List avoiding orientations

Peter Bradshaw, Yaobin Chen, Hao Ma, Bojan Mohar, Hehui Wu

March 30 2023

Introduction

Basic definitions

- G is an undirected simple graph. $V(G)$ is a set of vertices. $E(G)$ is a set of edges.

Introduction

Basic definitions

- G is an undirected simple graph. $V(G)$ is a set of vertices. $E(G)$ is a set of edges.
- D will be typically used for a directed graphs (or specific orientations)

Introduction

Basic definitions

- G is an undirected simple graph. $V(G)$ is a set of vertices. $E(G)$ is a set of edges.
- D will be typically used for a directed graphs (or specific orientations)
- $E_D(v)^-$, $E_D(v)^+$ denotes edges incoming/outgoing to v. Respectively we define $deg_D(v)^-$, $deg_D(v)^+$.

Introduction

Theorem, Frank and Gyárfás, 1976

For a graph G and two mappings $a, b : V(G) \rightarrow \mathbb{N}$ satisfying $a(v) \leq b(v)$ for every vertex v, G has an orientation D satisfying $a(v) \leq deg_D^+(v) \leq b(v)$ for every vertex v iff for each subset $U \in V(G)$:

Introduction

Theorem, Frank and Gyárfás, 1976

For a graph G and two mappings $a, b : V(G) \to \mathbb{N}$ satisfying $a(v) \leq b(v)$ for every vertex v, G has an orientation D satisfying $a(v) \leq deg_D^+(v) \leq b(v)$ for every vertex v iff for each subset $U \in V(G)$:

$$\sum_{v \in U} a(v) - e(U, \bar{U}) \leq |E(G[U])| \leq \sum_{v \in U} b(v)$$

Introduction

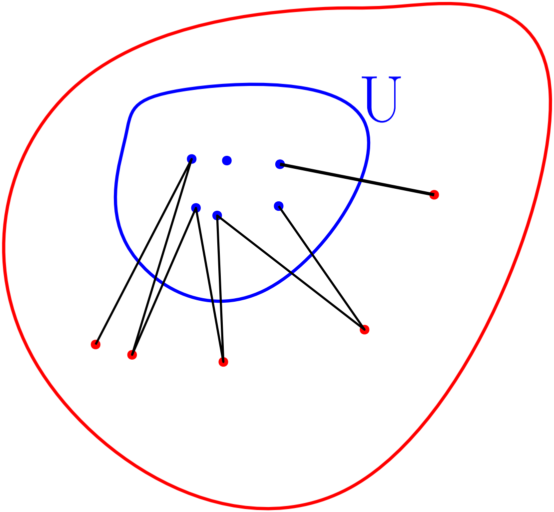
Theorem, Frank and Gyárfás, 1976

For a graph G and two mappings $a, b : V(G) \to \mathbb{N}$ satisfying $a(v) \leq b(v)$ for every vertex v, G has an orientation D satisfying $a(v) \leq deg_D^+(v) \leq b(v)$ for every vertex v iff for each subset $U \in V(G)$:

$$\sum_{v \in U} a(v) - e(U, \bar{U}) \leq |E(G[U])| \leq \sum_{v \in U} b(v)$$

Where $e(U, \bar{U})$ denotes the number of edges between U and $\bar{U}$ - that is $V(G) \setminus U$

Example



f - avoiding orientations

Definition

Given a graph G and a function $f : V(G) \to \mathbb{N}$, we say that an orientation D of G is *f-avoiding* if $deg^{+}D(v) \neq f(v)$ for each $v \in V(G)$.

f - avoiding orientations

Definition

Given a graph G and a function $f : V(G) \to \mathbb{N}$, we say that an orientation D of G is *f-avoiding* if $deg^+ D(v) \neq f(v)$ for each $v \in V(G)$.

