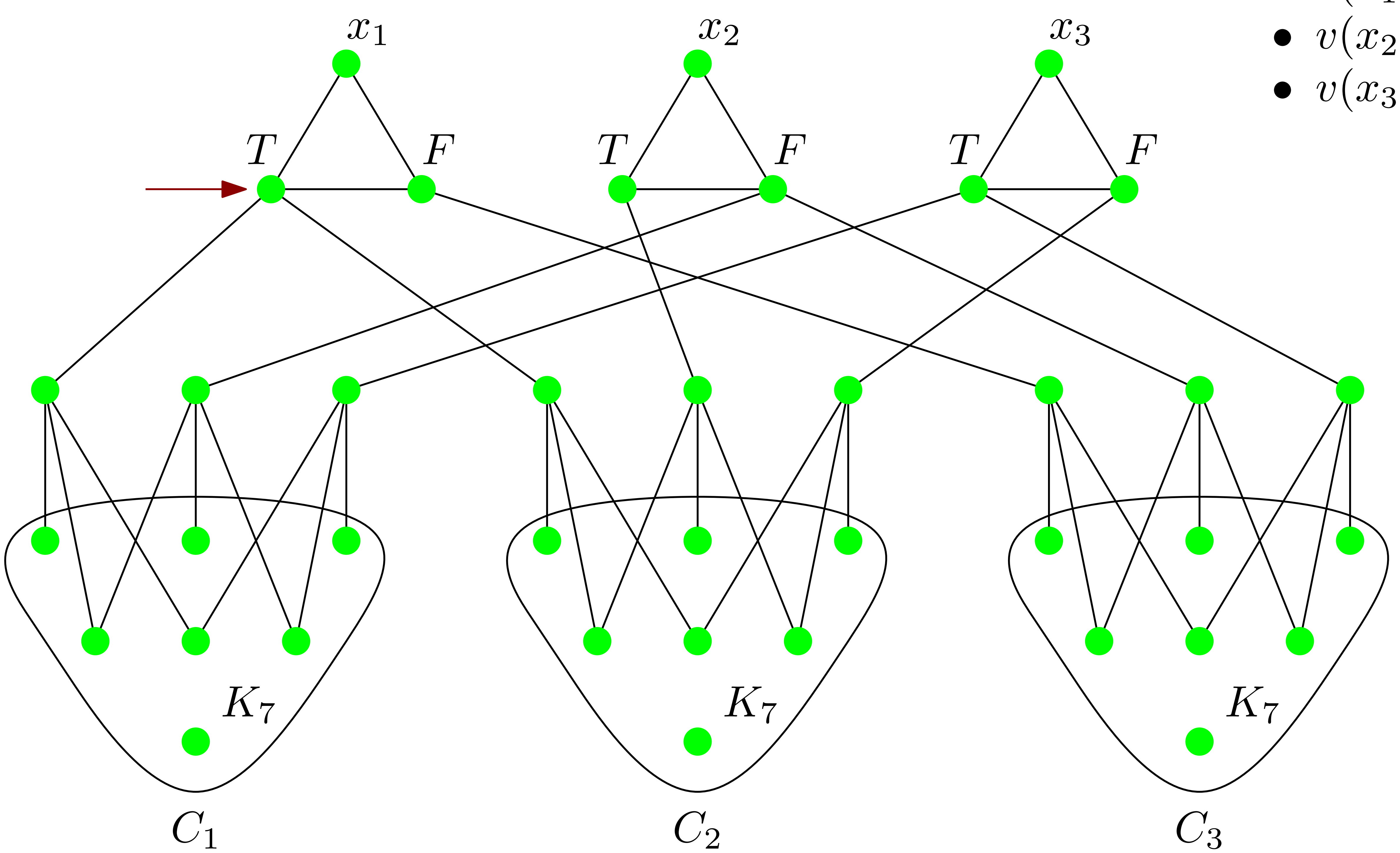

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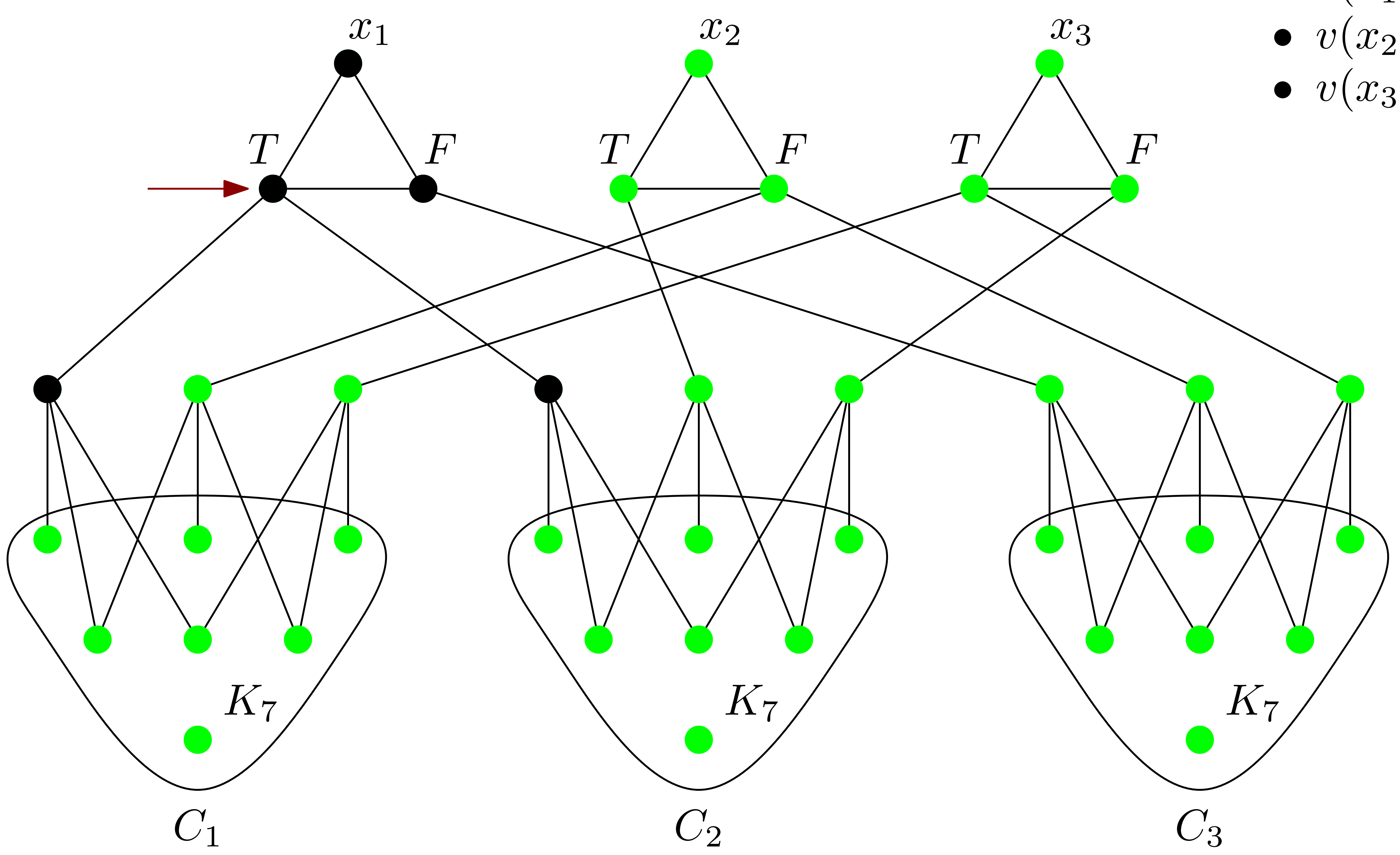

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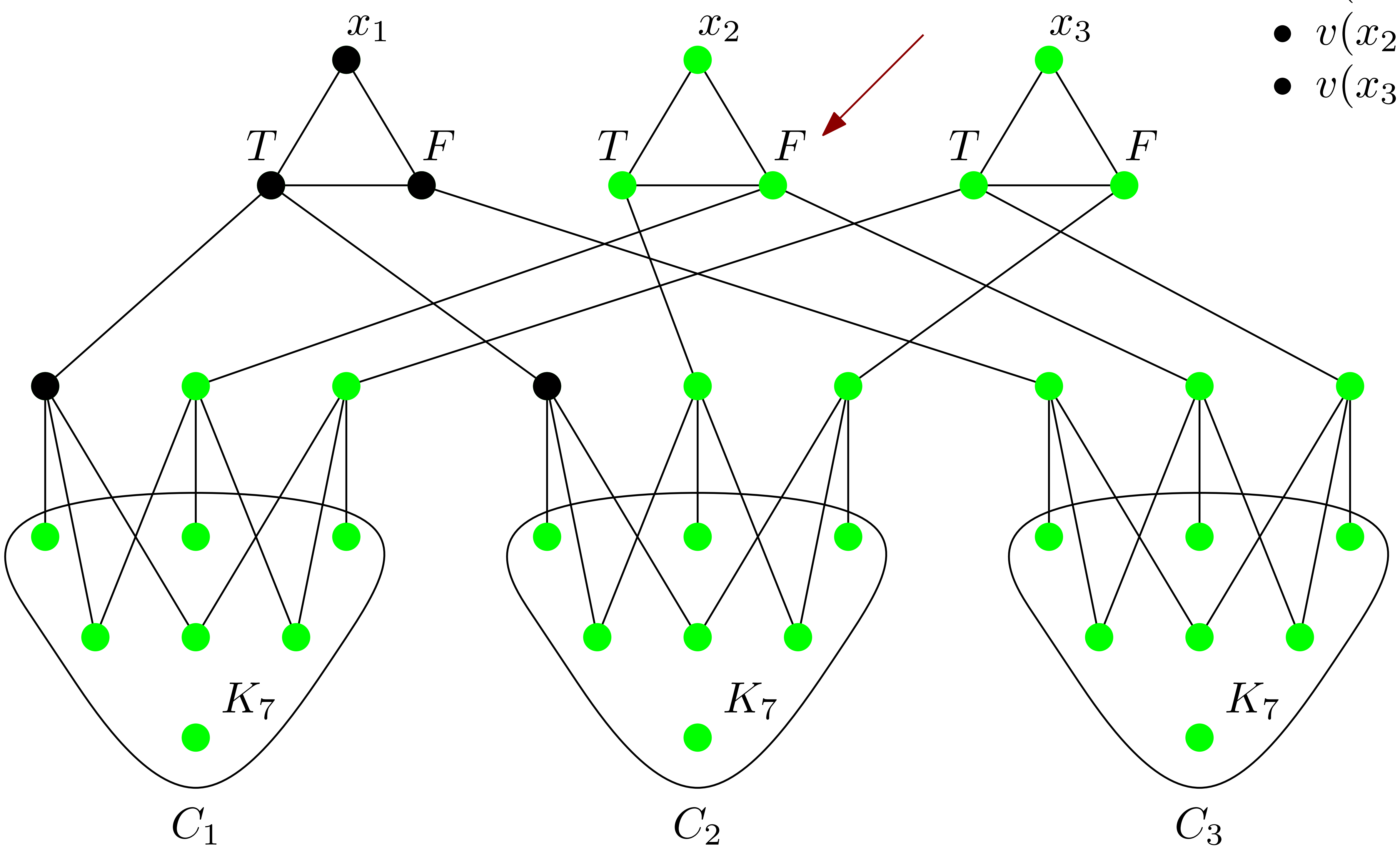

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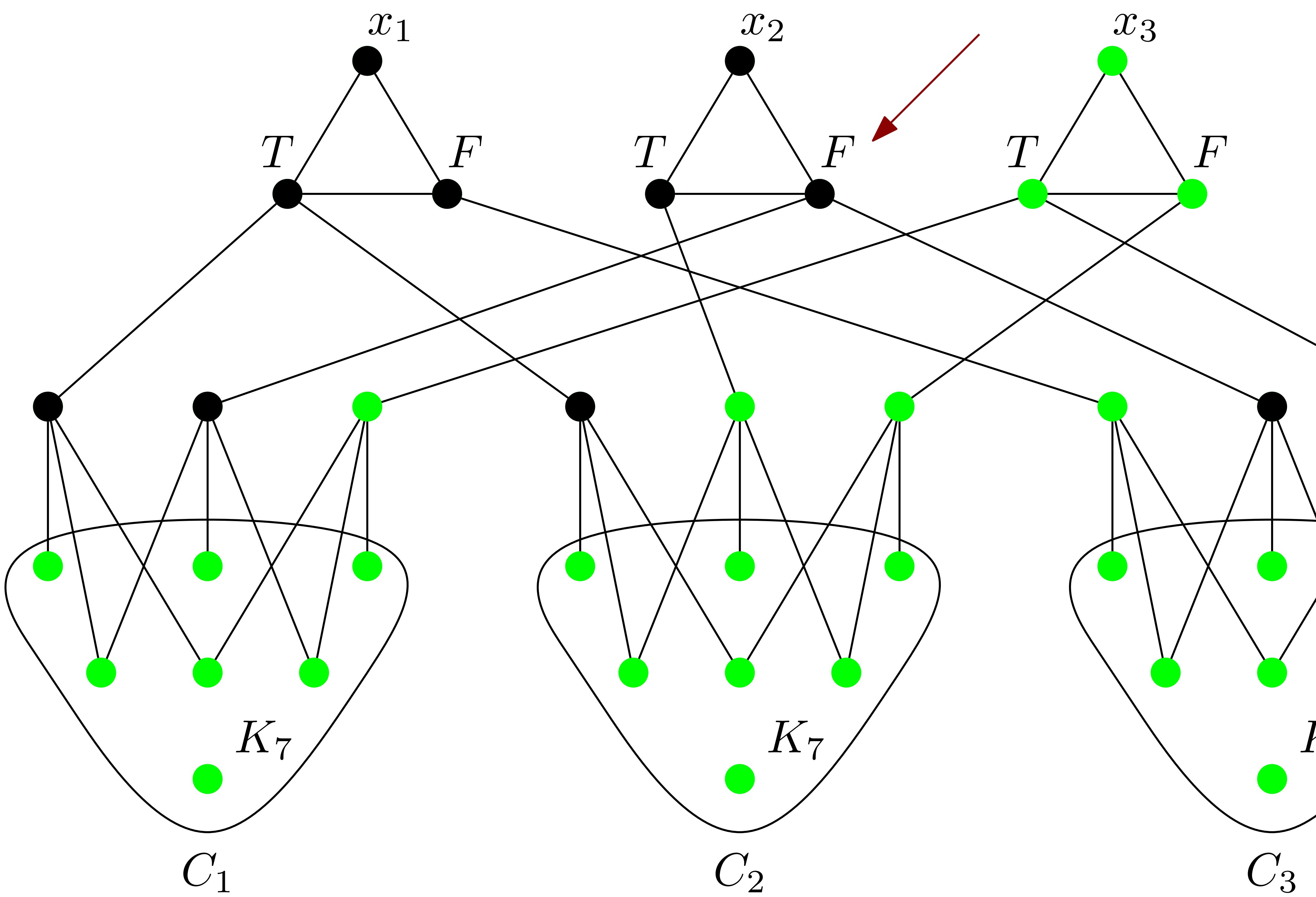

