Critically paintable, choosable or colorable graphs

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Based on A. Riasat and U. Schauz publication

Paintability

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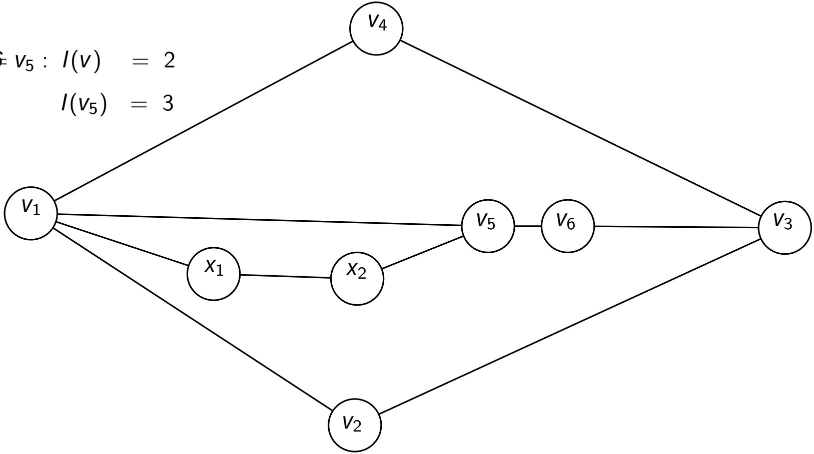
- Lister wins if some vertex has been marked more than $l(v)$ times.
- Painter wins if all vertices have been colored before Lister wins.
- G is l-paintable if Painter has the winning strategy.

Paintability example

$$\forall v \neq v_5 : \; l(v) = 2$$
$$l(v_5) = 3$$



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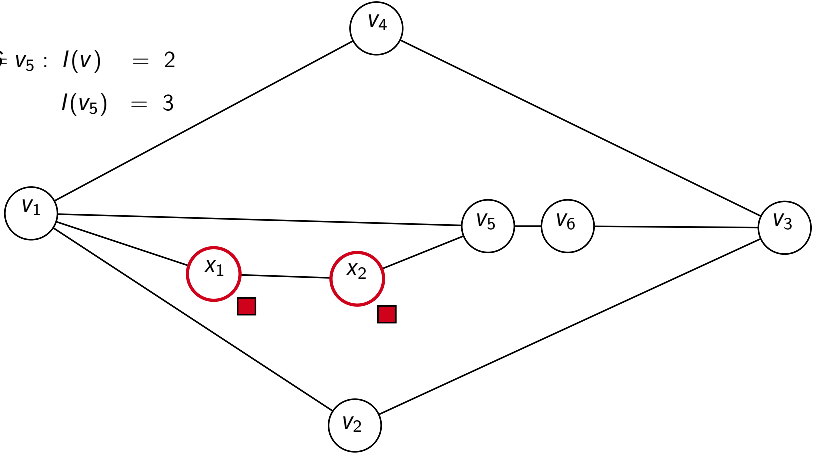

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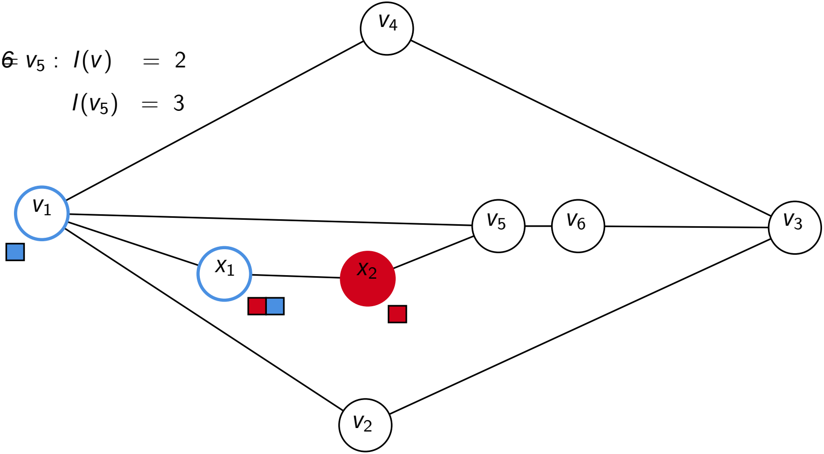