Theorem: S. Akbari, M. Dalirrooyfard, K. Ehsani, K. Ozeki, and R. Sherkati. [2020]

There is an *f-avoiding* orientation for every 2-connected graph G that is not an odd cycle and for every function $f : V(G) \to \mathbb{N}$, and that an odd cycle has an f-avoiding orientation if and only if $f(v) \neq 1$ for some vertex v of the cycle.

F - avoiding orientations

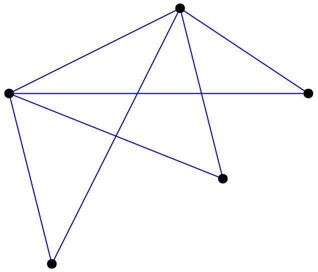
Definition

A graph G and a function $F : V(G) \to 2^{\mathbb{N}}$, an orientation D of G is said to be *F-avoiding* if $deg_D^+(v) \notin F(v)$ for each $v \in V(G)$.

F - avoiding orientations

Definition

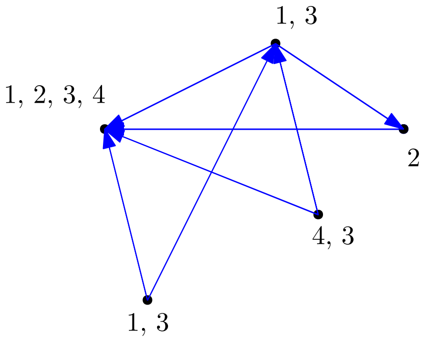
A graph G and a function $F : V(G) \to 2^{\mathbb{N}}$, an orientation D of G is said to be *F-avoiding* if $deg_D^+(v) \notin F(v)$ for each $v \in V(G)$.



F - avoiding orientations

Definition

A graph G and a function $F : V(G) \to 2^{\mathbb{N}}$, an orientation D of G is said to be *F-avoiding* if $deg_D^+(v) \notin F(v)$ for each $v \in V(G)$.



Conjectures

Conjecture 1

Let G be a graph, and let $F : V(G) \to 2^{\mathbb{N}}$. If $|F(v)| \leq \frac{1}{2}(deg_G(v) - 1)$ for each $v \in V(G)$, then G has an F - avoiding orientation.

Conjectures

Conjecture 1

Let G be a graph, and let $F : V(G) \to 2^{\mathbb{N}}$. If $|F(v)| \leq \frac{1}{2}(deg_G(v) - 1)$ for each $v \in V(G)$, then G has an F - avoiding orientation.

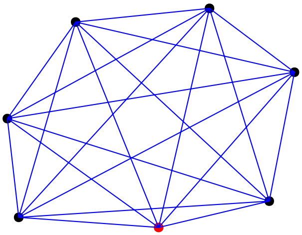
If that conjecture is true, the bound is tight. $2k$-regular graphs on n vertices with independence number less than $\frac{n}{k+1}$ and $F(v) = \{k, k+1, ..., 2k-1\}$ give sharpness. Eg. K_{2k+1}:

Conjectures

Conjecture 1

Let G be a graph, and let $F : V(G) \to 2^{\mathbb{N}}$. If $|F(v)| \leq \frac{1}{2}(deg_G(v) - 1)$ for each $v \in V(G)$, then G has an F - avoiding orientation.

If that conjecture is true, the bound is tight. $2k$-regular graphs on n vertices with independence number less than $\frac{n}{k+1}$ and $F(v) = \{k, k+1, ..., 2k-1\}$ give sharpness. Eg. K_{2k+1}:



$2k + 1 = 7$

$\mathrm{F(v)} = \{3,\ 4,\ 5\ \}$

Conjectures

Conjecture 2

Let G be a graph, and let $F : V(G) \to 2^{\mathbb{N}}$.If G has an orientation D such that $deg_D^+(v) \geq |F(v)| + 1$ for each $v \in V(G)$, then G has an F - avoiding orientation.