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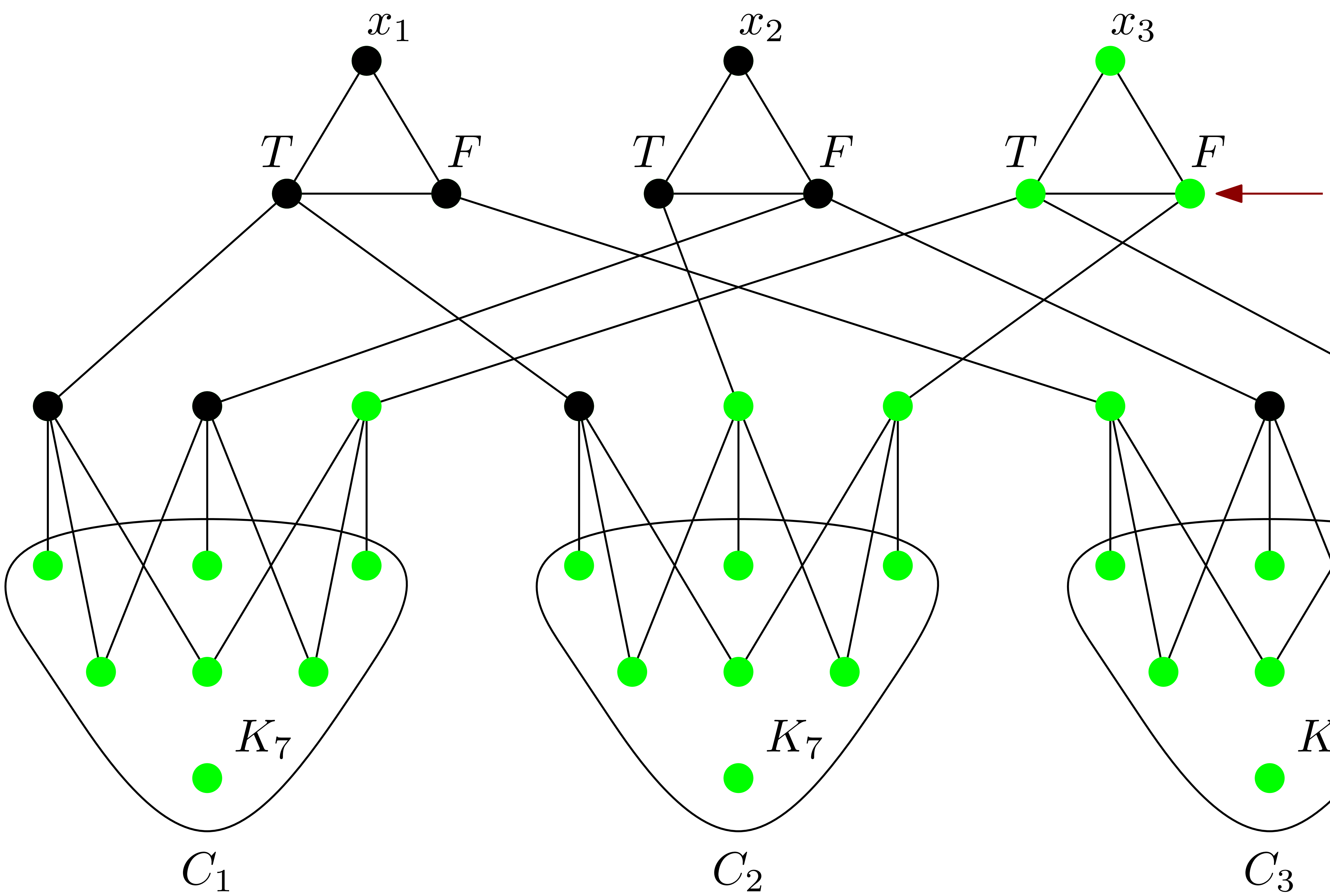

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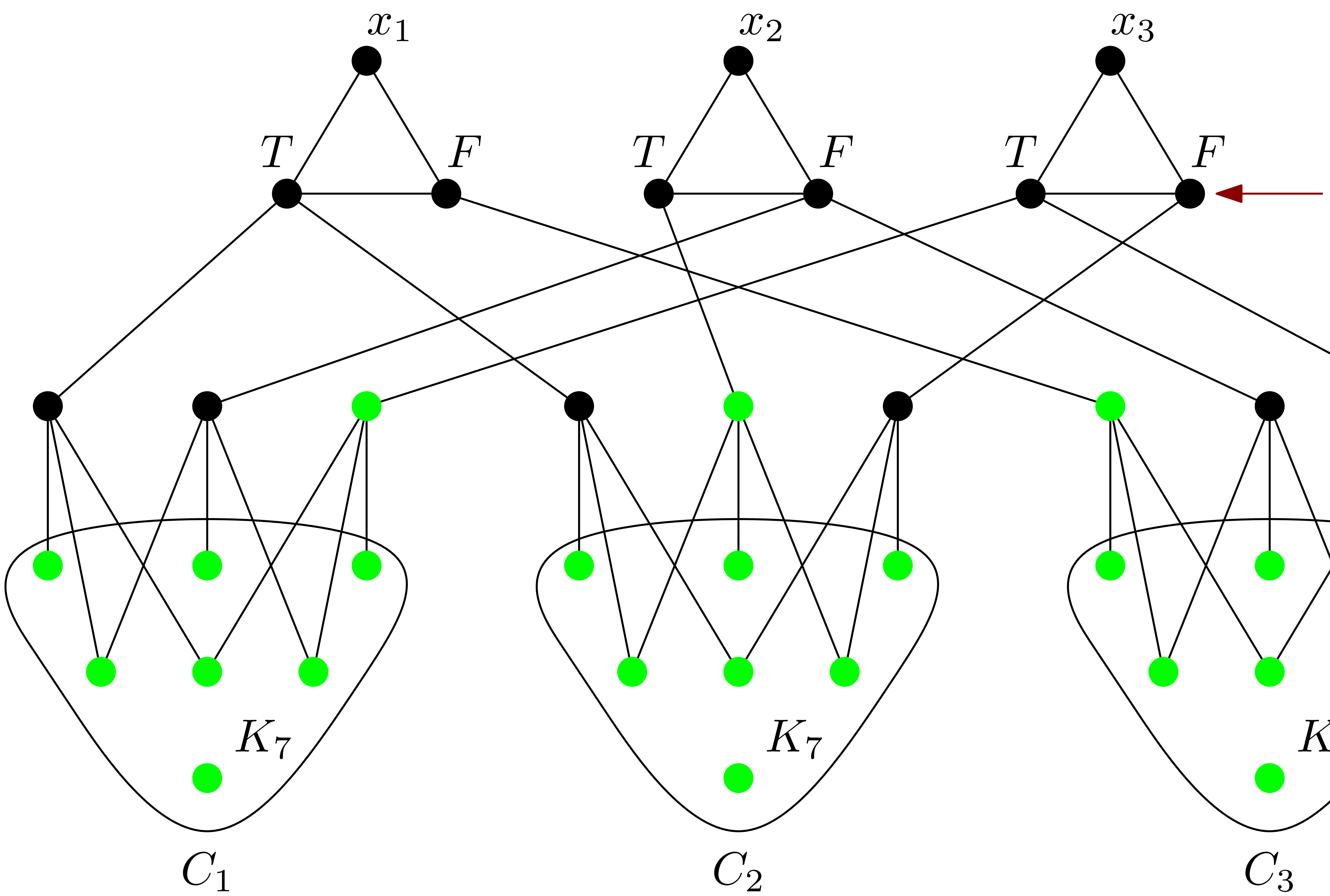

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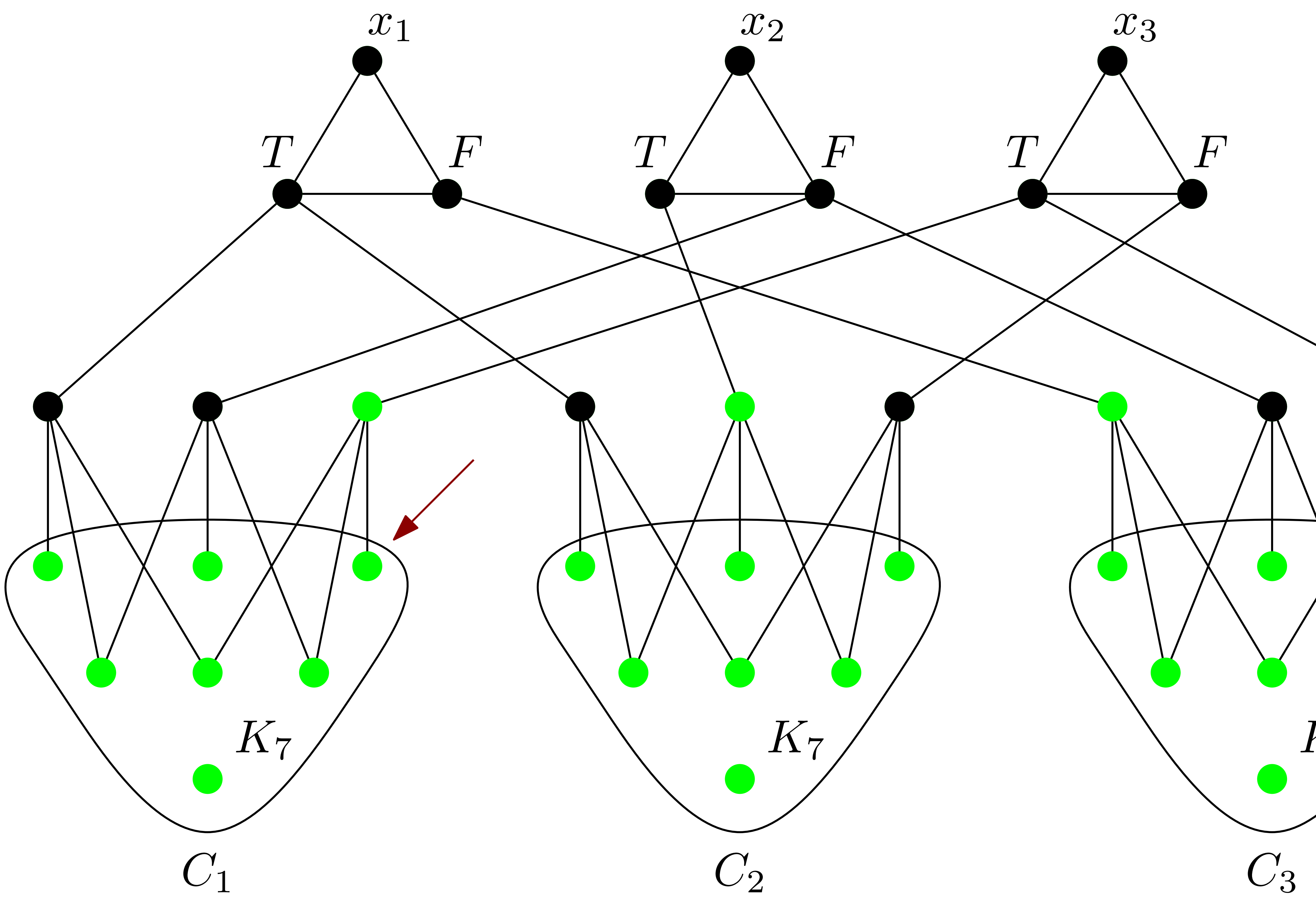

