Token sliding on graphs of girth five

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Notation

- $[n] = \{1, \ldots, n\}$
- All graphs are finite, simple and undirected
- *Open neighborhood* $N_G(v) = \{u \mid uv \in E(G)\}$
- *Closed neighborhood* $N_G[v] = N_G(v) \cup v$
- For $W \subseteq V(G)$, let $N_G(Q) = \bigcup_Q N_G(v) - Q$ and $N_G[Q] = N_G(Q) \cup Q$
- *Diameter* of G is $\mathrm{diam}(G) = \max_{v,u} \mathrm{dist}_G(v, u)$
- *Girth* of G is the length of the shortest cycle in G

Token sliding problem

Input: graph G and 2 independent k-sets $I_s, I_t \subseteq G$.
Question: whether there is a sequence of independent k-sets $(I_0, \ldots, I_l)$ such that

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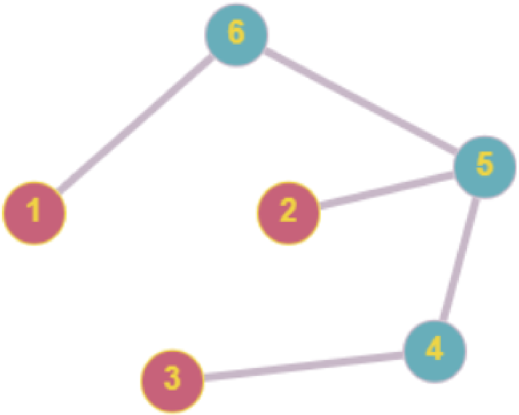
If we call vertices of I_i *tokens*, then every *move* from I_i to I_{i+1} is "sliding" one token along the edge maintaining the independence.

Token sliding problem



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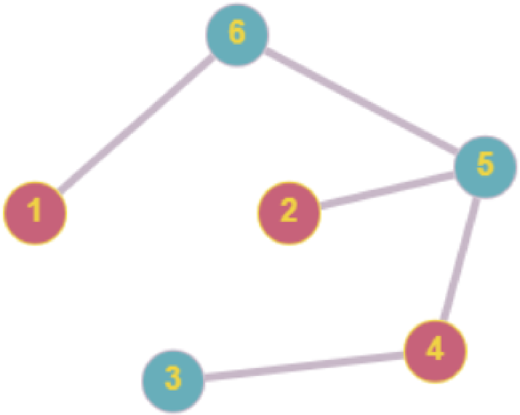

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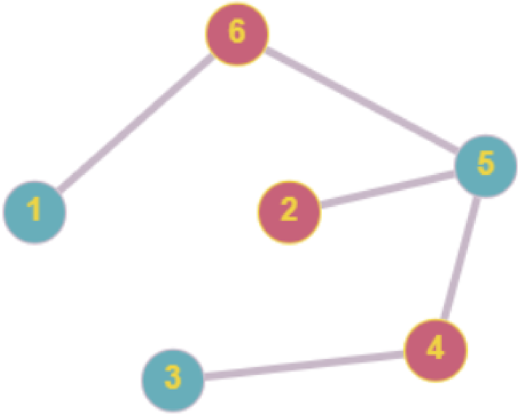

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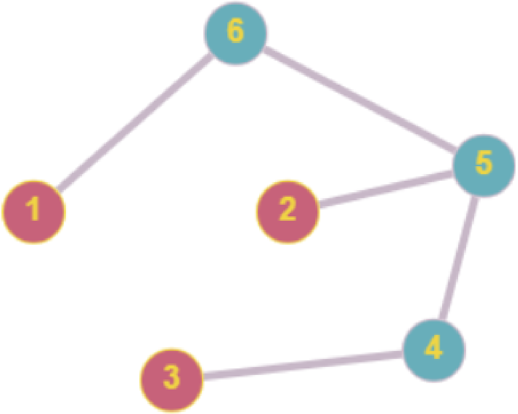

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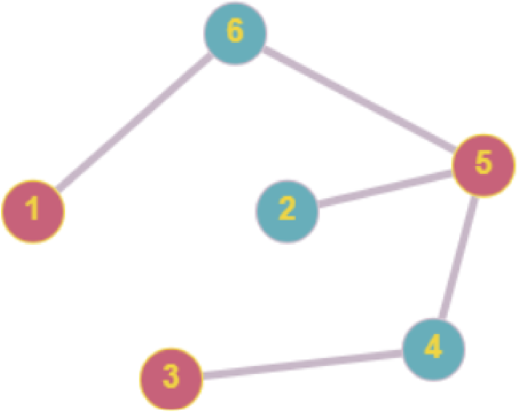

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TOKEN SLIDING can be solved naively by constructing a *reconfiguration graph* $\mathcal{R}(G, k)$, where vertices are independent k-sets of G, and edges correspond to *moves*. Then it's enough to verify if I_t is reachable from I_s - $O(n^k)$.

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Fixed-parameter tractable - $O(f(k) \cdot n^{O(1)})$.

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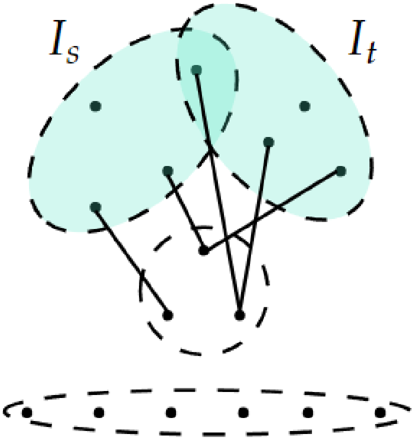
Goal: bound the size of G by $f(k)$, and apply the naive algorithm.