Conjectures

Conjecture 2

Let G be a graph, and let $F : V(G) \to 2^{\mathbb{N}}$.If G has an orientation D such that $deg_D^+(v) \geq |F(v)| + 1$ for each $v \in V(G)$, then G has an F - avoiding orientation.

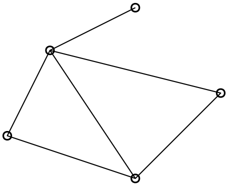
As every graph G has an orientation such that $v \in V(G)$ satisfies $deg_D^+(v) \geq \lfloor \frac{1}{2} deg_G(v) \rfloor$, Conjecture 2 (if true) implies Conjecture 1 with error at most 1.

Conjectures

Conjecture 2

Let G be a graph, and let $F : V(G) \rightarrow 2^{\mathbb{N}}$.If G has an orientation D such that $deg_D^+(v) \geq |F(v)| + 1$ for each $v \in V(G)$, then G has an F - avoiding orientation.

As every graph G has an orientation such that $v \in V(G)$ satisfies $deg_D^+(v) \geq \lfloor \frac{1}{2} deg_G(v) \rfloor$, Conjecture 2 (if true) implies Conjecture 1 with error at most 1.

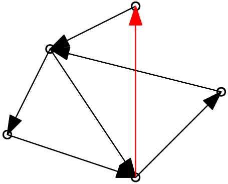


Conjectures

Conjecture 2

Let G be a graph, and let $F : V(G) \rightarrow 2^{\mathbb{N}}$.If G has an orientation D such that $deg_D^+(v) \geq |F(v)| + 1$ for each $v \in V(G)$, then G has an F - avoiding orientation.

As every graph G has an orientation such that $v \in V(G)$ satisfies $deg_D^+(v) \geq \lfloor \frac{1}{2} deg_G(v) \rfloor$, Conjecture 2 (if true) implies Conjecture 1 with error at most 1.

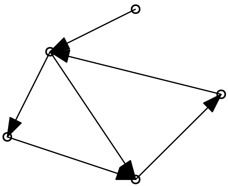


Conjectures

Conjecture 2

Let G be a graph, and let $F : V(G) \rightarrow 2^{\mathbb{N}}$.If G has an orientation D such that $deg_D^+(v) \geq |F(v)| + 1$ for each $v \in V(G)$, then G has an F - avoiding orientation.

As every graph G has an orientation such that $v \in V(G)$ satisfies $deg_D^+(v) \geq \lfloor \frac{1}{2} deg_G(v) \rfloor$, Conjecture 2 (if true) implies Conjecture 1 with error at most 1.



Central tool

Combinatorial Nullstellensatz

Let K be a field, and let f be a polynomial over the field $K[x_1, x_2, ..., x_n]$. Suppose that the degree of f is $\sum_{i=1}^{n} t_i$, where each t_i is a nonnegative integer, and suppose that the coefficient of $\prod_{i=1}^{n} x^{t_i}$ in f is nonzero. Then, if $S_1, ..., S_n$ are subsets of K each satisfying $|S_i| > t_i$, then there exist elements $s_1 \in S_1, ..., s_n \in S_n$ so that $f(s_1, ..., s_n) \neq 0$.

Central tool

Combinatorial Nullstellensatz

Let K be a field, and let f be a polynomial over the field $K[x_1, x_2, ..., x_n]$. Suppose that the degree of f is $\sum_{i=1}^{n} t_i$, where each t_i is a nonnegative integer, and suppose that the coefficient of $\prod_{i=1}^{n} x^{t_i}$ in f is nonzero. Then, if $S_1, ..., S_n$ are subsets of K each satisfying $|S_i| > t_i$, then there exist elements $s_1 \in S_1, ..., s_n \in S_n$ so that $f(s_1, ..., s_n) \neq 0$.