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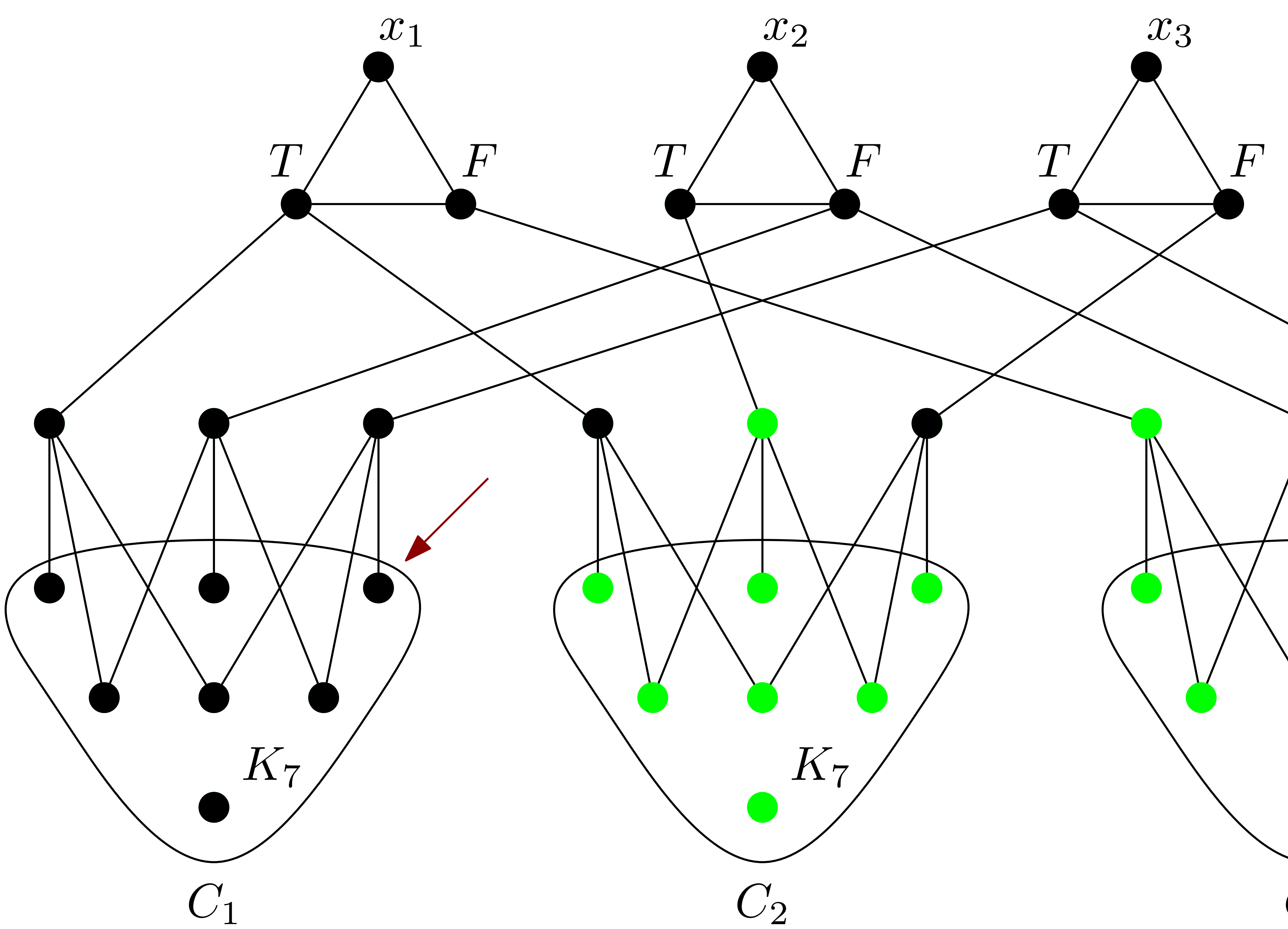

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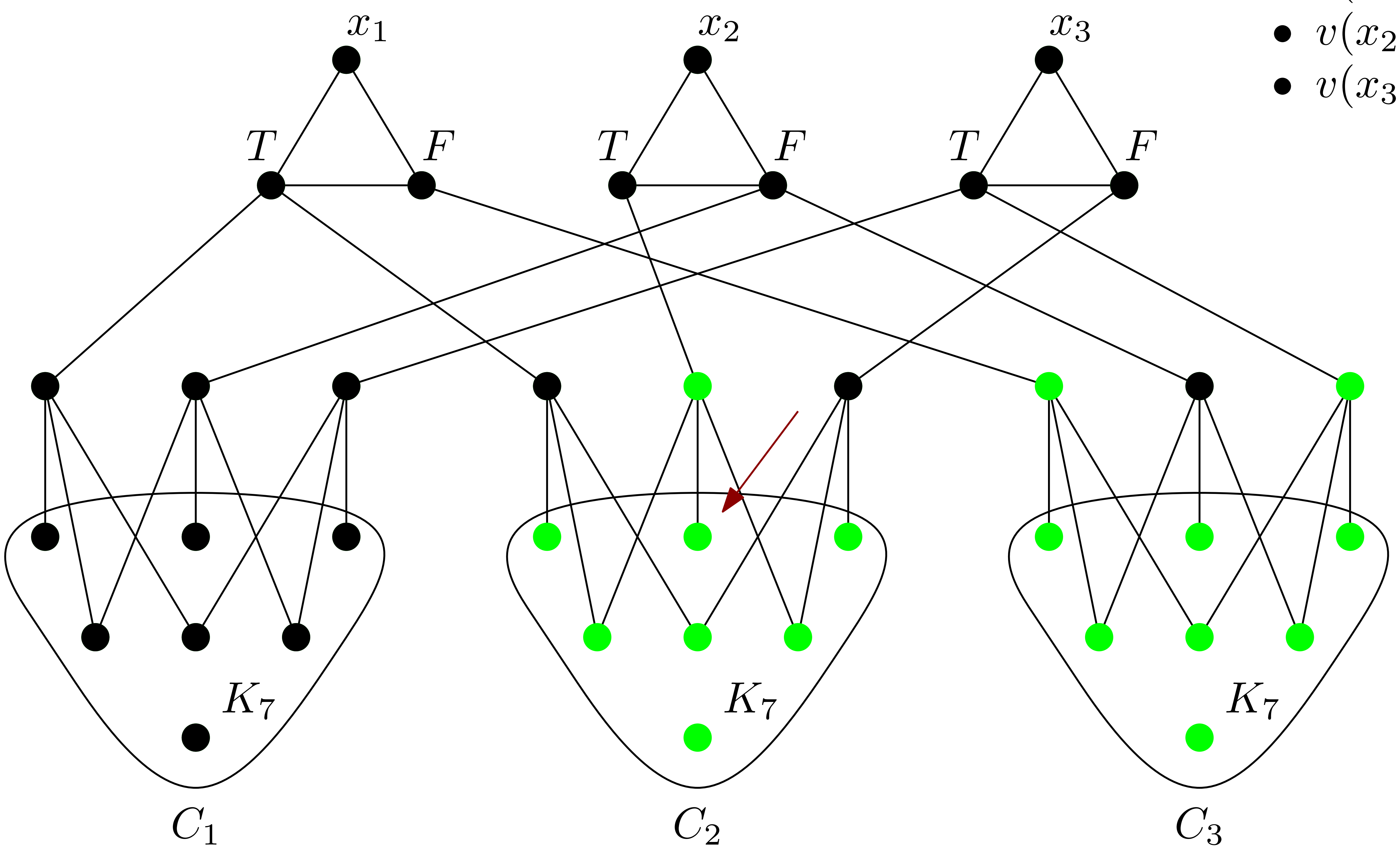

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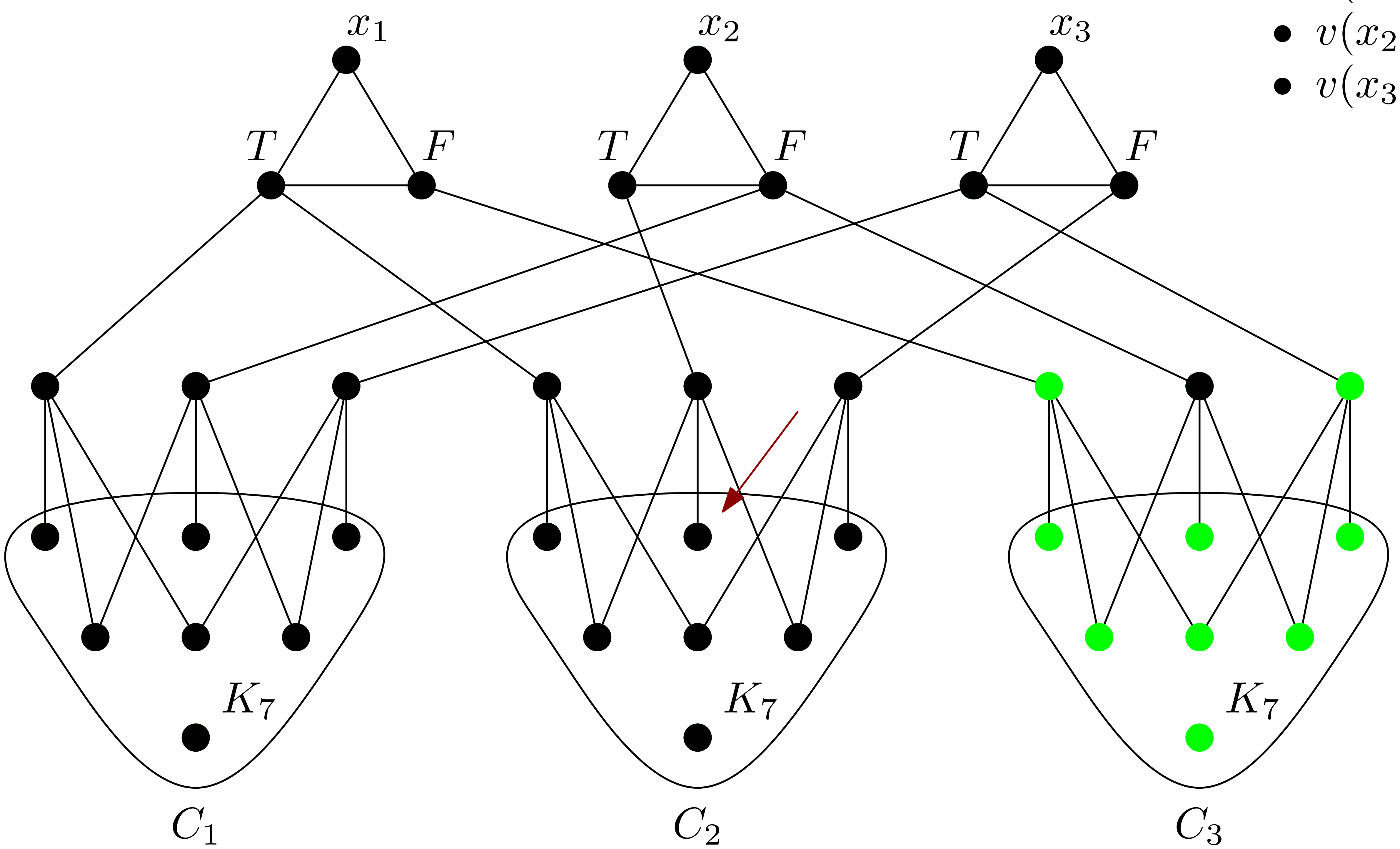

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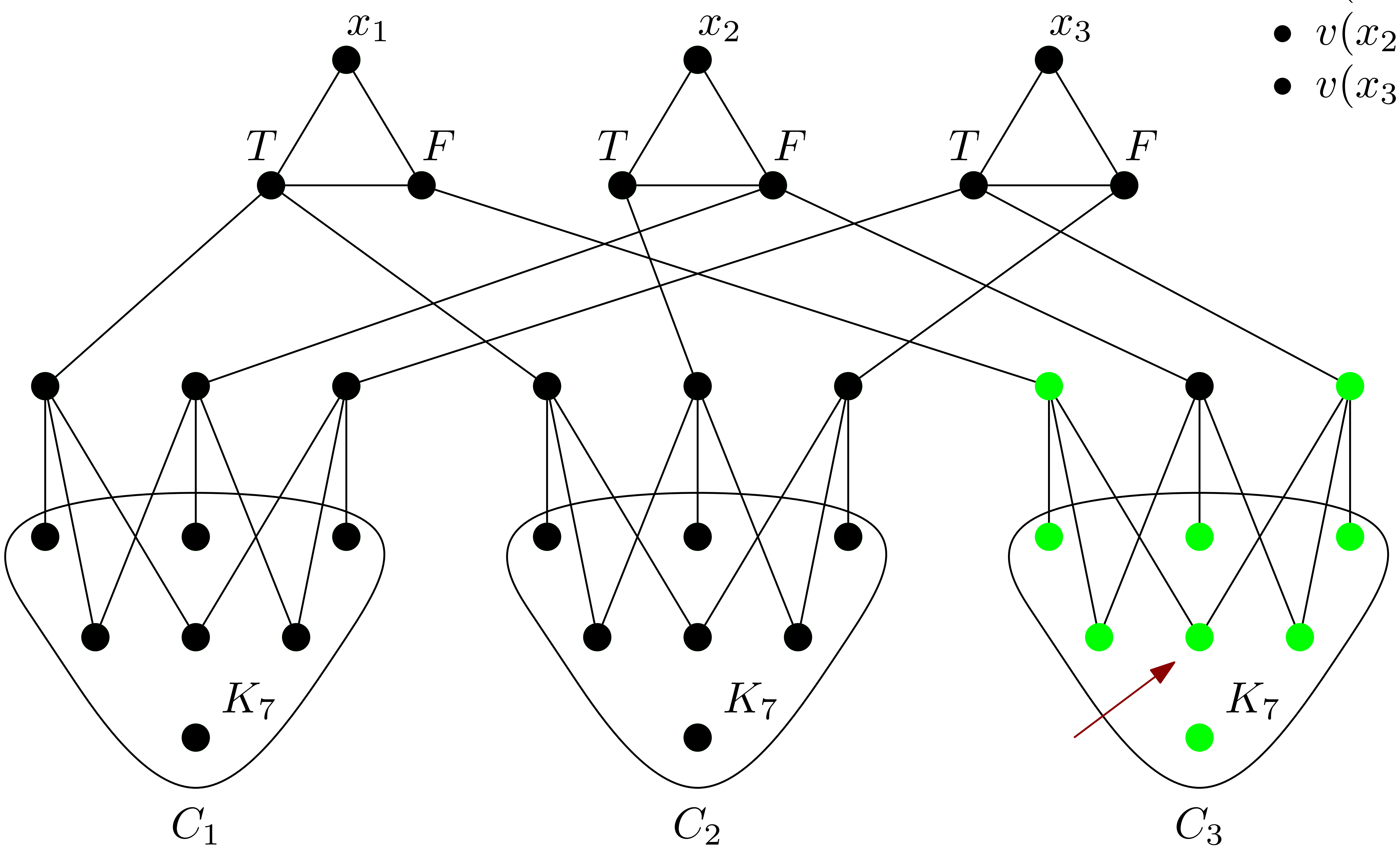