Partition G

- Let $L_1 = I_s \cup I_t$,



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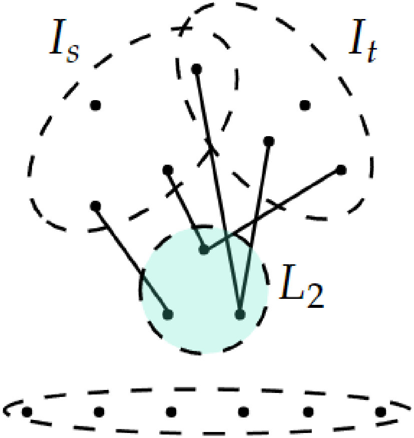

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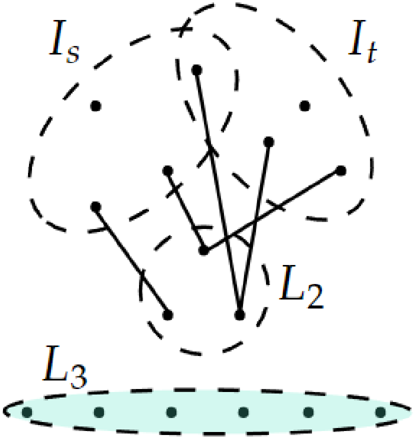
- $L_3 = V \setminus (L_1 \cup L_2)$



Partition G

Lemma

If $u \in L_2 \cup L_3$, then $|N_{L_1 \cup L_2}(u)| \leq 2k$.

4 component types

Let C be max connected component in $G[L_3]$.

Definition

C is diameter-safe if $diam(G[C]) > k^3$

Definition

C is degree-safe if $\exists u \in C.\ N_{G[C]}(u) > k^2$ and $|\{v \in N_{G[C]}(u) \mid \deg_{G[C]}(v) = 2\}| \geq k^2$

Definition

C is bounded if $diam(G[C]) \leq k^3$ and $\forall u \in C.\ \deg_{G[C]}(u) \leq k^2$

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C is bad otherwise

Bounded components

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Lemma

If C is a bounded component in $G[L_3]$, then $|V(C)| \leq k^{2k^3}$.

Safe components - informally

We will be trying to show that for a safe component C

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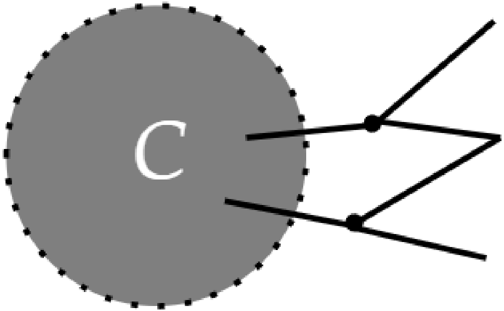

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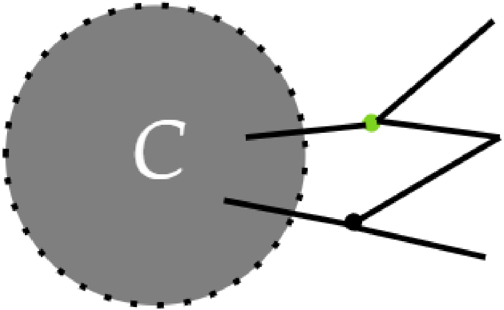

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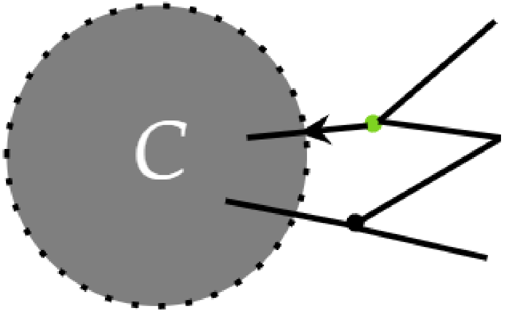

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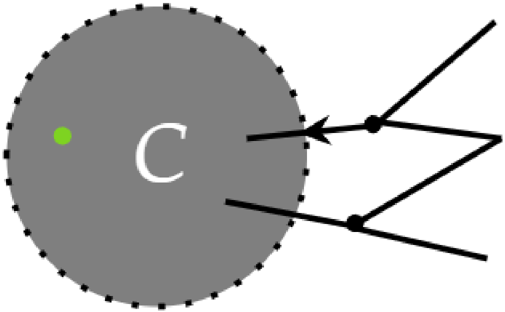

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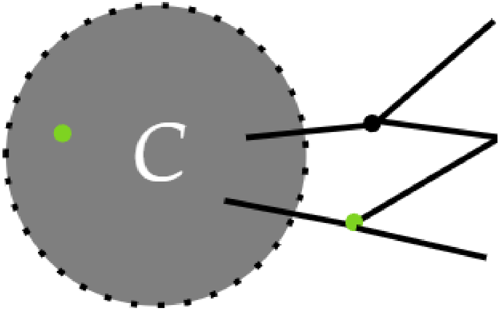

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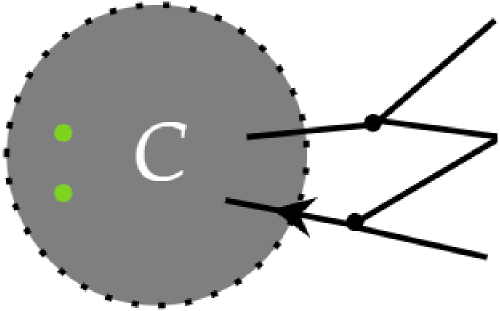

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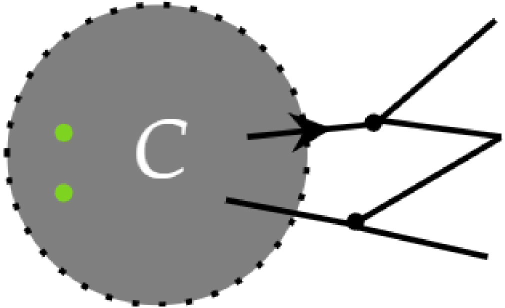

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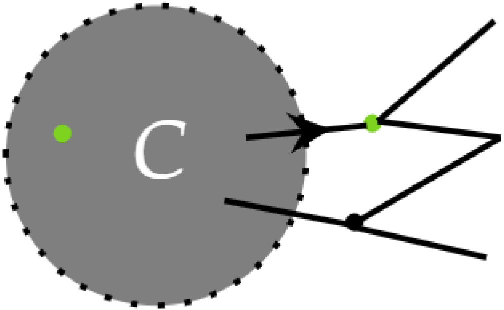