We define:

- matrix $M = (m_{v,e} : v \in V(G), e \in E(G))$ $m_{v,e} = 1$ if $e \in E_G^+(v_i)$, $m_{v,e} = 1$ if $e \in E_G^-(v_i)$, and $m_{v,e} = 0$

Central tool

Combinatorial Nullstellensatz

Let K be a field, and let f be a polynomial over the field $K[x_1, x_2, ..., x_n]$. Suppose that the degree of f is $\sum_{i=1}^{n} t_i$, where each t_i is a nonnegative integer, and suppose that the coefficient of $\prod_{i=1}^{n} x^{t_i}$ in f is nonzero. Then, if $S_1, ..., S_n$ are subsets of K each satisfying $|S_i| > t_i$, then there exist elements $s_1 \in S_1, ..., s_n \in S_n$ so that $f(s_1, ..., s_n) \neq 0$.

We define:

- matrix $M = (m_{v,e} : v \in V(G), e \in E(G))$ $m_{v,e} = 1$ if $e \in E_G^+(v_i)$, $m_{v,e} = 1$ if $e \in E_G^-(v_i)$, and $m_{v,e} = 0$
- $f_D^* = \prod_{e \in E(G)} (\sum_{v \in V(G)} m_{v,e} x_v)$

Central tool

Combinatorial Nullstellensatz

Let K be a field, and let f be a polynomial over the field $K[x_1, x_2, ..., x_n]$. Suppose that the degree of f is $\sum_{i=1}^{n} t_i$, where each t_i is a nonnegative integer, and suppose that the coefficient of $\prod_{i=1}^{n} x^{t_i}$ in f is nonzero. Then, if $S_1, ..., S_n$ are subsets of K each satisfying $|S_i| > t_i$, then there exist elements $s_1 \in S_1, ..., s_n \in S_n$ so that $f(s_1, ..., s_n) \neq 0$.

We define:

- matrix $M = (m_{v,e} : v \in V(G), e \in E(G))$ $m_{v,e} = 1$ if $e \in E_G^+(v_i)$, $m_{v,e} = 1$ if $e \in E_G^-(v_i)$, and $m_{v,e} = 0$
- $f_D^* = \prod_{e \in E(G)} (\sum_{v \in V(G)} m_{v,e} x_v)$
- $f_D^* = \prod_{e \in E(D)} (x_v - x_u)$

Central tool

Combinatorial Nullstellensatz

Let K be a field, and let f be a polynomial over the field $K[x_1, x_2, ..., x_n]$. Suppose that the degree of f is $\sum_{i=1}^{n} t_i$, where each t_i is a nonnegative integer, and suppose that the coefficient of $\prod_{i=1}^{n} x^{t_i}$ in f is nonzero. Then, if $S_1, ..., S_n$ are subsets of K each satisfying $|S_i| > t_i$, then there exist elements $s_1 \in S_1, ..., s_n \in S_n$ so that $f(s_1, ..., s_n) \neq 0$.

We define:

- matrix $M = (m_{v,e} : v \in V(G), e \in E(G))$ $m_{v,e} = 1$ if $e \in E_G^+(v_i)$, $m_{v,e} = 1$ if $e \in E_G^-(v_i)$, and $m_{v,e} = 0$
- $f_D^* = \prod_{e \in E(G)} (\sum_{v \in V(G)} m_{v,e} x_v)$
- $f_D^* = \prod_{e \in E(D)} (x_v - x_u)$

f_D^* is a classical graph polynomial. Graph coloring for instance might be translated to such polynomial naturally.