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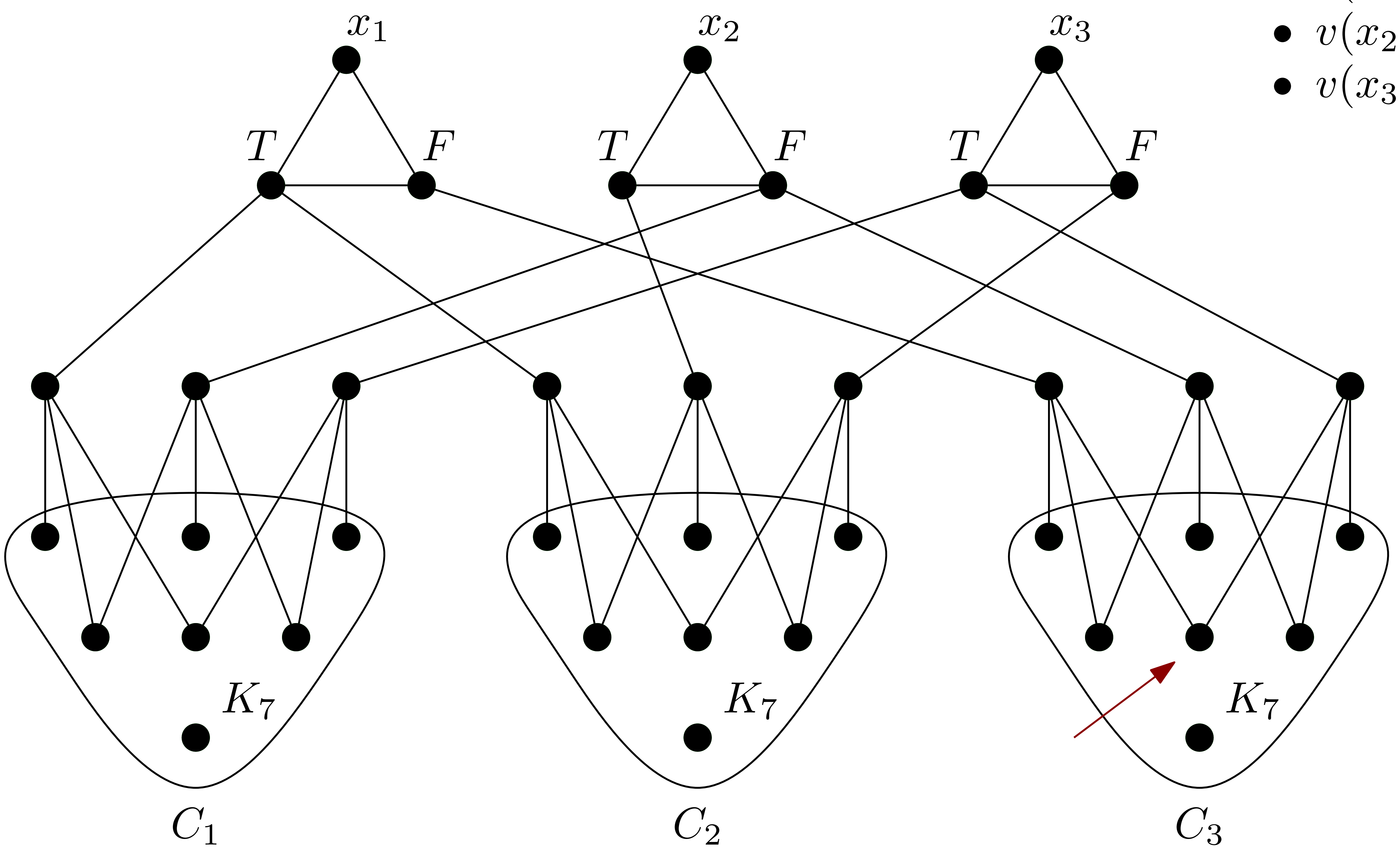

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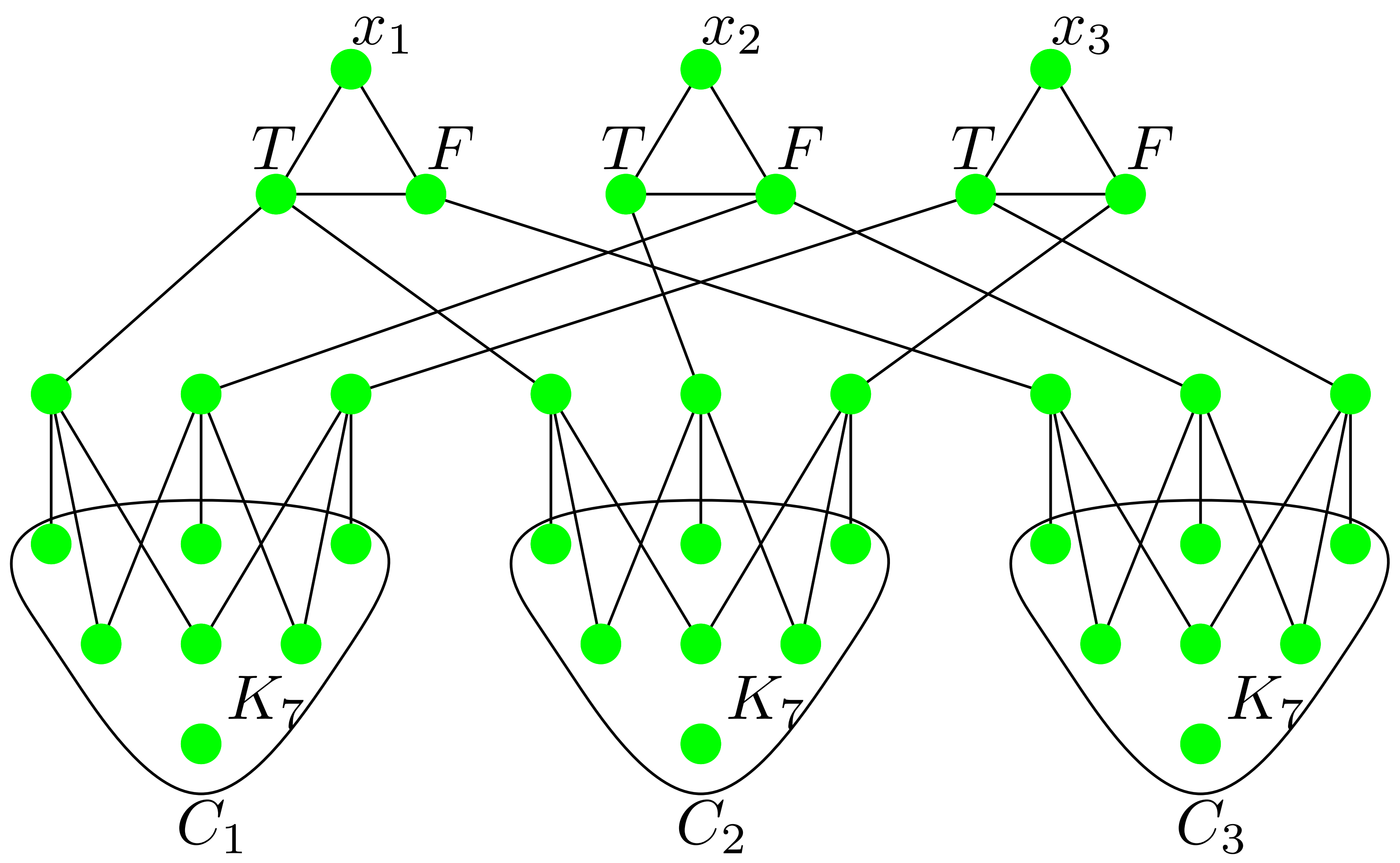

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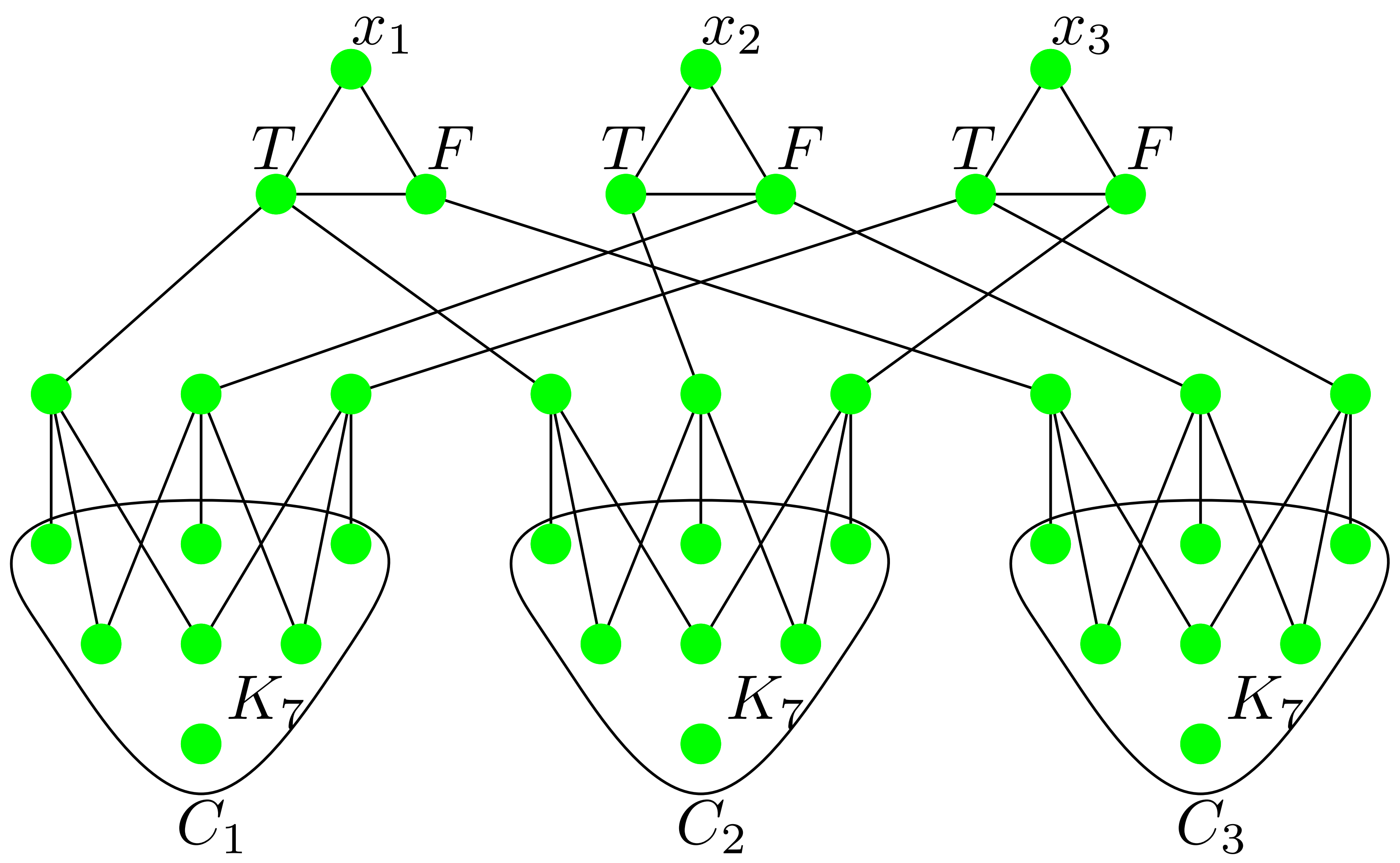