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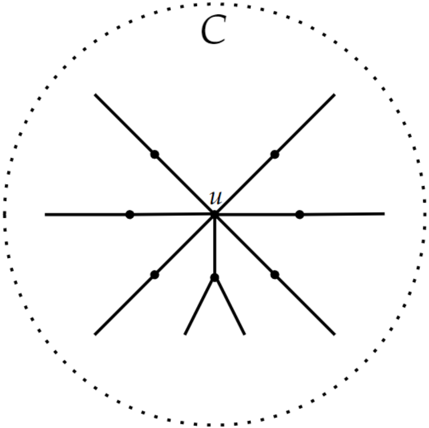

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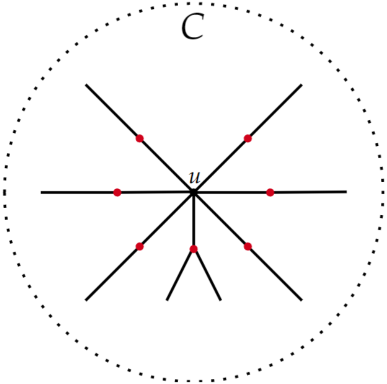

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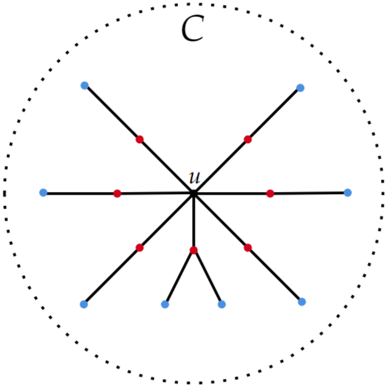

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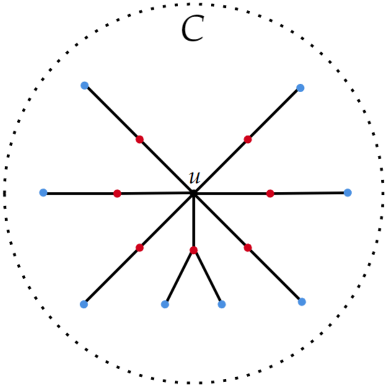

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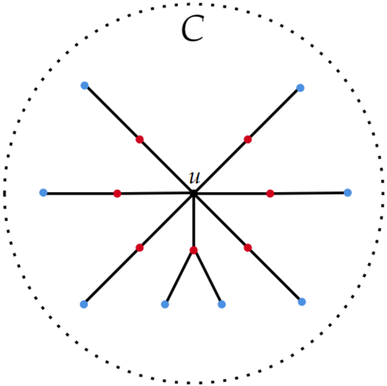

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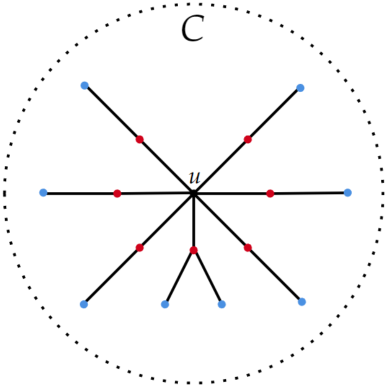

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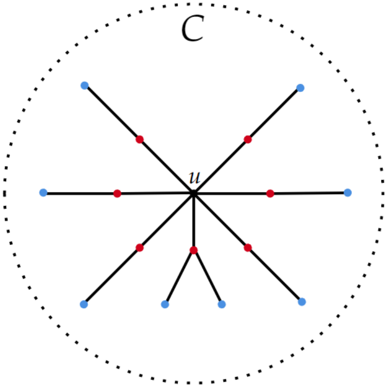

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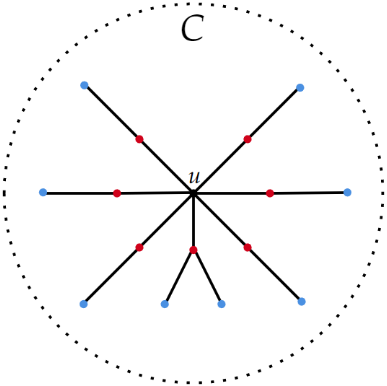

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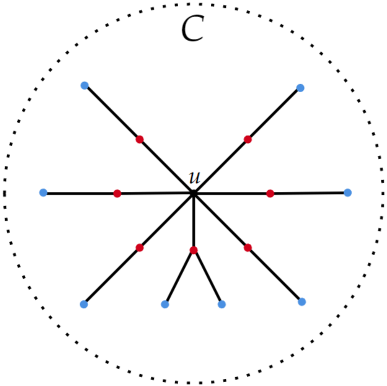

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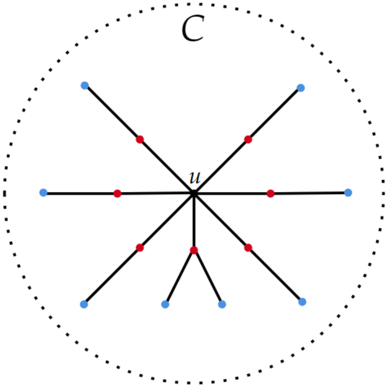
- This claim gives us the desired subdivided k-star



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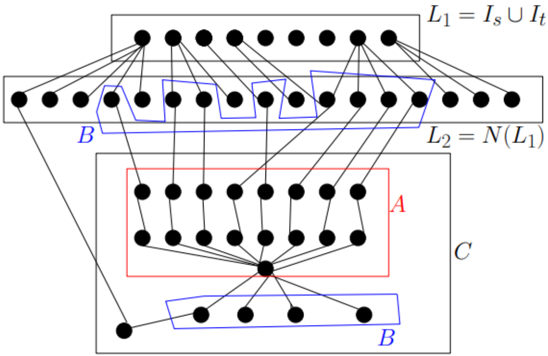

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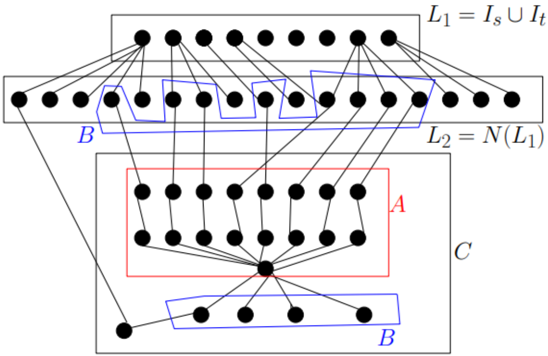