Main Construction

- We will consider $F : V(G) \to 2^{\mathbb{Z}}$ defined as *imbalance*, which is the difference $deg_D^+(v) \;\; deg_D(v)$. and F is avoiding if $deg_D^+(v)$ $deg_D(v) \notin F(v)$

Main Construction

- We will consider $F : V(G) \to 2^{\mathbb{Z}}$ defined as *imbalance*, which is the difference $deg_D^+(v)\ \ deg_D(v)$. and F is avoiding if $deg_D^+(v)$ $deg_D(v) \notin F(v)$
- incidence matrix M $=(m_{v,e} : v \in V(G), e \in E(G))$

Main Construction

- We will consider $F : V(G) \rightarrow 2^{\mathbb{Z}}$ defined as *imbalance*, which is the difference $deg_D^+(v) \;\; deg_D(v)$. and F is avoiding if $deg_D^+(v)$ $deg_D(v) \notin F(v)$
- incidence matrix M $=(m_{v,e} : v \in V(G), e \in E(G))$
- G with acyclic orientation such that each edge $v_i v_j$ is oriented from i to j if $i < j$

Main Construction

- We will consider $F : V(G) \to 2^{\mathbb{Z}}$ defined as *imbalance*, which is the difference $deg_D^+(v)$ $deg_D(v)$. and F is avoiding if $deg_D^+(v)$ $deg_D(v) \notin F(v)$
- incidence matrix M $=(m_{v,e} : v \in V(G), e \in E(G))$
- G with acyclic orientation such that each edge $v_i v_j$ is oriented from i to j if $i < j$
- For each vertex $v_i \in V(G)$, we let $E_G^R(v_i)$ denote the edges $v_i v_j \in E(G)$ with $j > i$, and we let $E_G^L(v_i)$ denote the edges if $i > j$. Similarly define $deg_G^R(v)$ and $deg_G^L(v)$

Main Construction

- We will consider $F : V(G) \to 2^{\mathbb{Z}}$ defined as *imbalance*, which is the difference $deg_D^+(v) \;\; deg_D(v)$. and F is avoiding if $deg_D^+(v)$ $deg_D(v) \notin F(v)$
- incidence matrix M $=(m_{v,e} : v \in V(G), e \in E(G))$
- G with acyclic orientation such that each edge $v_i v_j$ is oriented from i to j if $i < j$
- For each vertex $v_i \in V(G)$, we let $E_G^R(v_i)$ denote the edges $v_i v_j \in E(G)$ with $j > i$, and we let $E_G^L(v_i)$ denote the edges if $i > j$. Similarly define $deg_G^R(v)$ and $deg_G^L(v)$
- For each edge $e \in E(G)$, we consider a variable y_e. Given an orientation D of G, and given an edge e $= v_i v_j$ with $i < j$, we set $y_e = 1$ if e is oriented from v_i to v_j in D,and we set $y_e = -1$ otherwise.

Main Construction

- We will consider $F : V(G) \to 2^{\mathbb{Z}}$ defined as *imbalance*, which is the difference $deg^+_D(v) \;\; deg_D(v)$. and F is avoiding if $deg^+_D(v)$ $deg_D(v) \notin F(v)$
- incidence matrix M $=(m_{v,e} : v \in V(G), e \in E(G))$
- G with acyclic orientation such that each edge $v_i v_j$ is oriented from i to j if $i < j$
- For each vertex $v_i \in V(G)$, we let $E^R_G(v_i)$ denote the edges $v_i v_j \in E(G)$ with $j > i$, and we let $E^L_G(v_i)$ denote the edges if $i > j$. Similarly define $deg^R_G(v)$ and $deg^L_G(v)$
- For each edge $e \in E(G)$, we consider a variable y_e. Given an orientation D of G, and given an edge e $= v_i v_j$ with $i < j$, we set $y_e = 1$ if e is oriented from v_i to v_j in D,and we set $y_e = -1$ otherwise.