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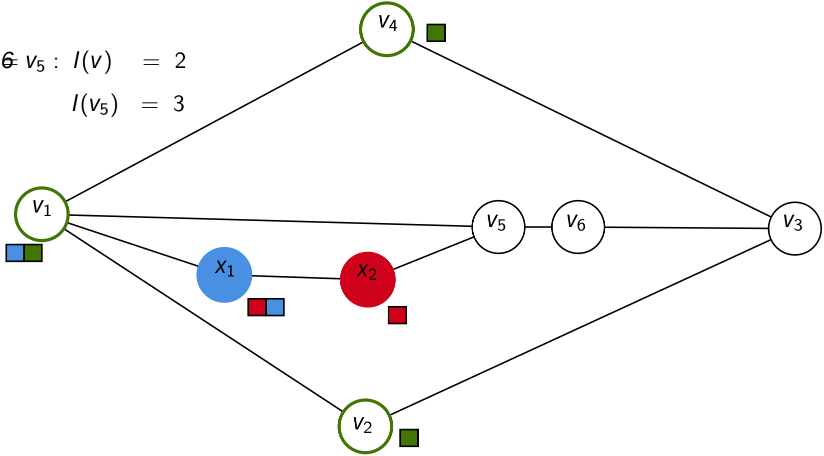

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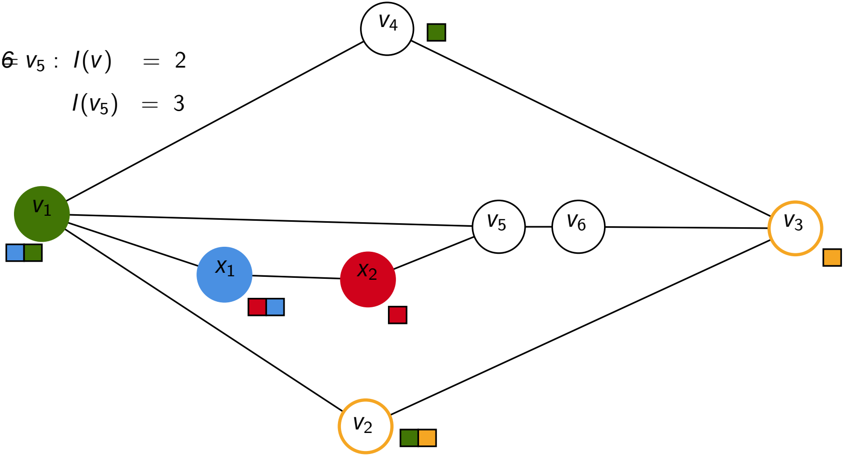

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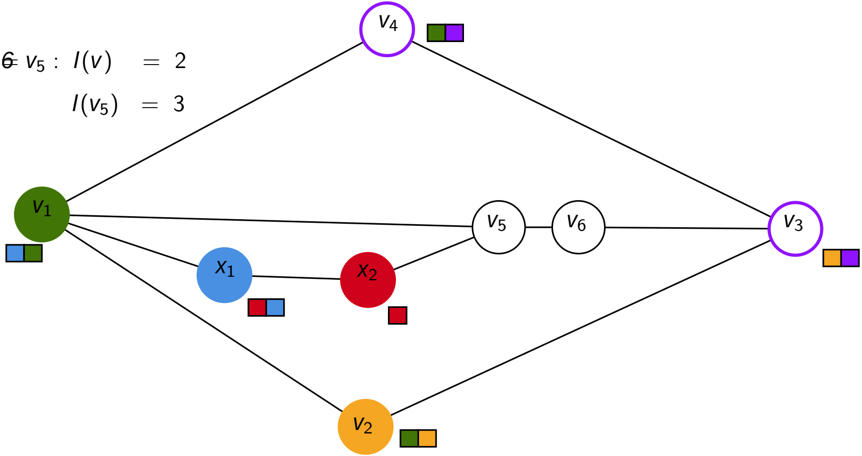

Relation between paintability

G is l-paintable $\implies$ G is l-choosable.

