A note on large induced subgraphs with prescribed residues in bipartite graphs

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March 2023

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Introduction

Some definitions

- Standard notation regarding undirected graphs: $V(G)$ - set of vertices, $E(G)$ - set of edges.

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- Order of G is its vertex set size: $|V(G)|$
- We denote by $\deg_G(v)$ the degree of a vertex v in G and by $N(v)$ the set of neighbors of v in G.
- H is an induced subgraph of G if $V(H) \subseteq V(G)$ and $\{u, v\} \in E(H) \iff u \in V(H) \wedge v \in V(H)$. We denote $G[X]$ as the induced subgraph on $X \subseteq V(G)$.

Induced subgraphs with prescribed residues

For a given graph G and integers $q > r \geqslant 0$ We define the function $f(G, r, q)$ as the maximum order of any induced subgraph H of G such that $\deg(v)_H \equiv r \mod q$, or 0 if no such H exists.

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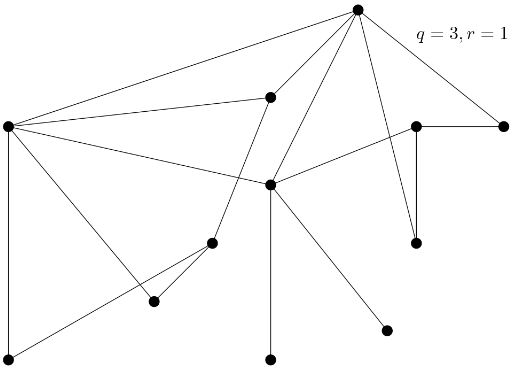

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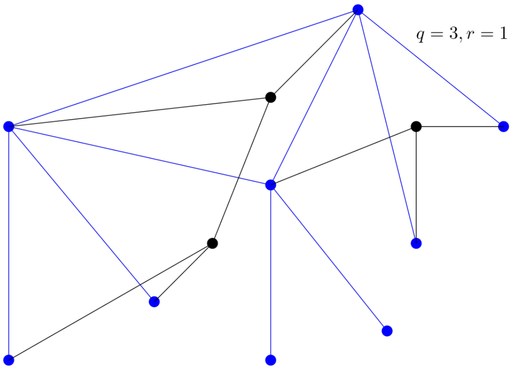

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Theorem: Gallai

For any graph $G = (V, E)$ there exists a partition $V = X \sqcup Y$ such that $G[X]$ and $G[Y]$ have all degrees even.

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Definition: $c(k)$

We define $c(k) = \inf_G \dfrac{f(G,1,k)}{|V(G)|}$ where the infimum ranges over all bipartite graphs G without isolated vertices.

Bounds

Theorem: Scott (2001)

$$\frac{1}{2^k + k + 1} \leq c(k) \leq \frac{1}{k}$$

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Consider the graph $K_{k,k}$. Then $V(K_{k,k}) = 2k$ and $f(K_{k,k}, 1, k) = 2$, so $c(k) \leq \frac{1}{k}$.

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In his paper, Scott mentions that the bound $c(k) \geqslant \frac{1}{2^k + k + 1}$ can be improved to $c(k) = \Omega\left(\frac{1}{k^2 \log(k)}\right)$.

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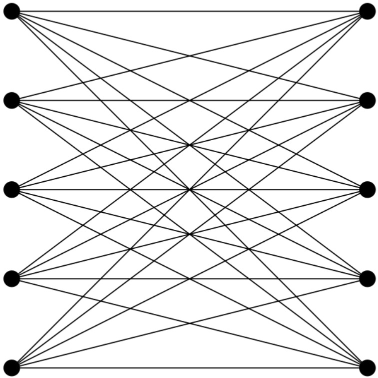
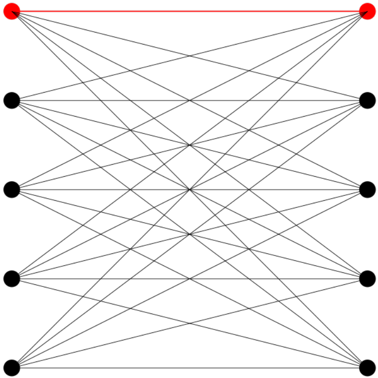
Lower bound

In his paper, Scott mentions that the bound $c(k) \geqslant \dfrac{1}{2^k + k + 1}$ can be improved to $c(k) = \Omega\left(\dfrac{1}{k^2 \log(k)}\right)$. **Main Theorem** of this article: $c(k) = \Omega\left(\frac{1}{k}\right)$, which is sharp up to a constant.

$K_{k,k}$





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Main Theorem: $c(k) = \Omega(1/k)$

Proof - Idea

The idea behind the proof of the Main Theorem is a slight modification of the proof of $c(k) = \Omega\left(\frac{1}{k^2 \log(k)}\right)$.