Main Construction

The Imbalance in every vertex is a linear polynomial:

$$deg_D^+(v) - deg_D(v) = \sum_{e \in E(G)} m_{v_e y_e} = \sum_{e \in E_G^R(v)} y_e - \sum_{e \in E_G^L(v)} y_e$$

Main Construction

The Imbalance in every vertex is a linear polynomial:

$$deg_D^+(v) - deg_D(v) = \sum_{e \in E(G)} m_{v_e y_e} = \sum_{e \in E_G^R(v)} y_e - \sum_{e \in E_G^L(v)} y_e$$

So the polynomial is defined as:

$$f_0 = \prod_{i=1}^{n} \prod_{a \in F(v_i)} \Big(\sum_{e \in E_G^R(v)} y_e - \sum_{e \in E_G^L(v)} y_e - a \Big)$$

We can use Combinatorial Nullstellensatz if there exists monomial with coefficient $\neq 0$ s. t. each y_e appears at most once in it, as $deg(f_0) = \sum_{i \in [n]} t_i$

Main Construction

Each y_e is present only in one term, because of the acyclic orientation. Def $f_0 = \sum_{i=1}^{n} t_i$, where $t_i = F(v_i)$, as we only care about the monomial of maximum degree:

Main Construction

Each y_e is present only in one term, because of the acyclic orientation. Def $f_0 = \sum_{i=1}^{n} t_i$, where $t_i = F(v_i)$, as we only care about the monomial of maximum degree:

$$f = \prod_{i \in [n]} \Big(\sum_{e \in E_G^R(v_i)} y_e - \sum_{e \in E_G^L(v_i)} y_e \Big)^{t_i} = \prod_{i \in [n]} \Big(\sum_{e \in E(G)} m_{ve} y_e \Big)^{t_i}$$

Main Construction

Each y_e is present only in one term, because of the acyclic orientation. Def $f_0 = \sum_{i=1}^{n} t_i$, where $t_i = F(v_i)$, as we only care about the monomial of maximum degree:

$$f = \prod_{i \in [n]} \Big(\sum_{e \in E_G^R(v_i)} y_e - \sum_{e \in E_G^L(v_i)} y_e \Big)^{t_i} = \prod_{i \in [n]} \Big(\sum_{e \in E(G)} m_{ve} y_e \Big)^{t_i}$$

The problem is to find a monomial with a nonzero coefficient in f of a form: $y_A = y_{e_1} y_{e_2} ... y_{e_{|A|}}$ for $A \in E(G)$ s. t. no y_e^2 does not divide that monomial.

Main Theorem

Theorem

Let $F : V(G) \to 2^{\mathbb{Z}}$ be an assignment of forbidden imbalances for a graph G. Suppose that there exists an ordering of $V(G)$ and a spanning subgraph H of G such that for each vertex $v \in V(G)$, it holds that $|F(v)| \leq deg_G^L(v) - 2deg_H^L(v) + deg_G^L(v)$. Then G has an F-avoiding orientation.

Main Theorem

Theorem

Let $F : V(G) \to 2^{\mathbb{Z}}$ be an assignment of forbidden imbalances for a graph G. Suppose that there exists an ordering of $V(G)$ and a spanning subgraph H of G such that for each vertex $v \in V(G)$, it holds that $|F(v)| \leq deg^{L}_{G}(v) - 2deg^{L}_{H}(v) + deg^{L}_{G}(v)$. Then G has an F-avoiding orientation.

One can prove it by considering the polynomial f and its prefix products

Main Theorem

Theorem

Let $F : V(G) \to 2^{\mathbb{Z}}$ be an assignment of forbidden imbalances for a graph G. Suppose that there exists an ordering of $V(G)$ and a spanning subgraph H of G such that for each vertex $v \in V(G)$, it holds that $|F(v)| \leq deg_G^L(v) - 2deg_H^L(v) + deg_G^L(v)$. Then G has an F-avoiding orientation.

One can prove it by considering the polynomial f and its prefix products

$$f_j = \prod_{i \in [j]} \Big(\sum_{e \in E(G)} m_{ve} y_e \Big)^{t_i}$$

Conjectures revisited

Conjecture 1: nearly $\frac{1}{2}$ approximation, S. Akbari, M. Dalirrooyfard, K. Ehsani, K. Ozeki, and R. Sherkati [2020]

Let G be a graph, and let $F : V(G) \to 2^{\mathbb{N}}$ be a map. If $|F(v)| \leq \frac{1}{4}deg(v)$ for each vertex $v \in V(G)$, then G has an F - avoiding orientation.