G is l-paintable $=)$ G is l-choosable.

Moreover, if $8v : l(v) = k$, then:
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Implications in other directions don't hold generally.

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Graph G is almost l-paintable (choosable, colorable), if it is not l-paintable (choosable, colorable), but *8v* : *G n fvg* is.

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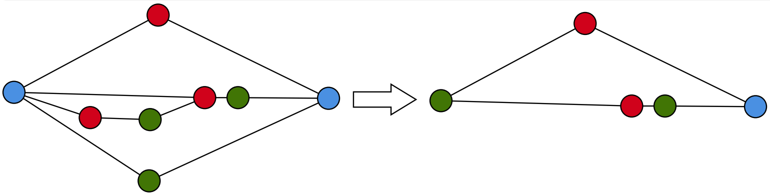
Lemma

If G doesn't contain an almost l-paintable (colorable, choosable) induced subgraph, then it is l-paintable (colorable, choosable).



Strong version of Brooks' Theorem

Theorem (Hladký, Král, Schauz; 2010)

For any connected graph:

- *if it is a Gallai Tree, then it is not degree-choosable,*
- *otherwise it is degree-paintable.*

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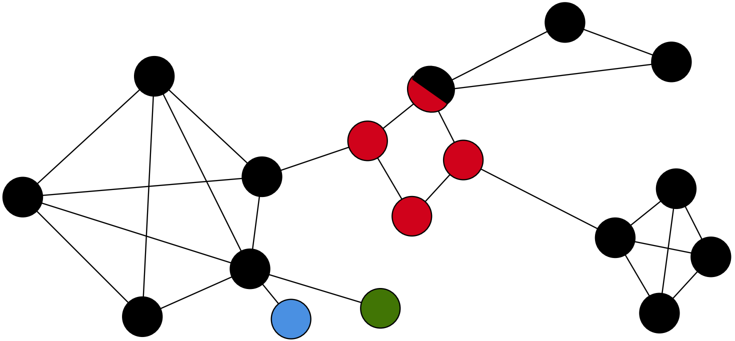

Cut Lemma

Lemma (Cut Lemma)

Let G = (U [W; E). Let 8u 2 U : (u) = jN(u) \ Wj.
If G[W] is l-paintable (choosable) and G[U] is (l   )-paintable (choosable), then G is l-paintable.

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Let $G = (U \; [\; W; E)$. Let $8u \; 2 \; U : \;\; (u) = jN(u) \setminus Wj$.
If $G[W]$ is l-paintable (choosable) and $G[U]$ is $(l \qquad)$-paintable (choosable), then G is l-paintable.

Consequence: Gallai Trees are almost degree-paintable and choosable.

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*Let $G = (U \cup W; E)$. Let $\forall u \in U: \lambda(u) = |N(u) \cap W|$.
If $G[W]$ is l-paintable (choosable) and $G[U]$ is $(l - \lambda)$-paintable (choosable), then G is l-paintable.*

Consequence: Gallai Trees are almost degree-paintable and choosable.

Moreover, almost l-paintable (choosable) graphs satisify $\forall v : d(v) \leq l(v)$.

Low-degree subgraphs

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The low-degree subgraph H of any almost paintable (choosable) graph G is a Gallai Forest.

Proof: if some biconnected component B of H is neither clique nor cycle, then it is degree-paintable (choosable). Using almost-paintability, $G n B$ is l-paintable, so from the cut lemma, G would also be.