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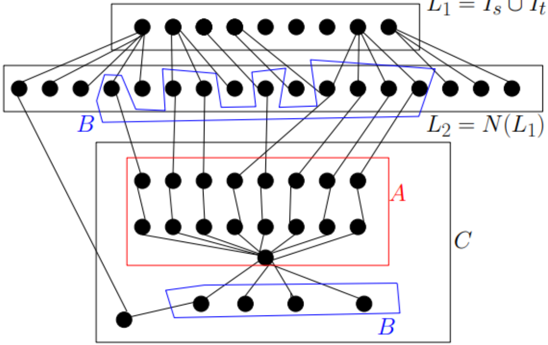

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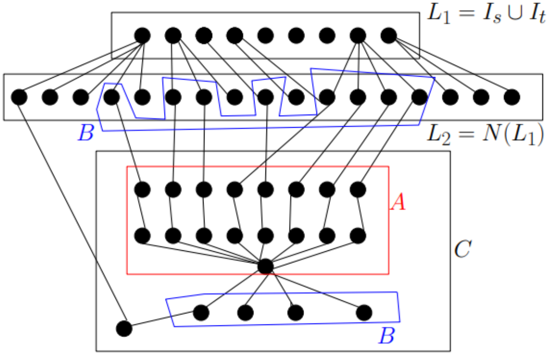

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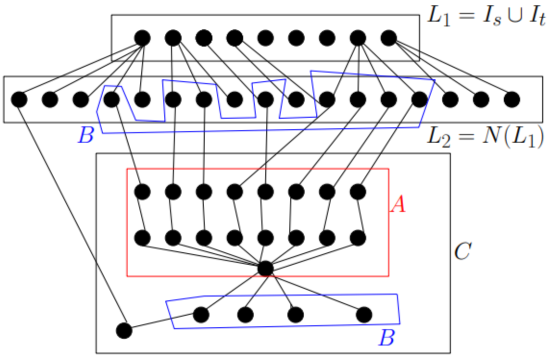

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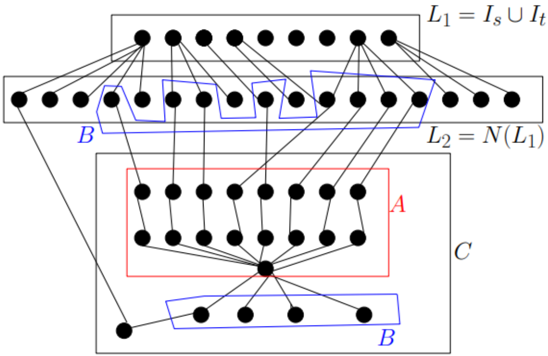

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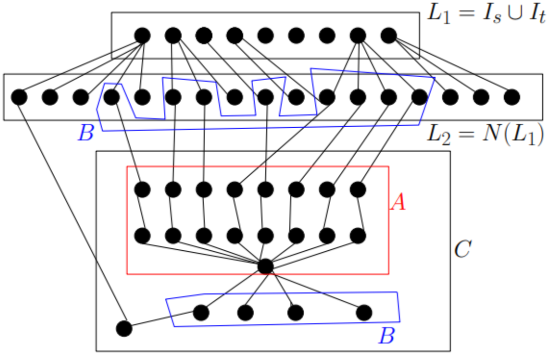

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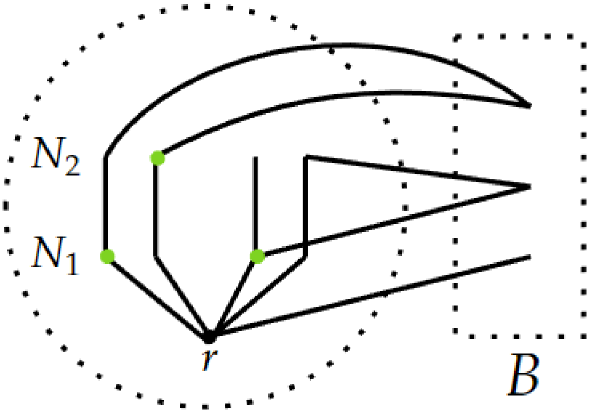
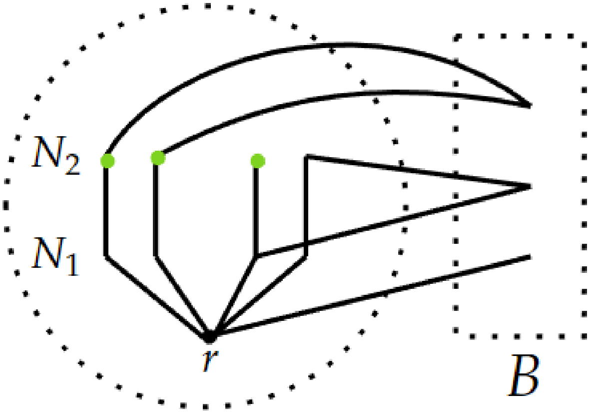


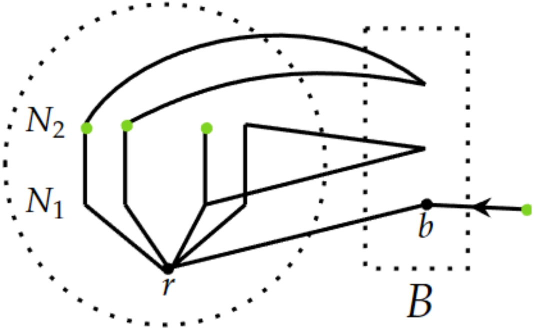
- Consider step I_{i-1}, right before some token enters vertex $b \in B$

Degree-safe components



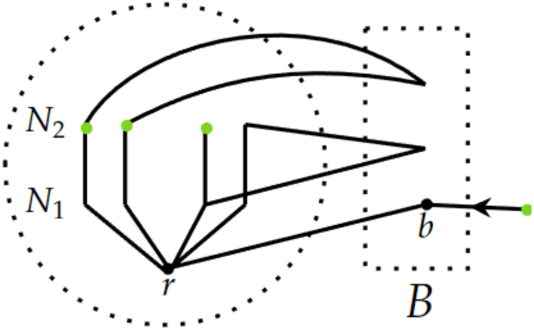
- Consider step I_{i-1}, right before some token enters vertex $b \in B$
- First, we move all tokens in A to N_2 - now every token has its own branch