Conjectures revisited

Conjecture 1: nearly $\frac{1}{2}$ approximation, S. Akbari, M. Dalirrooyfard, K. Ehsani, K. Ozeki, and R. Sherkati [2020]

Let G be a graph, and let $F : V(G) \to 2^{\mathbb{N}}$ be a map. If $|F(v)| \leq \frac{1}{4}deg(v)$ for each vertex $v \in V(G)$, then G has an F - avoiding orientation.

Using the main Theorem, we can show a better approximation

Conjectures revisited

Conjecture 1: nearly $\frac{1}{2}$ approximation, S. Akbari, M. Dalirrooyfard, K. Ehsani, K. Ozeki, and R. Sherkati [2020]

Let G be a graph, and let $F : V(G) \to 2^{\mathbb{N}}$ be a map. If $|F(v)| \leq \frac{1}{4}deg(v)$ for each vertex $v \in V(G)$, then G has an F - avoiding orientation.

Using the main Theorem, we can show a better approximation

Conjecture 1: a better approximation

Let G be a graph, and let $F : V(G) \to 2^{\mathbb{N}}$ be a map. If $|F(v)| \leq \frac{1}{3}deg(v) - 1$ for each vertex $v \in V(G)$, then G has an F - avoiding orientation.

Conjectures revisited

Conjecture 1: nearly $\frac{1}{2}$ approximation, S. Akbari, M. Dalirrooyfard, K. Ehsani, K. Ozeki, and R. Sherkati [2020]

Let G be a graph, and let $F : V(G) \to 2^{\mathbb{N}}$ be a map. If $|F(v)| \leq \frac{1}{4}deg(v)$ for each vertex $v \in V(G)$, then G has an F - avoiding orientation.

Using the main Theorem, we can show a better approximation

Conjecture 1: a better approximation

Let G be a graph, and let $F : V(G) \to 2^{\mathbb{N}}$ be a map. If $|F(v)| \leq \frac{1}{3}deg(v) - 1$ for each vertex $v \in V(G)$, then G has an F - avoiding orientation.

Moreover we can show a $\frac{2}{3}$ approximation of Conjecture 2.

Conjecture approximations

Conjecture 1: better approximation

Let G be a graph, and let $F : V(G) \to 2^{\mathbb{N}}$ be a map. If $|F(v)| \leq \frac{1}{3}deg(v) - 1$ for each vertex $v \in V(G)$, then G has an F - avoiding orientation.

Conjecture approximations

Conjecture 1: better approximation

Let G be a graph, and let $F : V(G) \rightarrow 2^{\mathbb{N}}$ be a map. If $|F(v)| \leq \frac{1}{3}deg(v) - 1$ for each vertex $v \in V(G)$, then G has an F -avoiding orientation.

Moreover we can show a $\frac{2}{3}$ approximation of Conjecture 2.

Conjecture 2: $\frac{2}{3}$ approximation

Let G be a graph, and let $F : V(G) \rightarrow 2^{\mathbb{N}}$. If G has an orientation D such that $|F(v)| \leq \frac{2}{3}deg_D^+(v) - 1$ for each v V (G), then G has an F -avoiding orientation.

Proof sketch

Technical lemma

Given a graph $G = (V, E)$, let $M = (m_{ve} : v \in V, e \in E)$ be a real-valued matrix in which $m_{ve} \neq 0$ only if $v \in e$. Let $y \in [0, 1]^E$ be a vector, and let $x = My$. Then, there exists a 01-vector $y' \in \{0, 1\}^E$ such that $x' = My'$ satsfies $x'_v \geq x_v - b_v$ for each $v \in V(G)$, where $b_v = max\{|m_{ve}| : e \in E\}$. Furthermore, we may choose y so that $x_v > x_v - b_v$ whenever $b_v > 0$.