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Technical Lemma

Let X_i be independent and identically distributed (i.i.d) random variables such that $\mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = 1/2$. If $n \geqslant k^3$, then

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i \equiv 1 \mod k\right) \geqslant (1 - o_k(1))/k$$

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Lemma intuition

If we flip a coin $n \gg k$ times, then we expect the number of heads mod k to have almost equal probability of being $\{0, 1, \dots, k-1\}$.

$c(k) = \Omega(1/k)$ - Proof sketch

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Minimal dominating set

We say that A is a minimal dominating set of B if A is a minimal with respect to inclusion set satisfying $N(v) \cap A \neq \emptyset$ for all $v \in B$.

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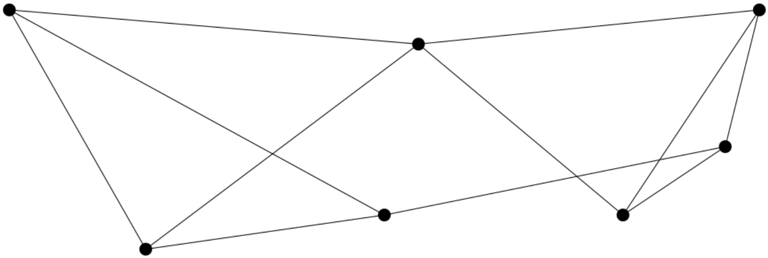

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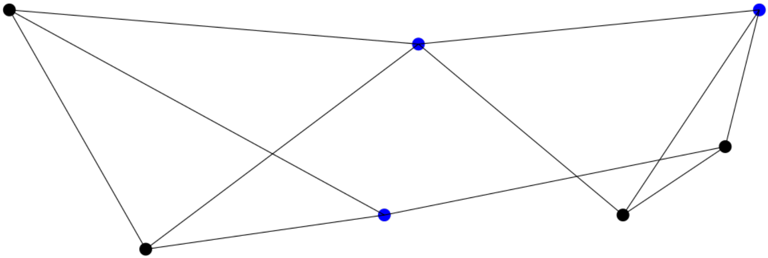

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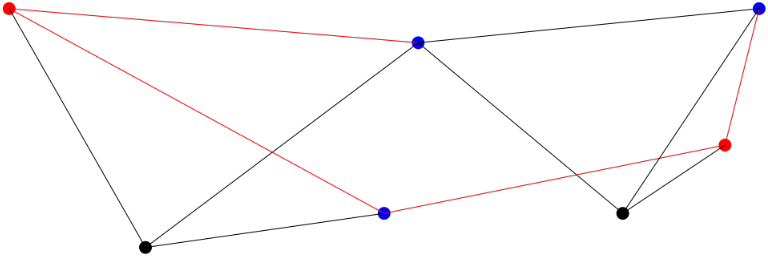

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Observation

If A is a minimal dominating set of B, then for every element $a \in A$ there exists an element $b_a \in B$ such that $N(b_a) \cap A = \{a\}$. We denote $B_A = \{b_a : a \in A\}$

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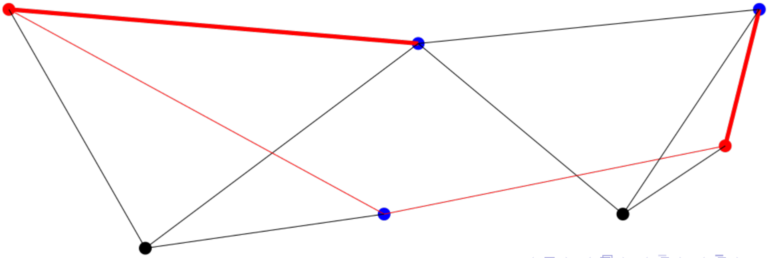

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Setting

We will be working with a bipartite graph $G = (X \sqcup Y, E)$ with $|X| \geqslant |Y|$ and $|X| + |Y| = n$. We assume that G has no isolated vertices ($\delta(G) \geqslant 1$).

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Construction

Let $W_1 \subseteq Y$ be a minimal dominating set of X and let $S_1 = X_{W_1}$. Then $G[W_1 \cup S_1]$ is a matching (every vertex has degree 1), so $f(G, 1, k) \geqslant 2|W_1|$.