Degree-safe components



- **Case** $b \in N(r)$:

Degree-safe components



- **Case** $b \in N(r)$:
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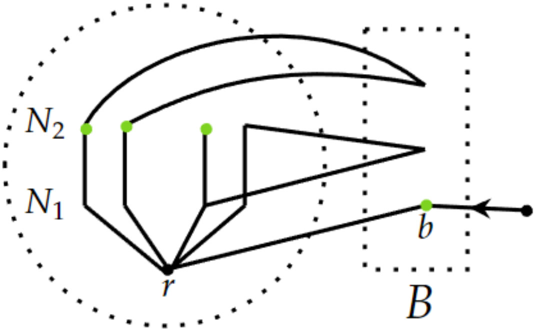
Degree-safe components



- **Case** $b \in N(r)$:
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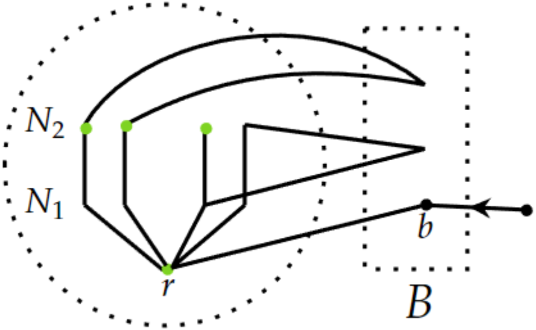
Degree-safe components



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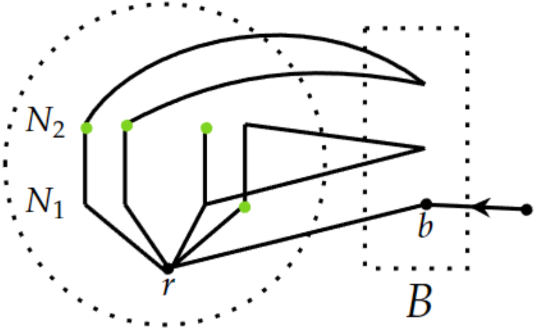

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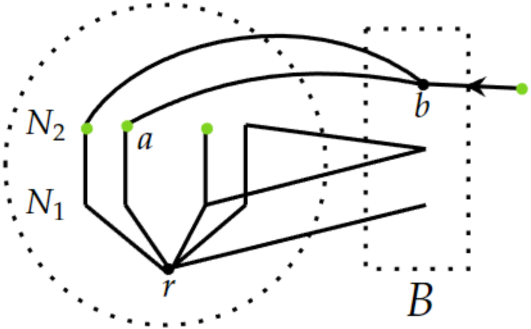

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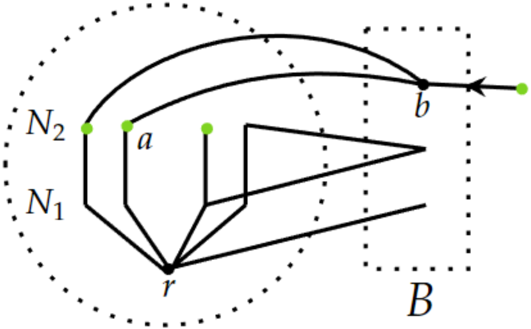

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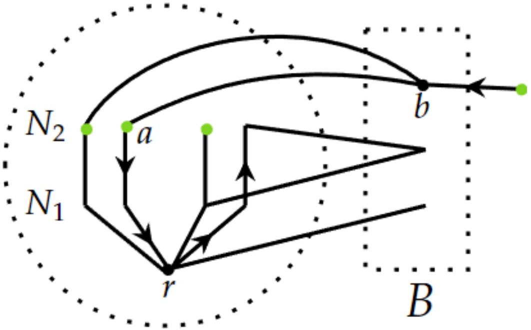

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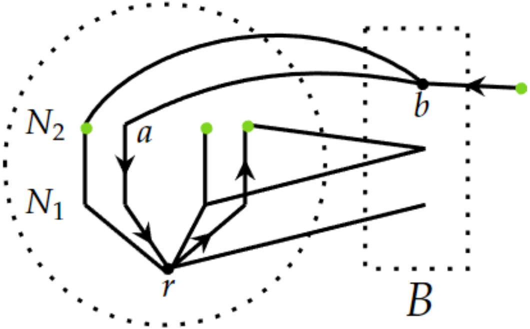

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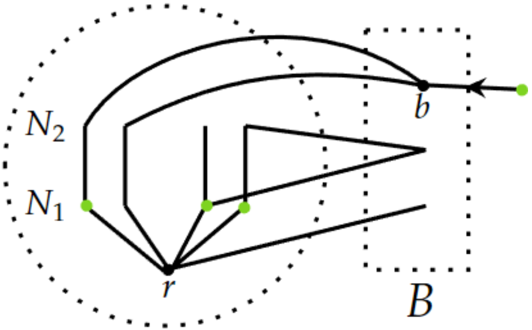

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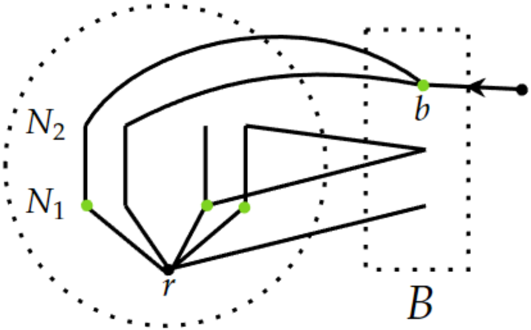

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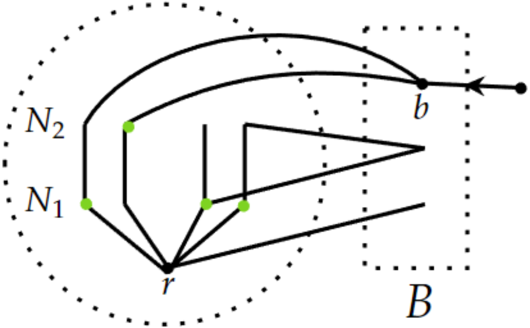