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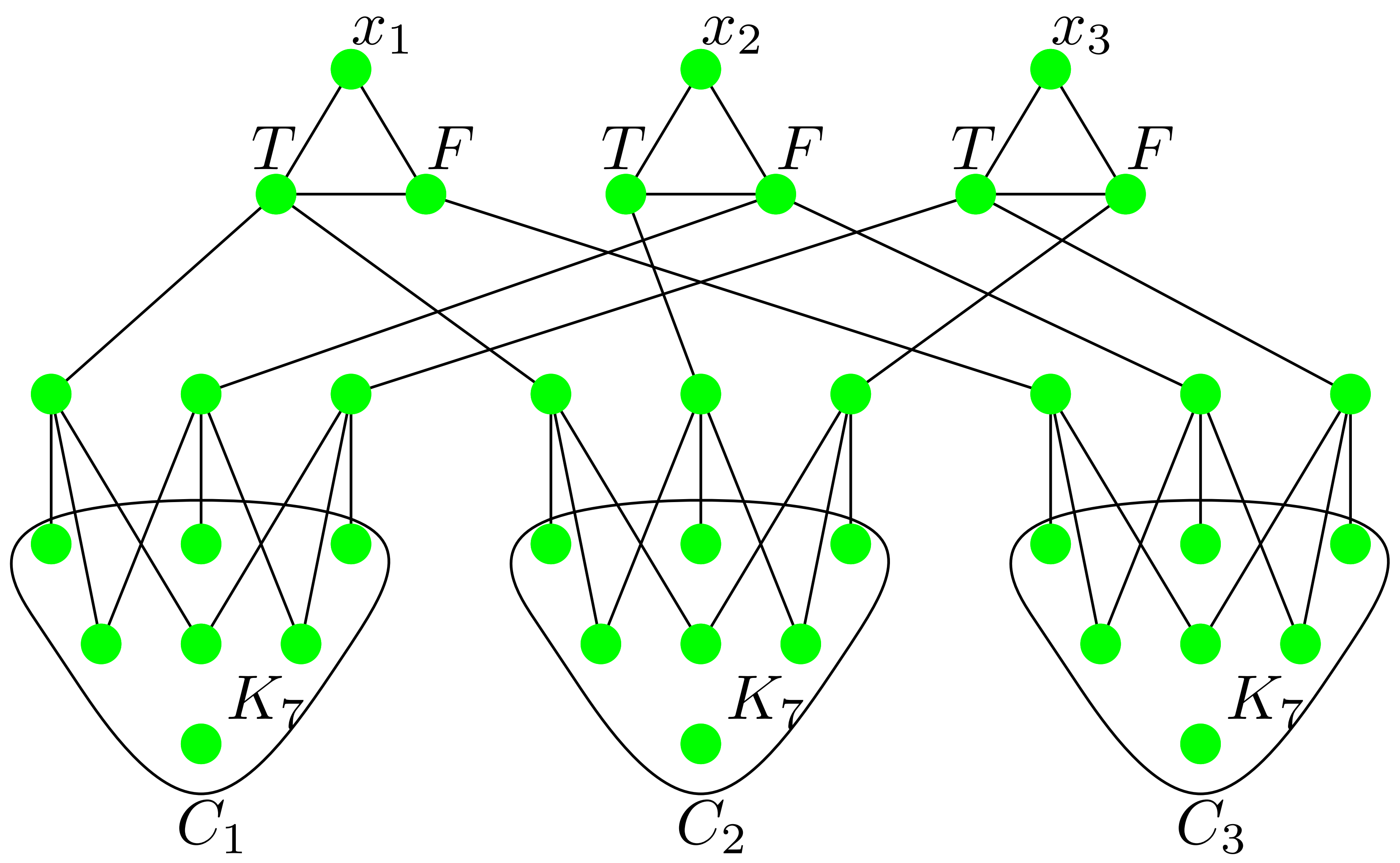

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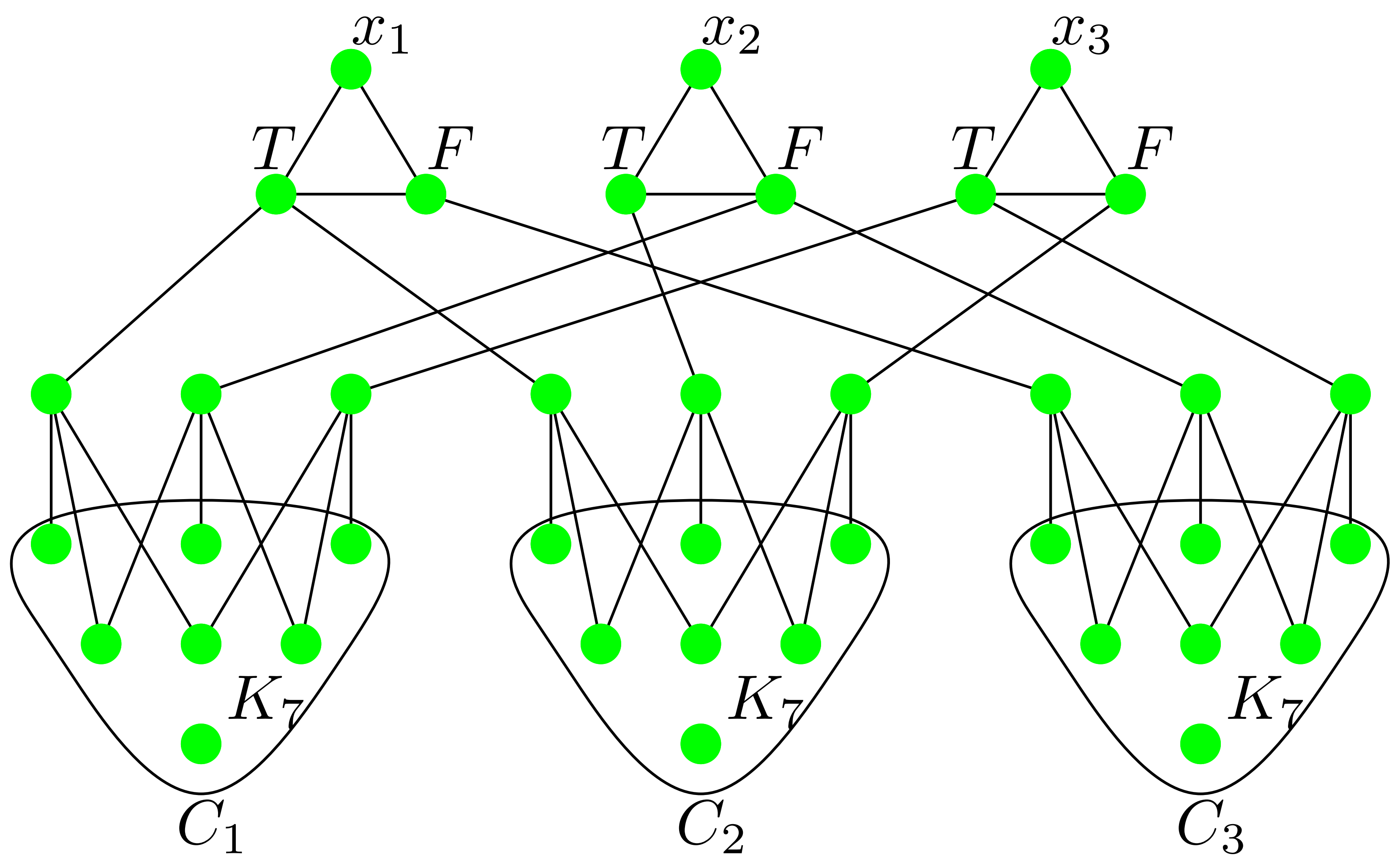

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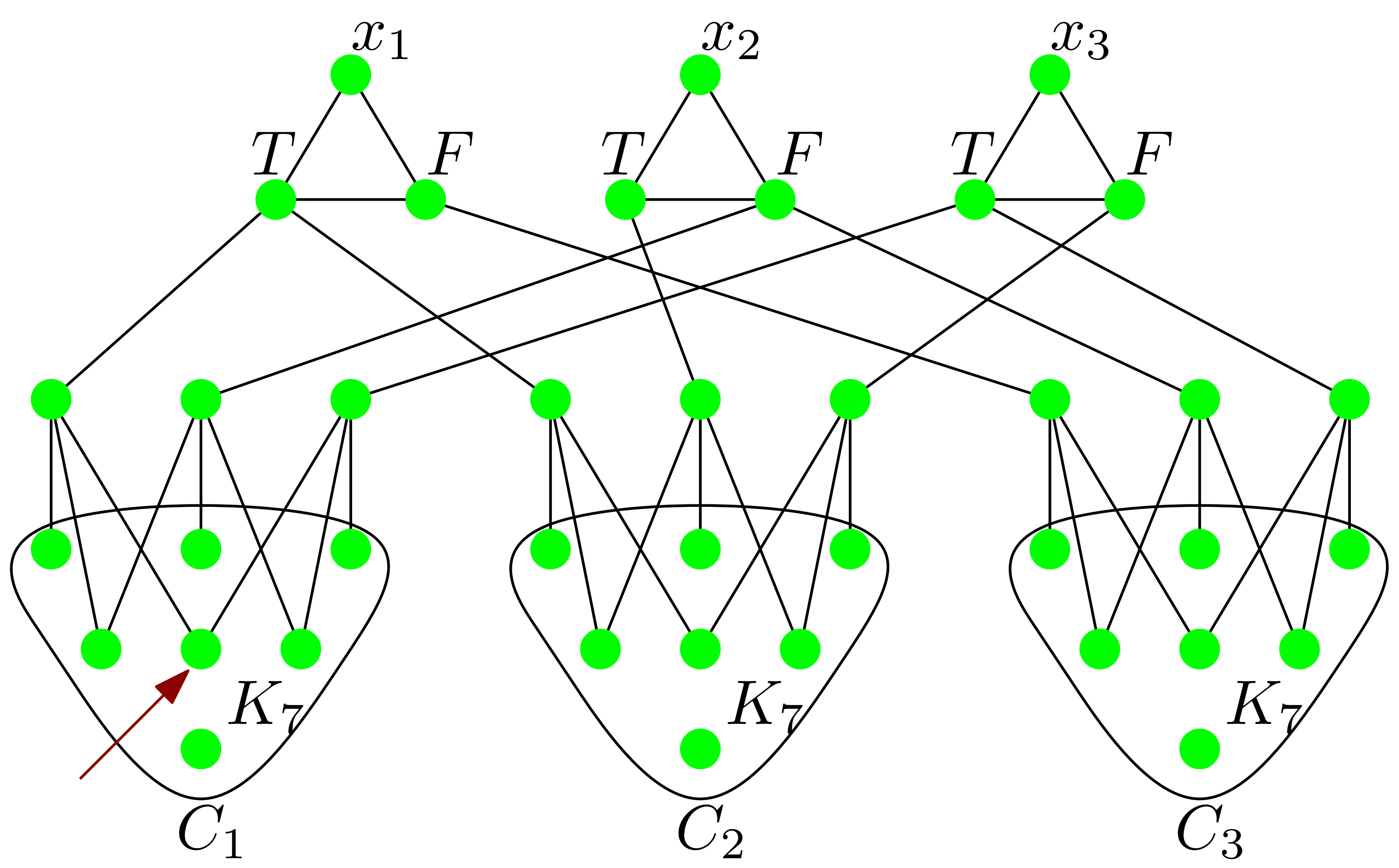

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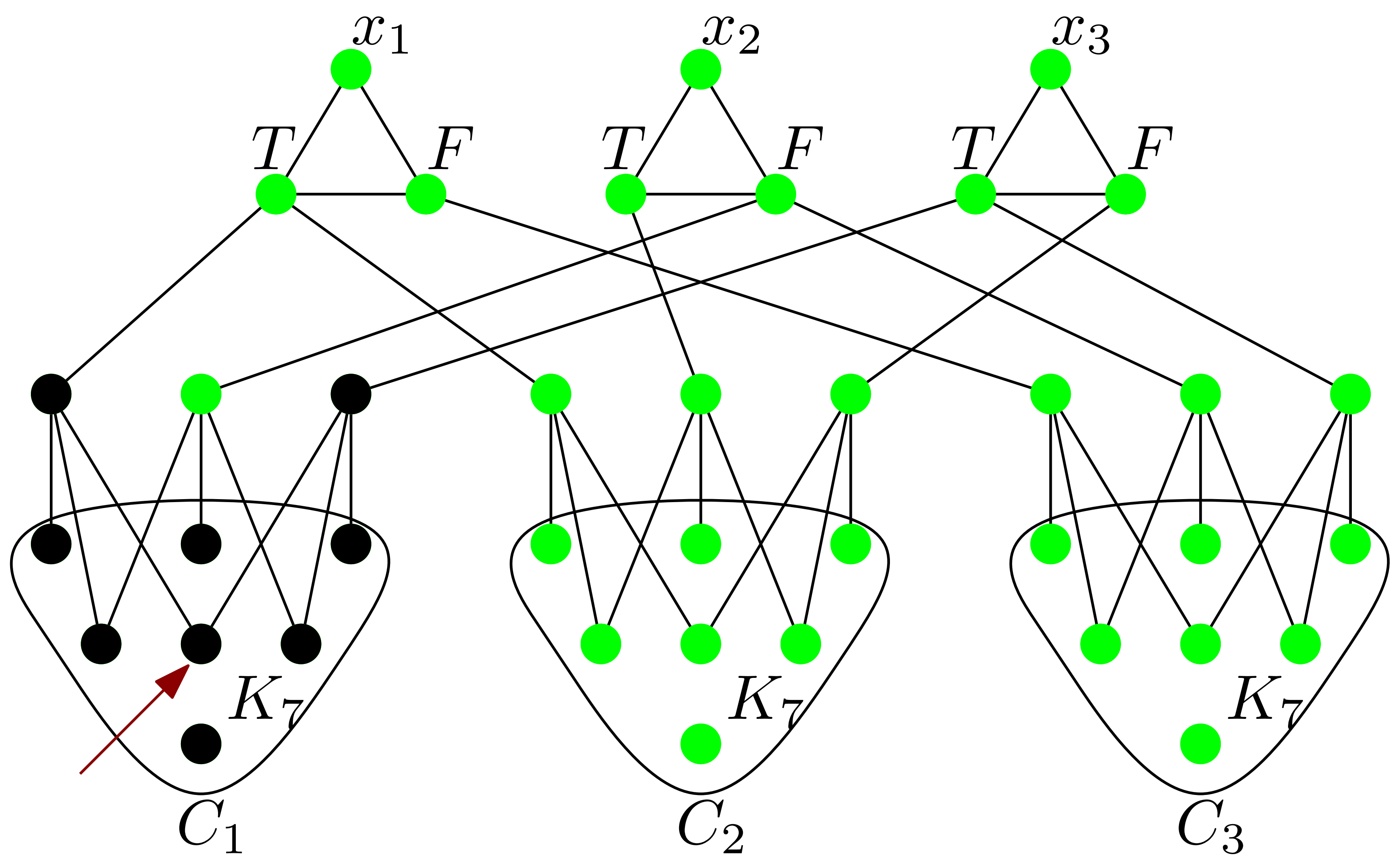
$(G, n+m) \in k-$ALLOFF $\rightarrow \phi$ is satisfiable



Observations:

- at least one activation needed in each variable gadget
- at least one activation needed in each clique

$n + m$ activations $\rightarrow$
exactly one activation in each variable gadget and each clique and no activations in literal nodes

Since no clique node activation can cover all of the clause's literals, it must be that for every clique, there exists a variable gadget node that activated it.