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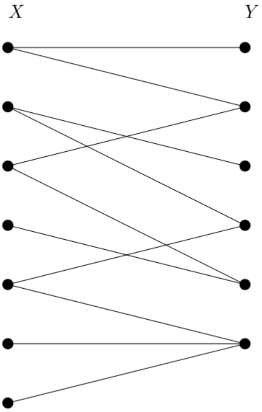

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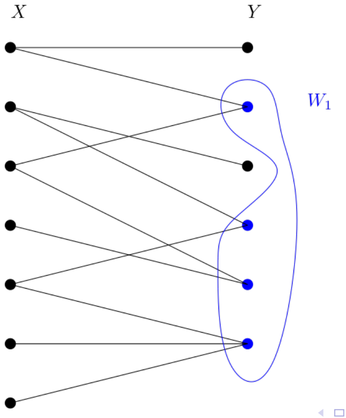

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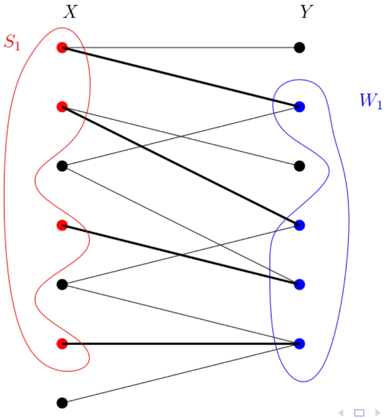

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Next steps

We now choose a global constant c_1. Graphs G for which $|W_1| \geqslant c_1|X|/k$ cannot be counterexamples to the Main Theorem. Assume then that $|W_1| < c_1|X|/k$.

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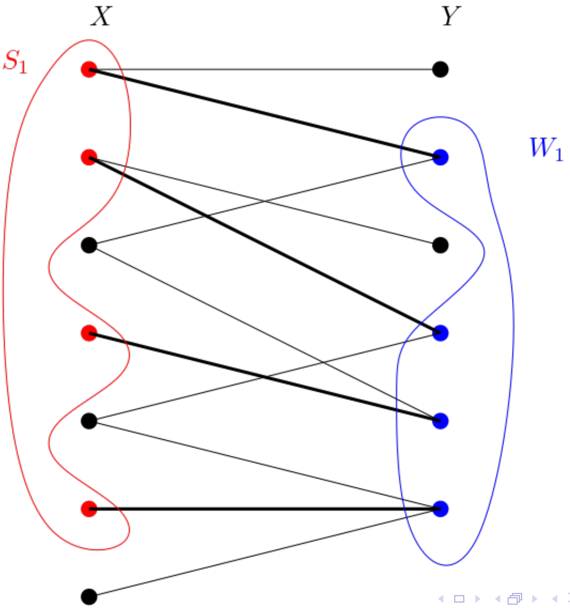

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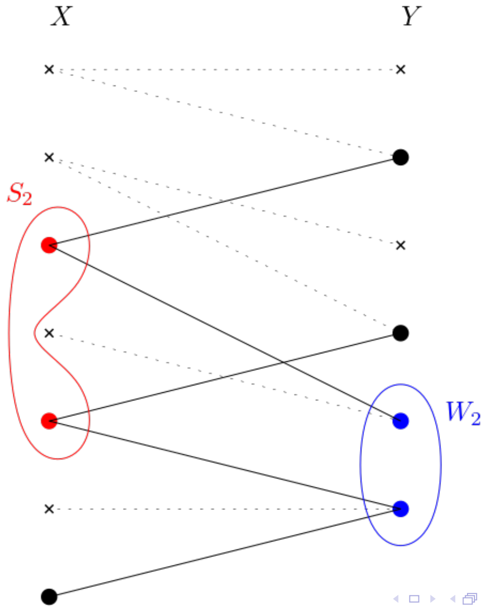

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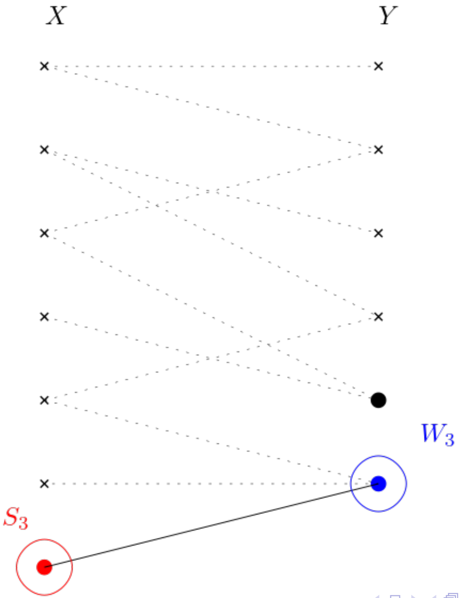

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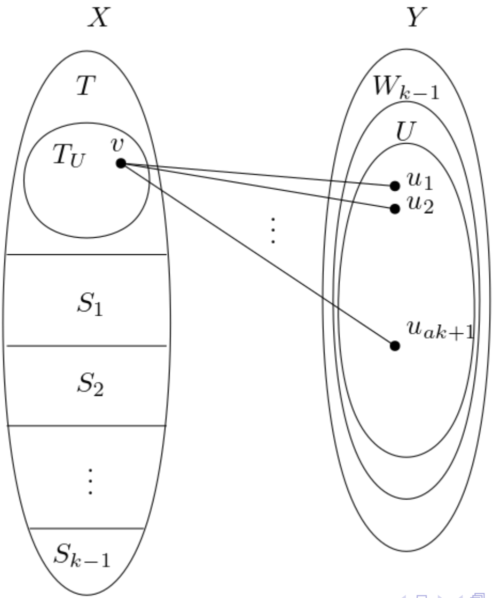
Observation

Let $T = X \setminus \bigcup_{i=1}^{k-1} S_i$. Then $|T| \geqslant (1-c_1)|X|$, because $|W_1| < c_1|X|/k$.