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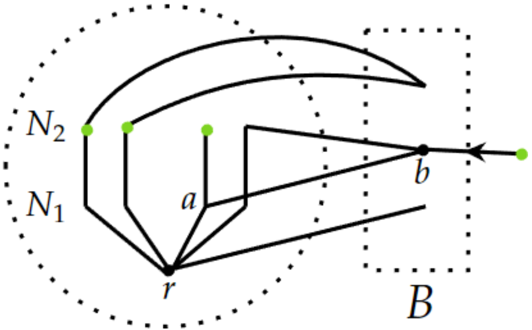

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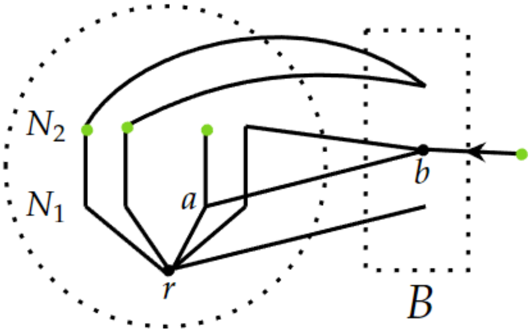

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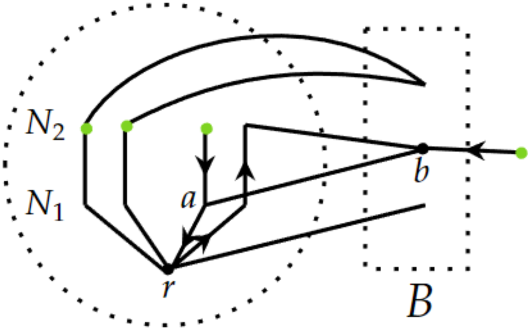

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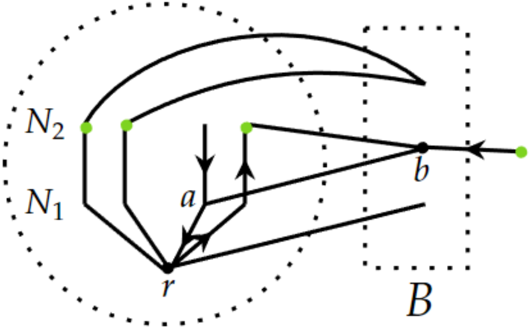

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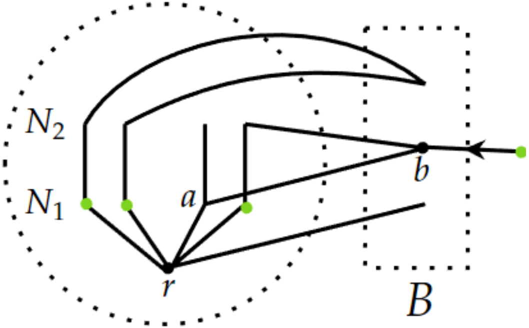

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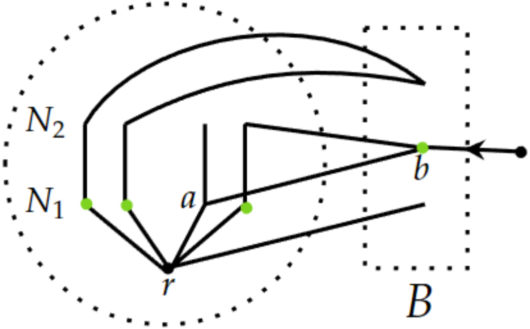

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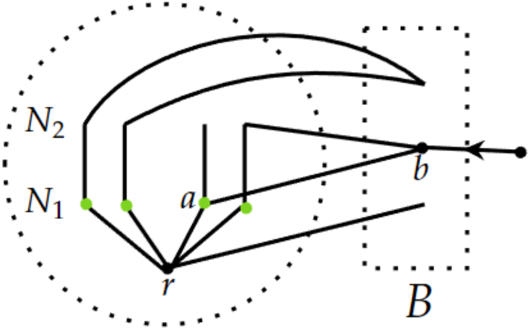

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- The step right before a token exits B - repeat the above procedure in reverse

Corollary

Let C be a degree-safe component. If a sequence from I_s to I_t exists, then also a sequence such that $N(C)$ never has > 1 tokens exists.

Proof.

Follow the path P from $c \in N(C)$ to r and apply the previous lemma when c enters B. We can always find such a path P that $N[P]$ contains no tokens, because all of them have been absorbed by C. □

Degree-safe components

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Diameter-safe components

Definition

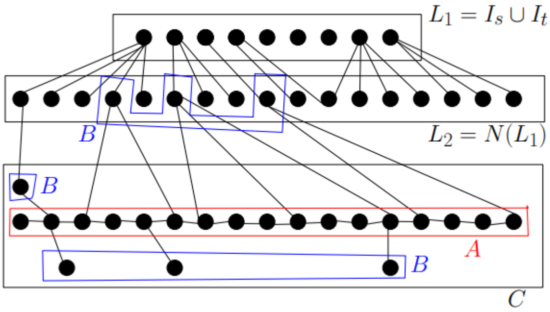
C is diameter-safe if $diam(G[C]) > k^3$

Diameter path A of a diameter-safe component C is the longest shortest path $u \to v$ in C.

Diameter-safe components

Lemma

Let A be a diameter path of C, and $B = N_G(A)$. If a sequence from I_s to I_t exists, then also a sequence such that B never has > 1 tokens exists.