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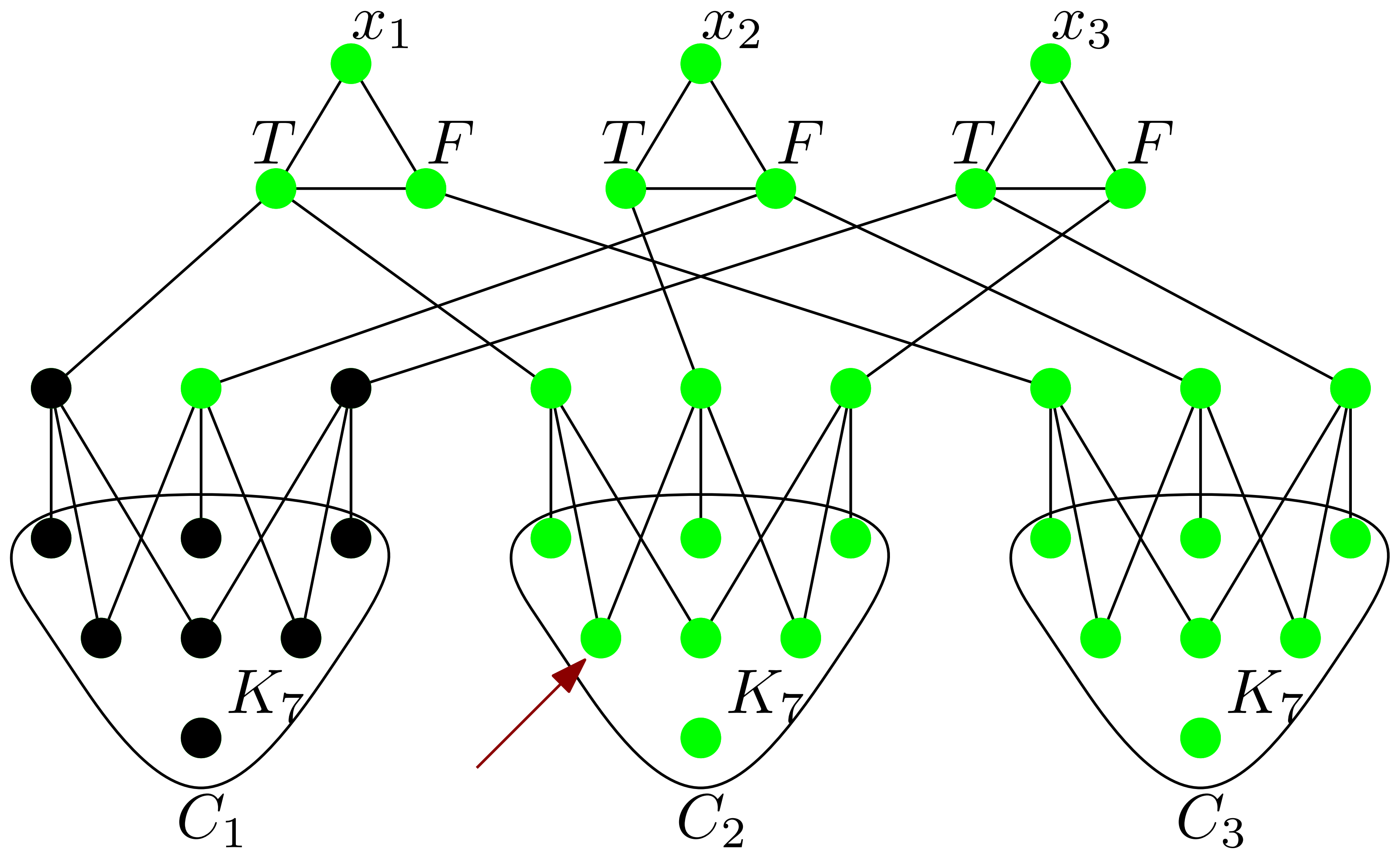

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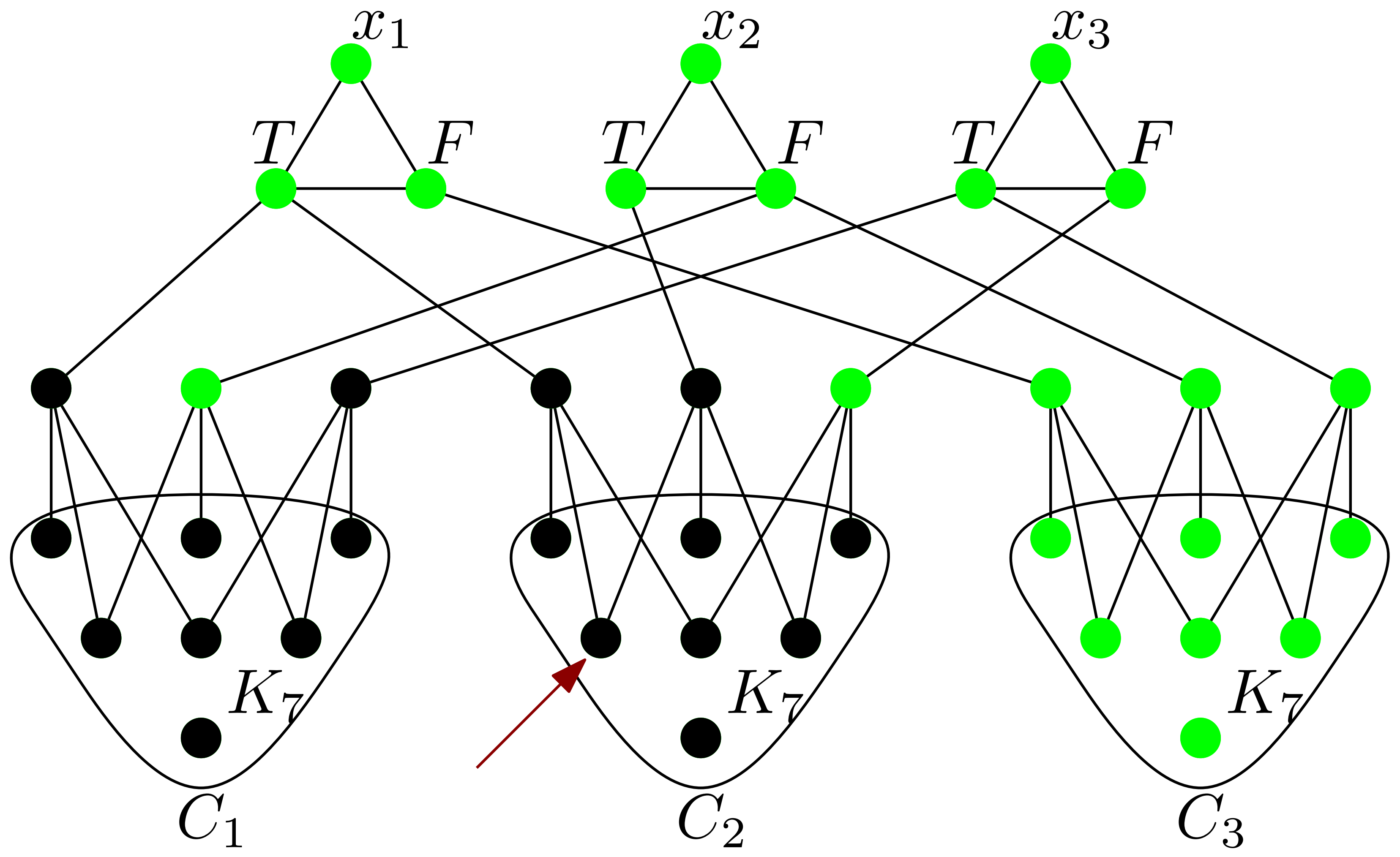

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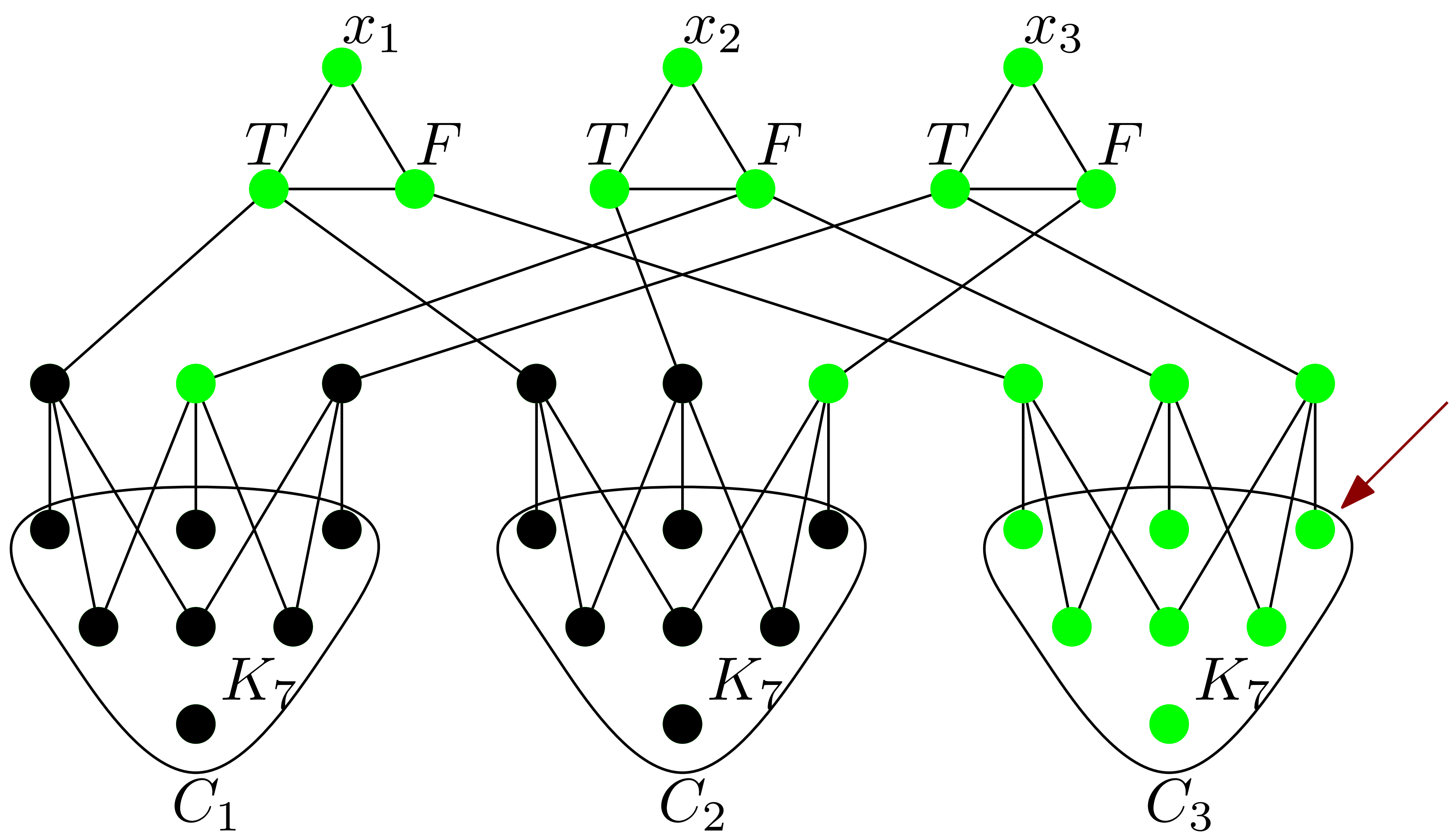