U and T_U

Let $U \subseteq W_{k-1}$ where each $u \in W_{k-1}$ is included in U with probability $1/2$.
Let $T_U = \{v \in T : |N(v) \cap U| \equiv 1 \pmod{k}\}$.

Visualization



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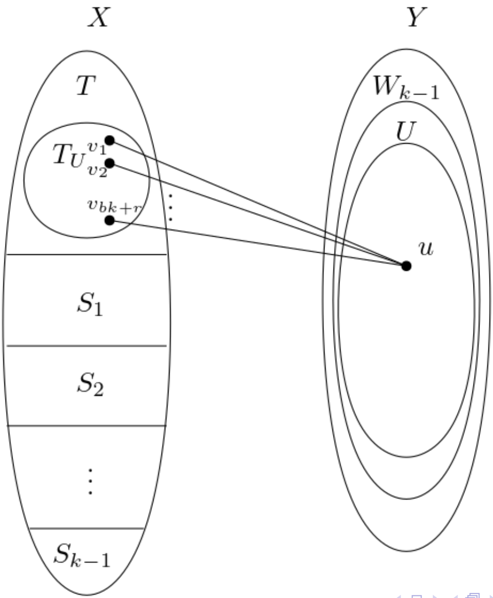

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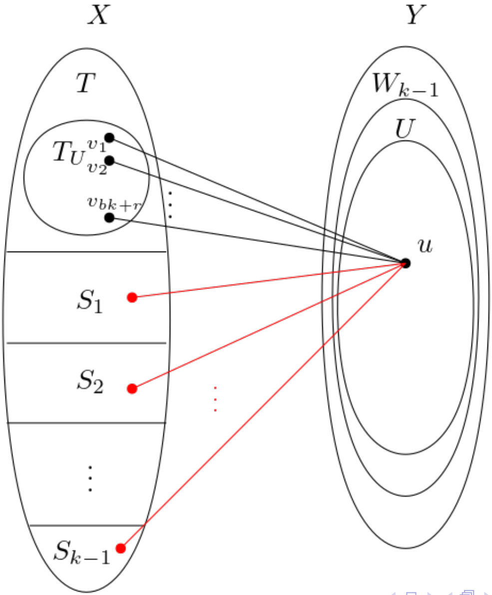

$c(k) = \Omega(1/k)$ - Proof sketch

Observation

We can choose $S \subseteq \bigcup_{i=1}^{k-1} S_i$ such that $|N(u) \cap (T_U \cup S)| \equiv q \pmod{k}$.
Therefore $f(G, 1, k) \geqslant |S \cup T_U \cup U| \geqslant |T_U|$.

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Bounds

Let $T^* = \{v \in T : |N(v) \cap W_{k-1}| \geqslant k^3\}$ and let c_2 be a global constant. If $|T^*| \geqslant c_2|X|/k$ for a given graph G, then our technical lemma implies that $|T_U| \geqslant (c_2 - o(1))|X|/k$ and so G isn't a counterexample to our Main Theorem.

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Observation

Assume then, that $|T^*| < c_2|X|/k$ and thus $T \setminus T^*$ has size at least $(1 - c_1 - c_2)|X|/k$.

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Observation

$T \setminus T^*$ has size at least $(1 - c_1 - c_2)|X|/k$.

Pigeonhole principle

There exists some $0 \leqslant p \leqslant \log_2(k^3) = \mathcal{O}(\log(k))$ such that

$$|T_p| = |\{v \in T : 2^p \leqslant |N(v) \cap W_{k-1}| < 2^{p+1}\}| \geqslant |T \setminus T^*|/\mathcal{O}(\log(k))$$

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Final steps

Similarly as before, we define $U \subseteq W_{k-1}$ to be a random subset. This time every vertex in W_{k-1} has probabiliry $1/2^p$ to be included, instead of $1/2$. We further define T_U and S as before.

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Open questions

Exact values of $c(k)$

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Behaviour of $f(G, 1, k)$

Not much is known about the behaviour of $f(G, 1, k)$ for $k > 2$ in the general case (non-bipartite graphs). Is there a linear lower bound on $f(G, 1, k)$ if we remove the restriction to bipartite graphs?