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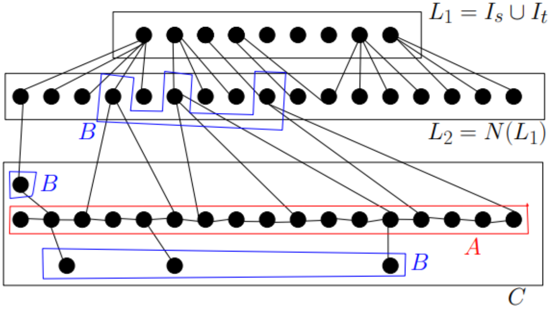

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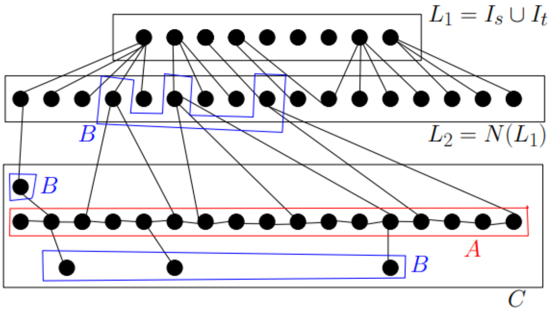
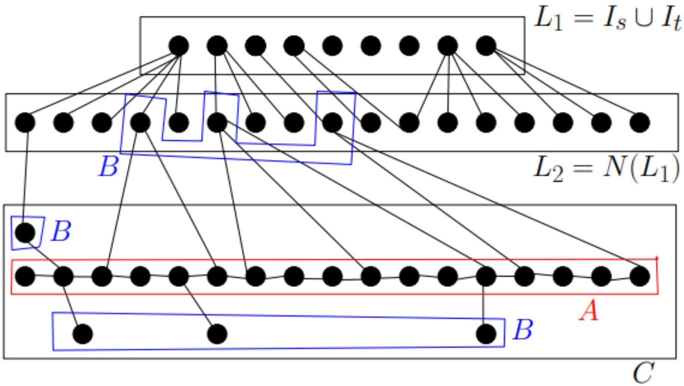
- Note that $I_s \cap B = I_t \cap B = \emptyset$
- Like before, we will convert a sequence $\hat{I}$ from I_s to I_t into a sequence $\hat{I}'$ such that
  1. B never has > 1 tokens in $\hat{I}'_i$,
  2. at any step #tokens in $A \cup B$ is same and
  3. positions of tokens in $\overline{A \cup B}$ are same

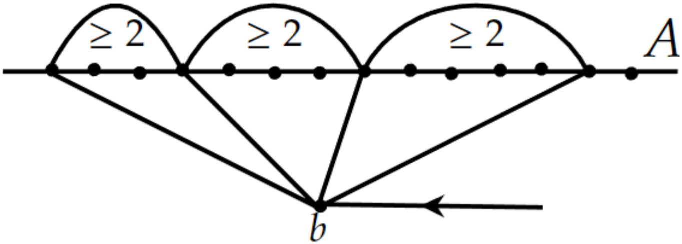


Diameter-safe components



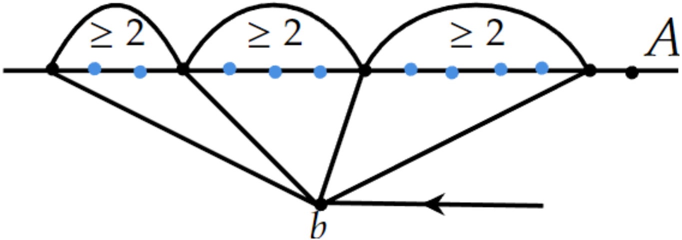
- No 2 non-consecutive vertices in A are adjacent, as A is a shortest path
- Consider step I_{i-1}, right before some token enters vertex $b \in B$

Diameter-safe components



- $\forall x, y \in N_A(b)\ d_A(x, y) \geq 3$, as G has no 4-cycles

Diameter-safe components



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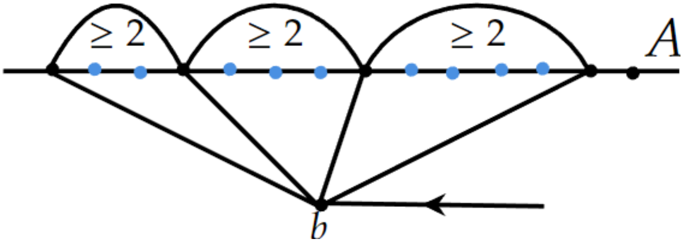

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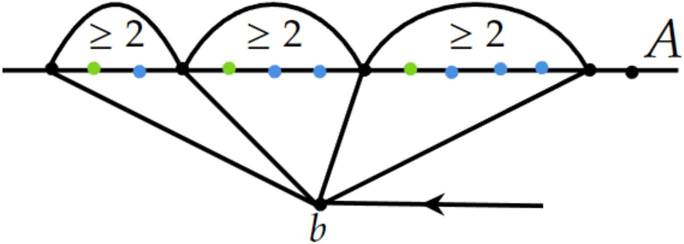

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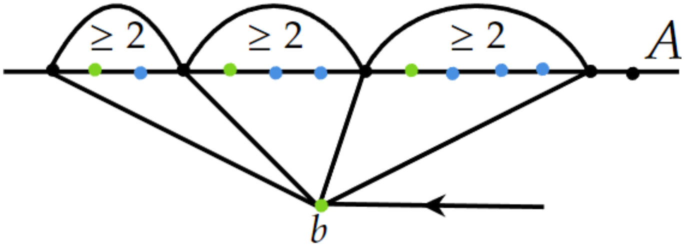
Diameter-safe components



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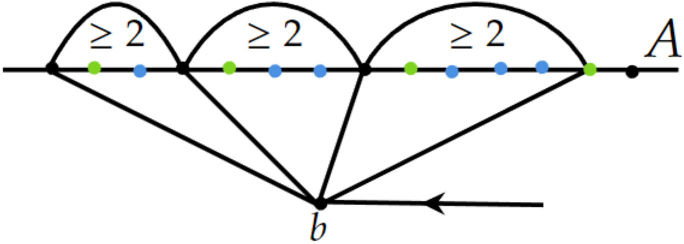

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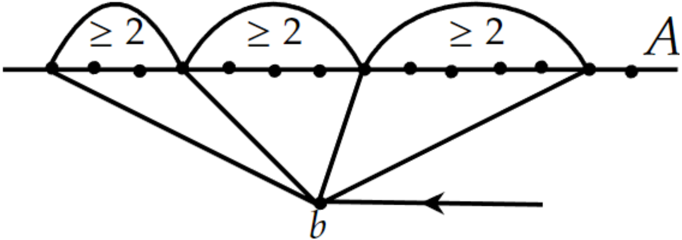