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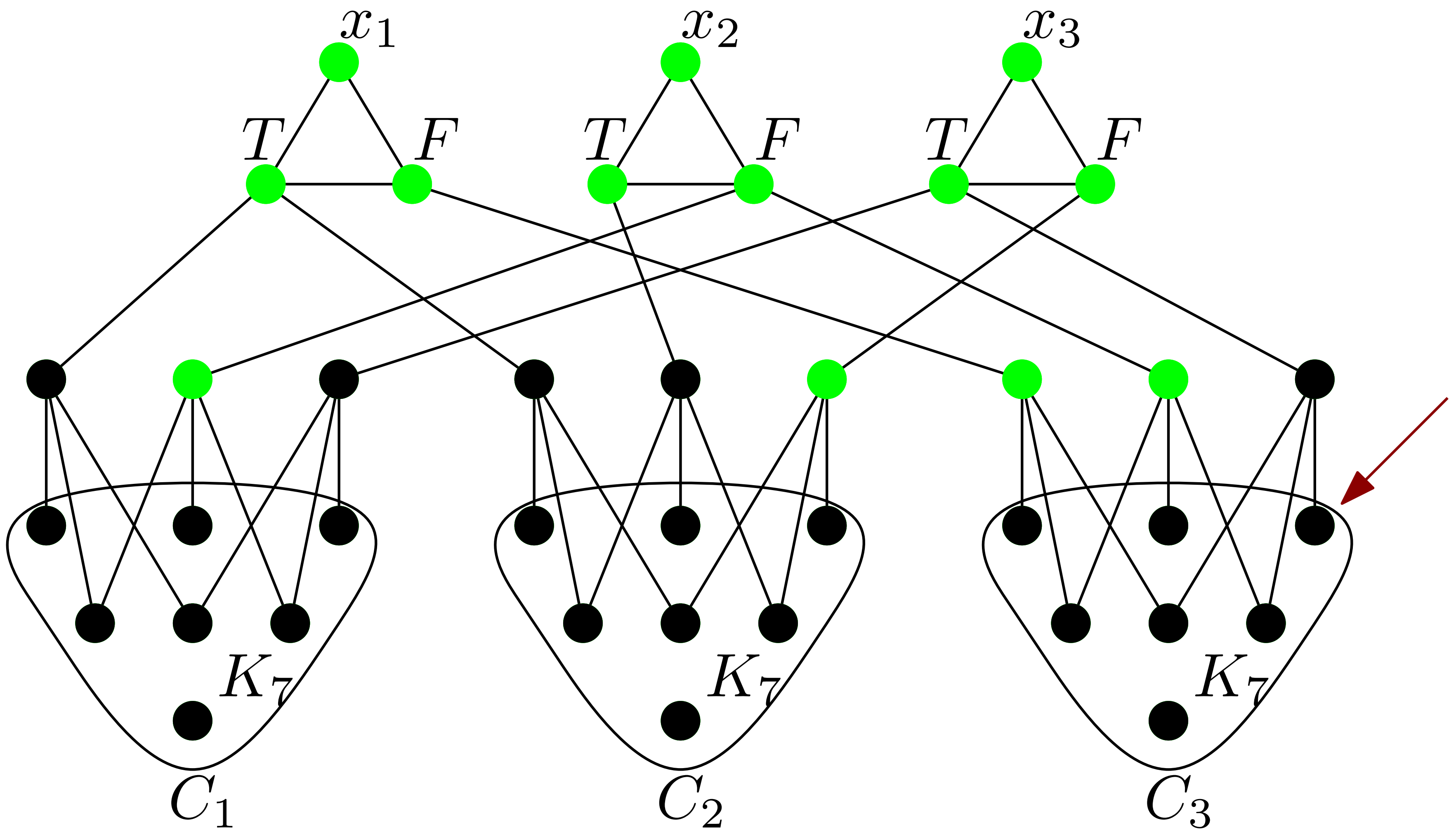

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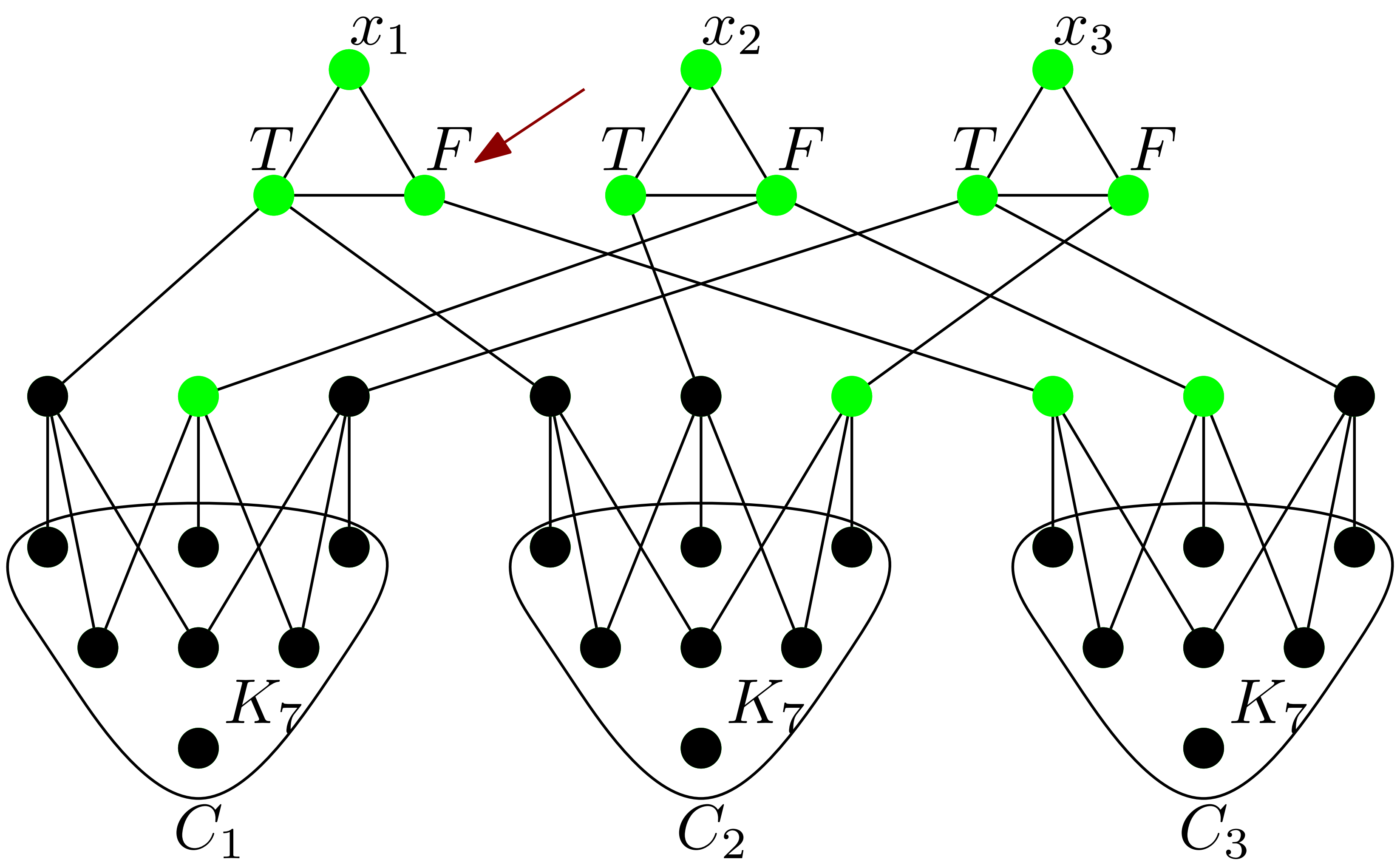

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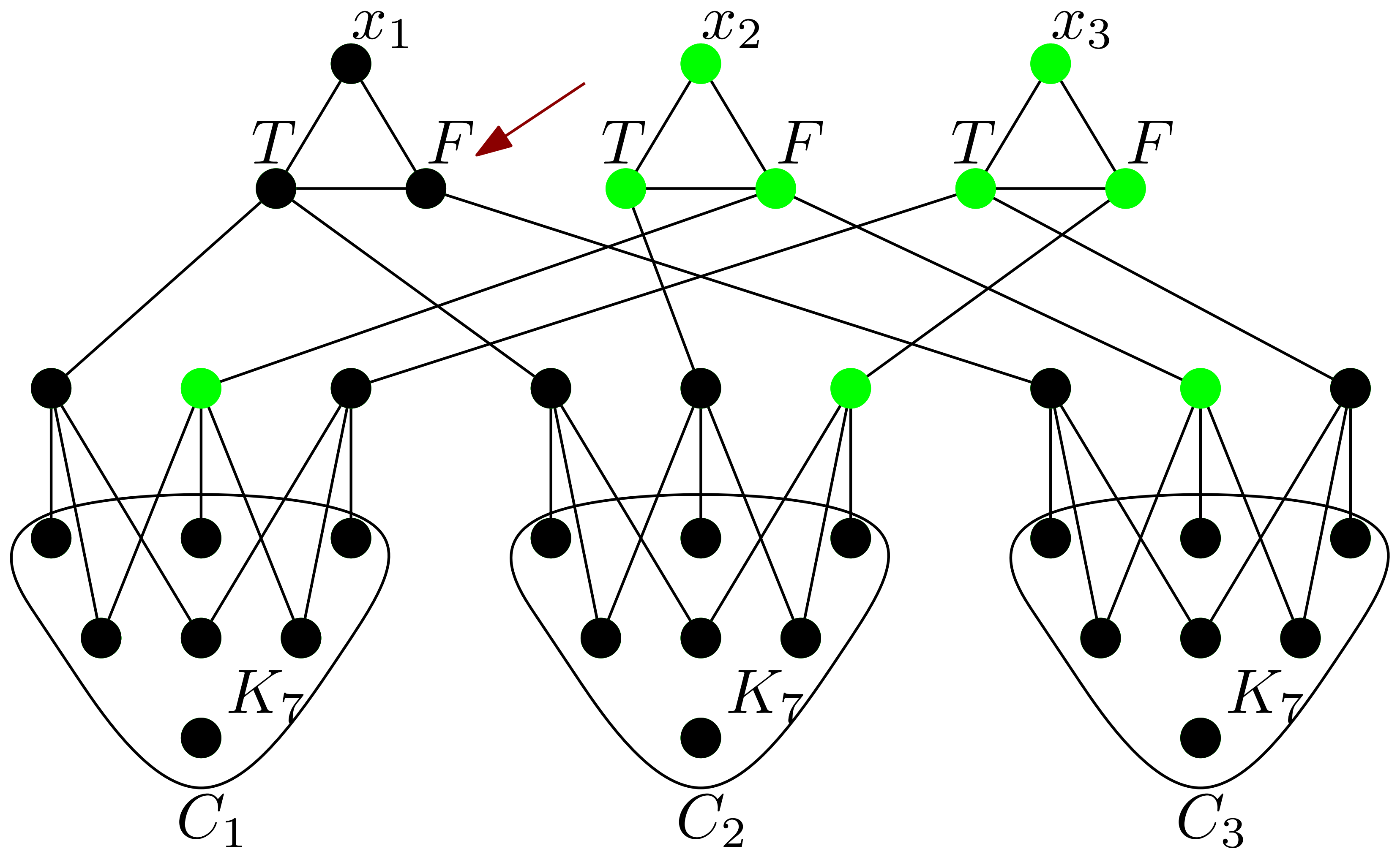

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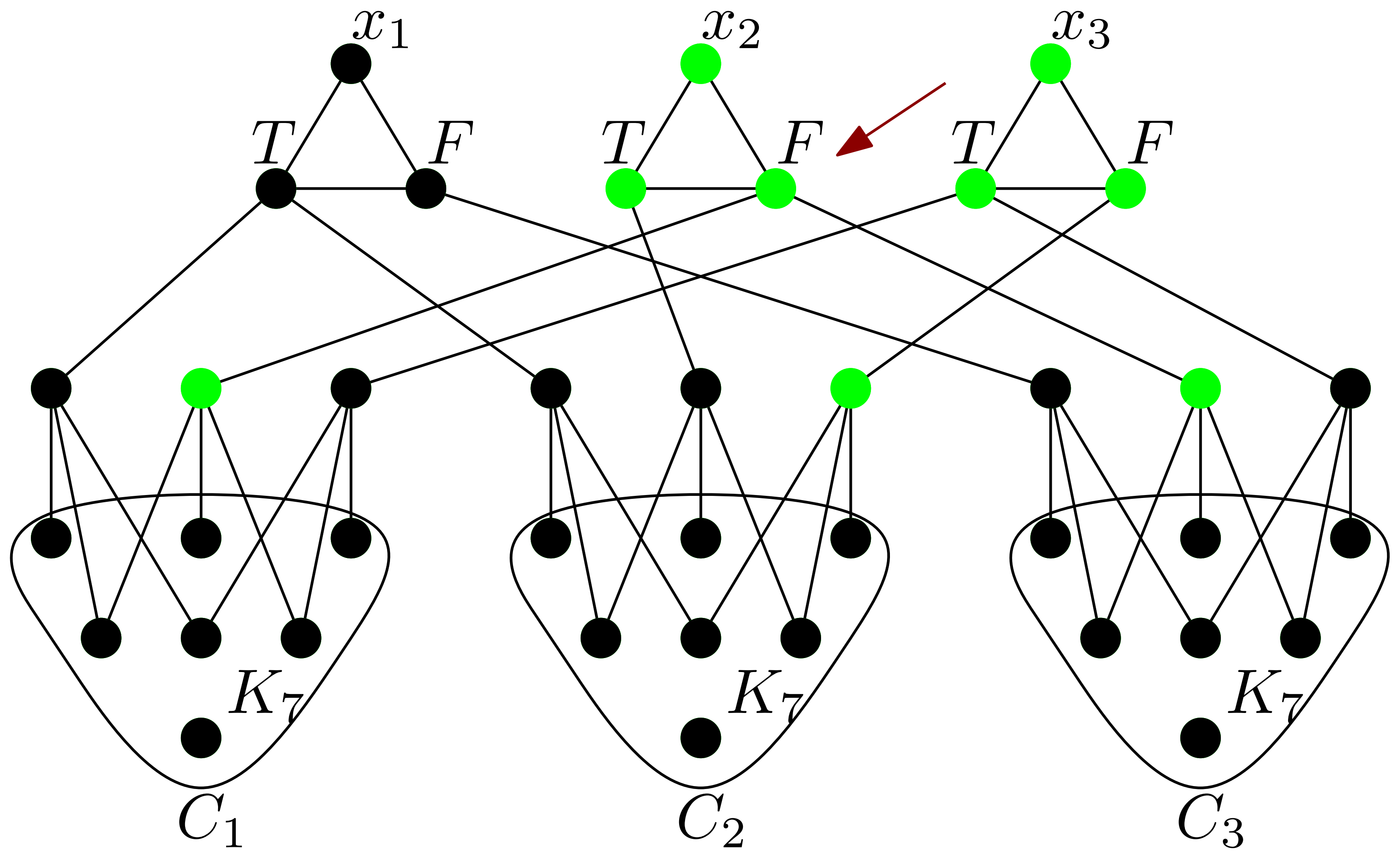