Edge density lower bound

Lemma (Gallai, Kritische)

For a Gallai Tree $G = (V; E)$ *di erent from* $K_{\Delta+1}$ *and with* $\Delta \quad 3$*:*

$$\frac{jEj}{jVj} < \frac{\Delta \quad 1}{2} + \frac{1}{\Delta}$$

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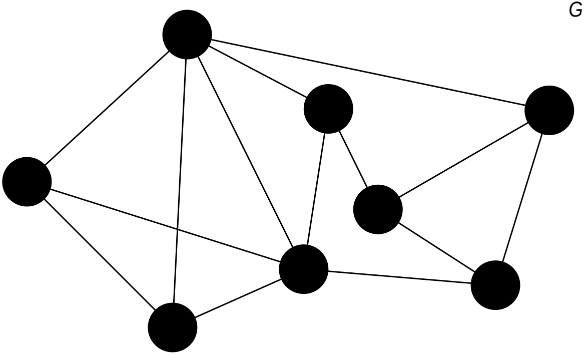

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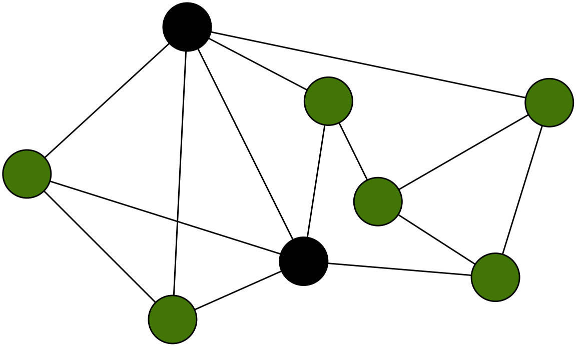

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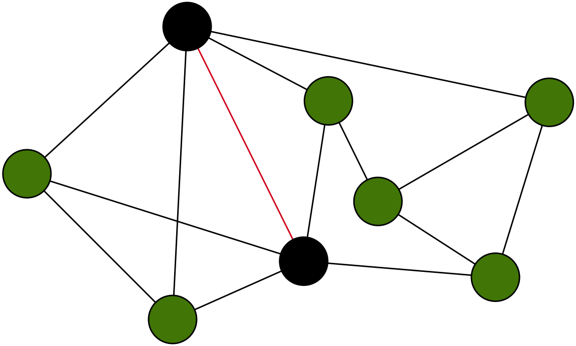

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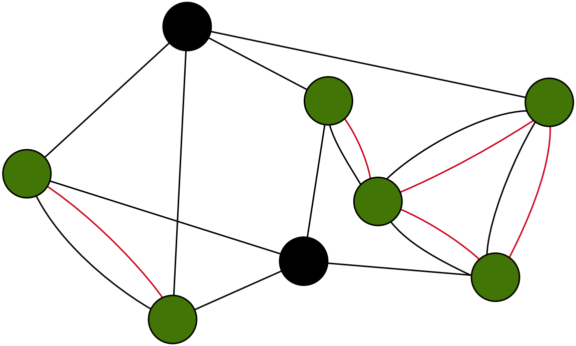

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$H = G[fu : d_G(u) = \; g]$

$jE(G)j \quad jE(G)j \quad jE(G \; n \; H)j$

$= \quad jV(H)j \quad jE(H)j$

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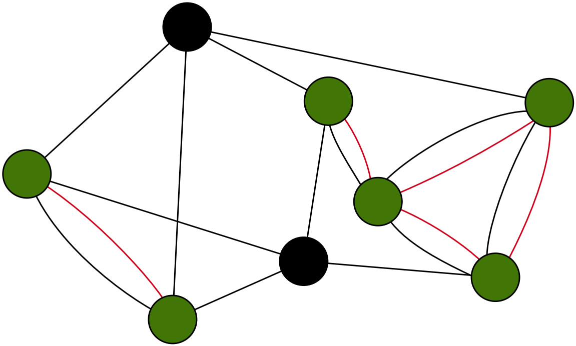

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$H = G[\{u : d_G(u) = \ \}]$

$|E(G)| \geq |E(G)| - |E(G \setminus H)|$

$= |V(H)| \quad |E(H)|$

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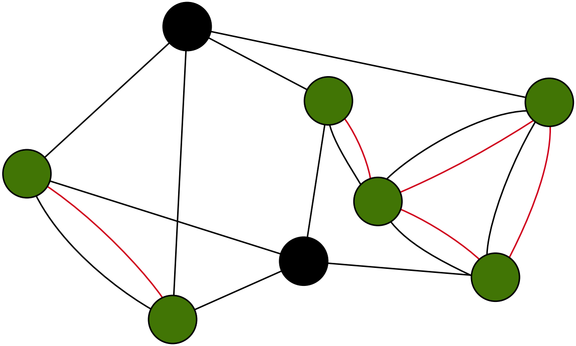

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Edge density - cont.