- $\forall x, y \in N_A(b)$ $d_A(x, y) \geq 3$, as G has no 4-cycles
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- **Case** $|N_A(b)| \leq k$: then there are $\leq k-1$ gap intervals $+$ maybe 2 intervals at both ends of A. Hence there exists an interval i of size at least

$$\alpha = \frac{\mathrm{diam}(C) - |N_A(b)|}{|N_A(b)| + 1} \geq \frac{k^3 - k}{k+1} \geq 2k$$

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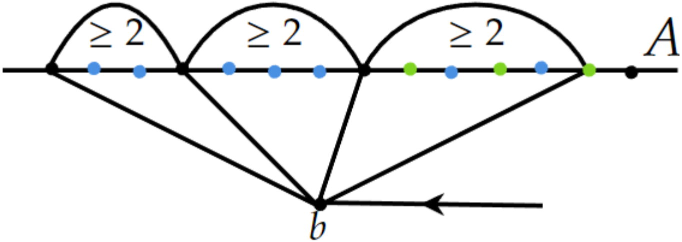

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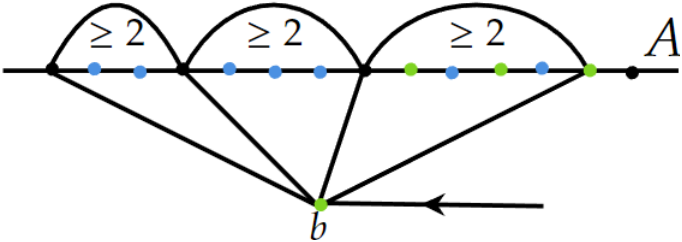

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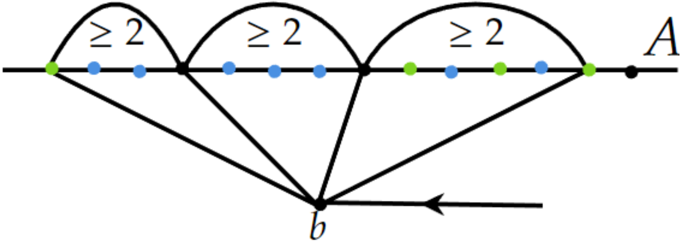

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Corollary

Let C be a diameter-safe component. If a sequence from I_s to I_t exists, then also a sequence such that $N(C)$ never has > 1 tokens exists.

Proof.

Follow the path P from $c \in N(C)$ to the closest vertex in diameter-path A and apply the previous lemma when c enters B. □

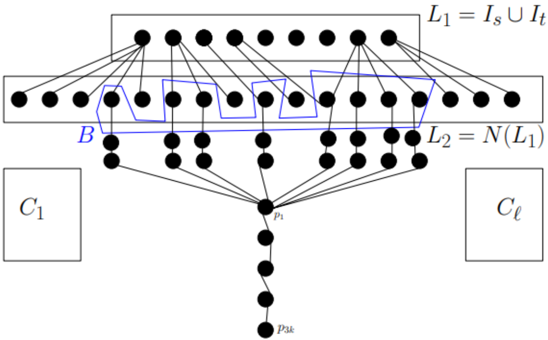
Safe components: replacement gadget

Lemma

Let C be a safe component in $G[L_3]$ and G' be the graph obtained from G as follows:

- *delete C*
- *$\forall v \in N(C)$ add new vertices $v \to v' \to v''$*
- *add a path $p_1, \ldots p_{3k}$*
- *add edges $v'' \to p_1$*

Then $(G, I_s, I_t) \equiv (G', I_s, I_t)$.



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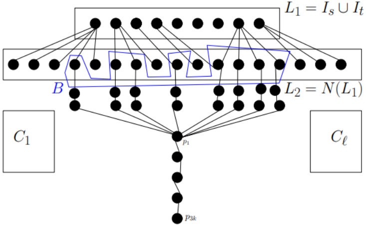
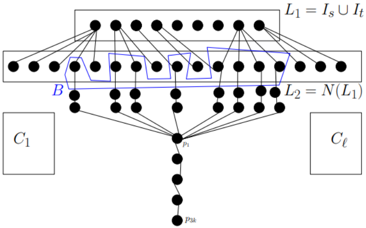


Proof.

- From previous corollaries, there exist a sequence $\hat{I}$ from I_s to I_t with at most 1 token in $N(C)$ at all times

Safe components: replacement gadget



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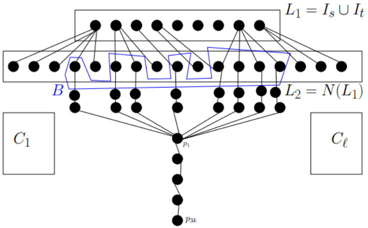

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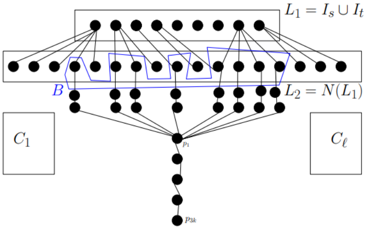

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- Note that size of replacement component is $3k + 2|N(C)|$, and $N(C) \subseteq L_2$

Bounding the size of G

Lemma

Assume $u \in L_1$ *with* $\deg(u) > 2k^2$ *(WLOG* $u \in I_s$*). Then there exists* I'_s *such that* $I_s \Delta I'_s = \{u, u'\} \in E(G)$ *and* $\deg(u') \leq 2k^2$*.*

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Corollary

For (G, I_s, I_t), there exists equivalent (G', I_s, I_t) such that $|L_2| \leq 4k^3$, and each safe component is replaced in G' by replacement component of size at most $3k + 8k^3 = O(k^3)$

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From now on, we refer to both bounded and bad components as bounded components.

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TOKEN SLIDING is fixed-parameter tractable when parameterized by k on graphs of girth ≥ 5.

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- whether the problem remains tractable if we exclude odd cycles

References

Valentin Bartier, Nicolas Bousquet, Jihad Hanna, Amer E. Mouawad and Sebastian Siebertz (2022)
Token sliding on graphs of girth five
arXiv

The End