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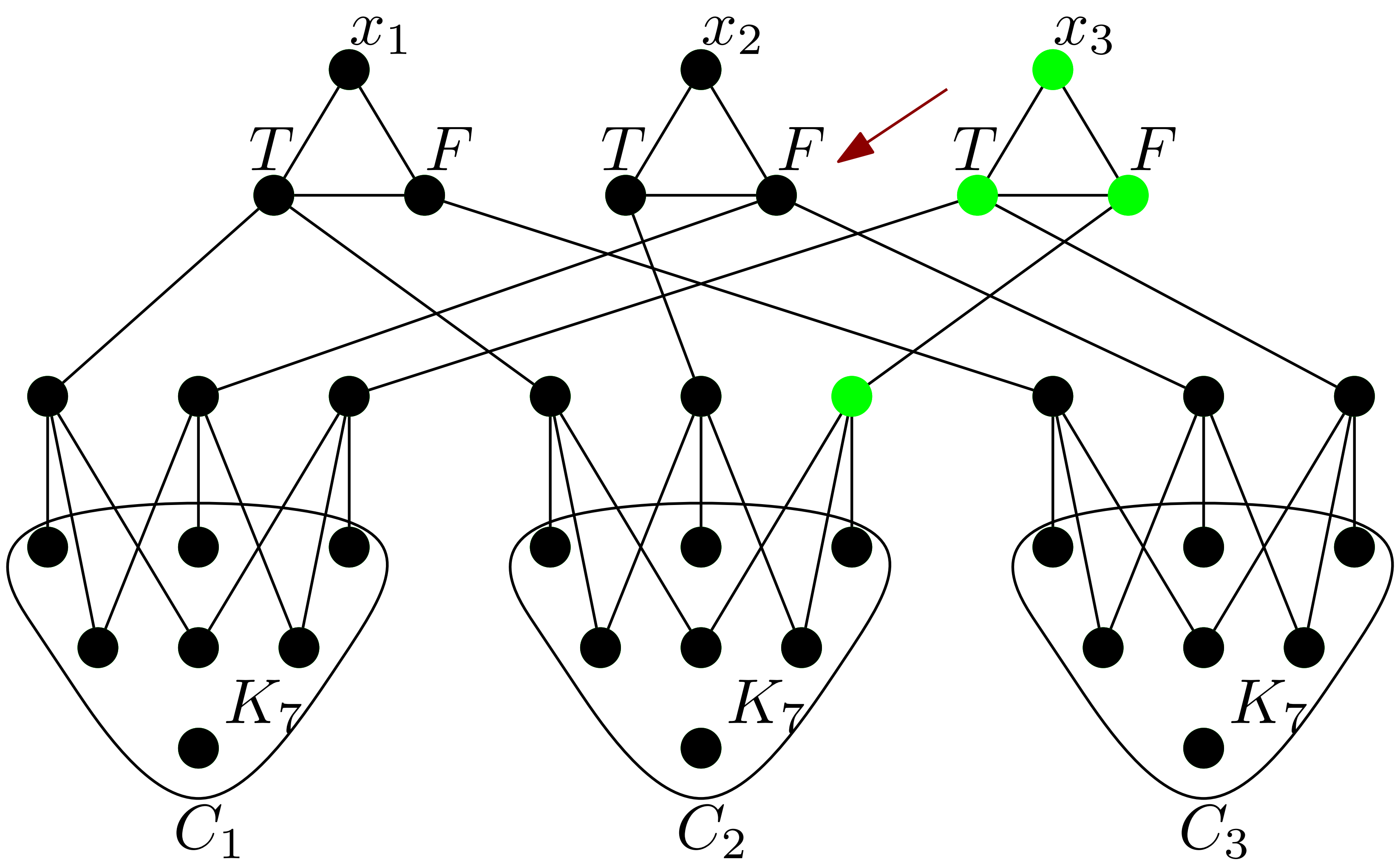

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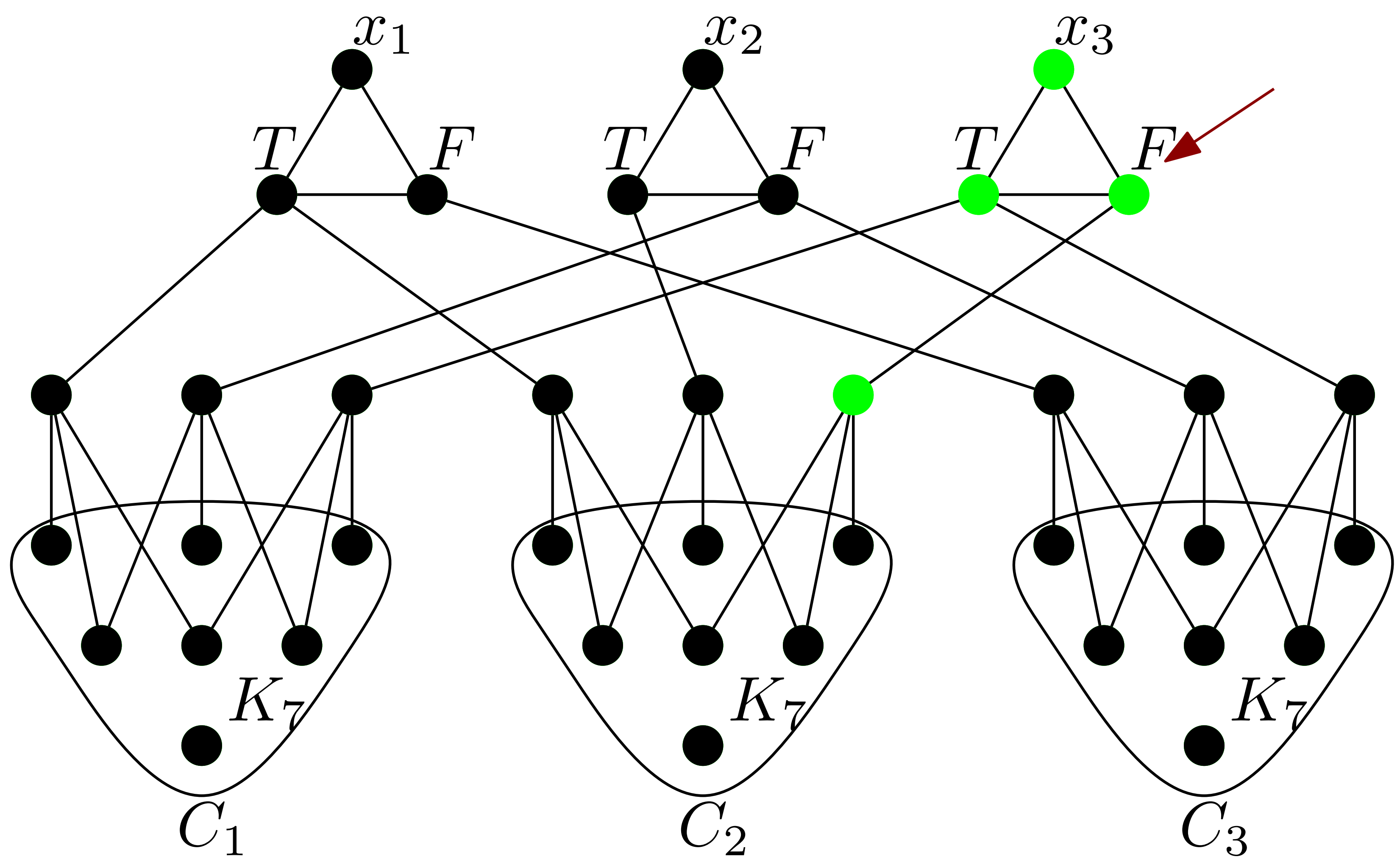

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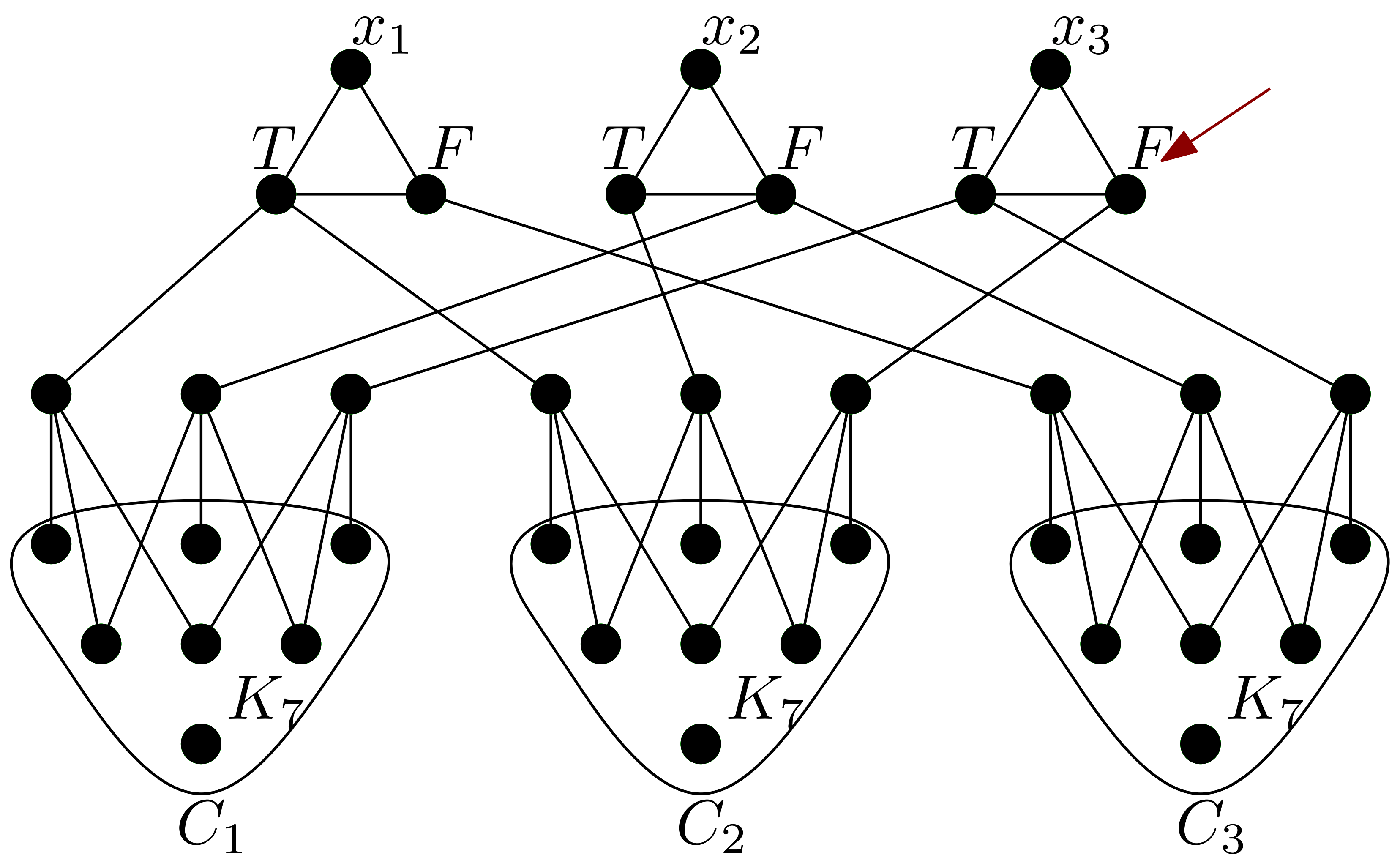

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Optimization Problems

Definition MOS (Maximizing Off Switches) - find a set of nodes that, when activated, would lead to the least possible amount of lights being still on.

Theorem MOS is $NP - complete$. Further, there exists a constant $\epsilon > 0$ such that no polynomial time algorithm for MOS can achieve an approximation ratio better than $1 + \epsilon$, unless $P = NP$.

Proof Reduction from a variant of $MAX - 3SAT$ where each variable appears in exactly five clauses.