On the other side:

$$2jEj \quad (\;\; + 1)jV(G \; n \, H)j + \;\; jV(H)j = (\;\; + 1)jVj \quad jV(H)j$$

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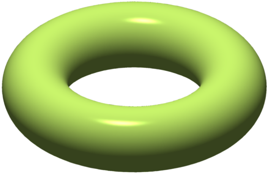
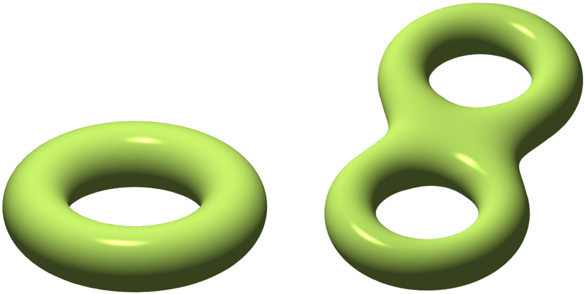
Lemma

If $G \neq K_{k+1}$ is almost l-paintable (l-choosable, l-colorable), where $k := min(l(v)) \quad 3$, then:

$$2\frac{|E|}{|V|} > k + \frac{k \quad 2}{k^2 + 2k \quad 2}$$

Graphs on surfaces





Lemma (Euler's formula)

For any connected graph G drawn on a surface with a genus g:

$2 - g = |V(G)| - |E(G)| + |F(G)|$

Moreover:

$2 - g \le |V(G)| - \frac{1}{3}|E(G)|$

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So, the graph is $H(g) \quad 1$-degenerate, and from the cut lemma also $H(g)$-paintable.

Heawood's Map-Coloring Theorem

[redacted]

A graph G on a surface with genus g 1 is $H(g)$-paintable, choosable, and colorable. Except for the Klein Bottle, this is the best possible bou[redacted]

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Likely, it's also true for paintability – proven for all g except $g \in \{1; 3\}$.

Critical graphs on surfaces

Let G be an almost-paintable graph on a surface with genus g.

Recall, that $6(2 \quad g) \quad 6|V| \quad 2|E| \quad (6 \quad\quad)|V|:$

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$$|V| \quad \frac{6(g \quad 2)}{\quad 6} \quad \frac{6(g \quad 2)}{\min(l) \quad 6}$$

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And if $\min(l) = 6$ and $G \neq K_7$:

$$6(2 \quad g) \quad 6|V| \quad 2|E| \quad 6 \quad 6 + \frac{6 \quad 2}{6^2 + 2 \quad 6 \quad 2} \quad |V| \quad \frac{2}{23}|V|$$

Critical graphs on surfaces

Let G be an almost-paintable graph on a surface with genus g.

Recall, that $6(2-g) \leq 6|V| - 2|E| \leq (6-\delta)|V|$.

If $\delta \geq 7$, then, using the fact, that $\forall v \in V : l(v) \leq d(v)$:

$$|V| \leq \frac{6(g-2)}{\delta - 6} \leq \frac{6(g-2)}{\min(l) - 6}$$

And if $\min(l) = 6$ and $G \neq K_7$:

$$6(2-g) \leq 6|V| - 2|E| \leq \left(6 - \left(6 + \frac{6-2}{6^2+2\cdot 6-2}\right)\right)|V| \leq -\frac{2}{23}|V|$$

In both cases $|V(G)| < 69(g-2)$, so if $\min(l) \geq 6$, then there are only finitely many pairs (G, l) of almost l-paintable (choosable) graphs.

Almost 2-paintable and choosable graphs

If G has a vertex v with $d(v) = 1, l(v) \geq 2$, then G is l-paintable iff $G \setminus \{v\}$ is so.

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We'll show, that:
$T_{ch} := C_{2a} \cup \quad_{2;2;2}$ and $T_p := C_2 \quad \cup \{ \quad_{2;2;2} \}$
are exactly the classes of 2-choosable (paintable) graphs.

Proof sketch:

Taking vertex minors (taking subgraph, then contracting vertices = contracting all edges incident to it) preserves 2-choosability and 2-paintability.

2-choosability and paintability identi cation

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- G is either C _ ; ; or contains $C_3; K_{3;3}$, , , or a subdivided K_4.

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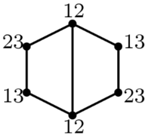
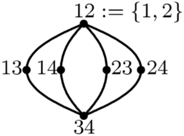
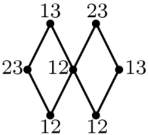
5. Identify minimal elements w.r.t vertex minors.



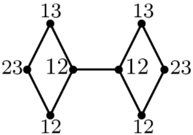
$\Theta_{1,3,3}$



$K_{2,4} = \Theta_{2,2,2,2}$



$0\!\!-\!\!0_{4,0,4}$



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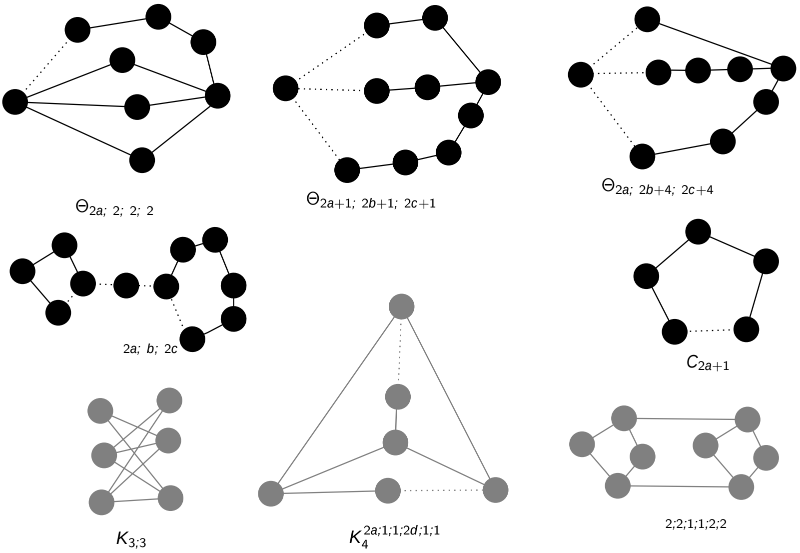
Almost 2-choosable graphs w.r.t. vertex deletion:



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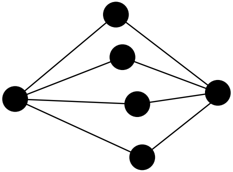
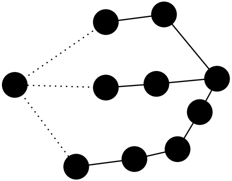
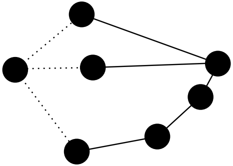


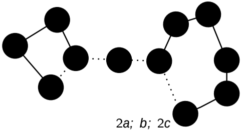
$K_{2;4}$



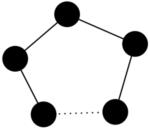
$\Theta_{2a+1;\ 2b+1;\ 2c+1}$



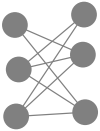
$\Theta_{2a;\ 2b;\ 2c+4}$



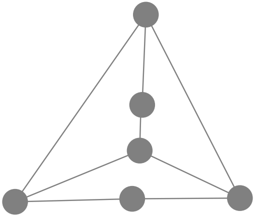
$2a;\ b;\ 2c$



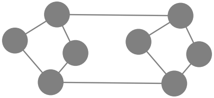
C_{2a+1}



$K_{3;3}$



$K_4^{2;1;1;2;1;1}$



$2;2;1;1;2;2$