Proof sketch

Technical lemma

Given a graph $G = (V, E)$, let $M = (m_{ve} : v \in V, e \in E)$ be a real-valued matrix in which $m_{ve} \neq 0$ only if $v \in e$. Let $y \in [0, 1]^E$ be a vector, and let $x = My$. Then, there exists a 01-vector $y' \in \{0, 1\}^E$ such that $x' = My'$ satsfies $x'_v \geq x_v - b_v$ for each $v \in V(G)$, where $b_v = max\{|m_{ve}| : e \in E\}$. Furthermore, we may choose y so that $x_v > x_v - b_v$ whenever $b_v > 0$.

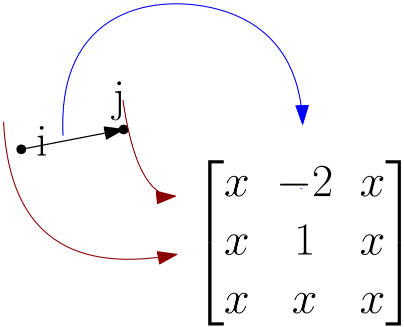
$$\begin{bmatrix} 1/2 \\ 3/7 \\ 2/5 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Proof sketch

- Using the main Theorem we get
  $deg_G^L(v) - 2deg_H^L(v) + deg_H^R(v) \geq \lfloor \frac{1}{3} deg(v) \rfloor - 1.$

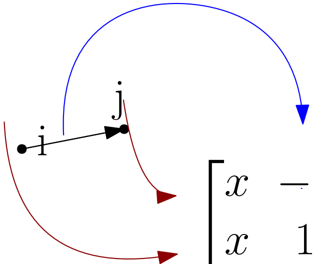
Proof sketch

- Using the main Theorem we get
  $deg_G^L(v) - 2deg_H^L(v) + deg_H^R(v) \geq \lfloor \frac{1}{3} deg(v) \rfloor - 1.$



Proof sketch

- Using the main Theorem we get $deg_G^L(v) - 2deg_H^L(v) + deg_H^R(v) \geq \lfloor \frac{1}{3} deg(v) \rfloor - 1$.

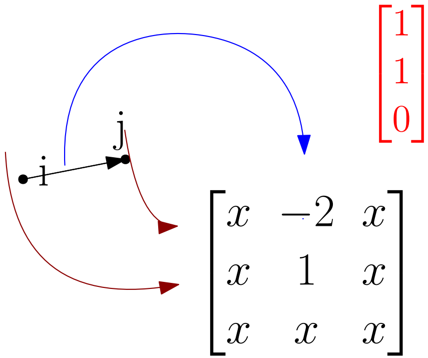


$$\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$\begin{bmatrix} x & -2 & x \\ x & 1 & x \\ x & x & x \end{bmatrix}$$

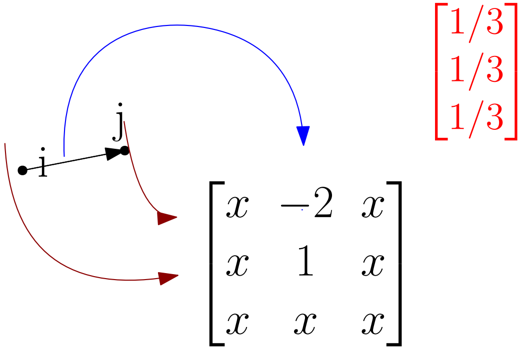
Proof sketch

- Using the main Theorem we get
  $deg_G^L(v) - 2deg_H^L(v) + deg_H^R(v) \geq \lfloor \frac{1}{3} deg(v) \rfloor - 1$.



Proof sketch

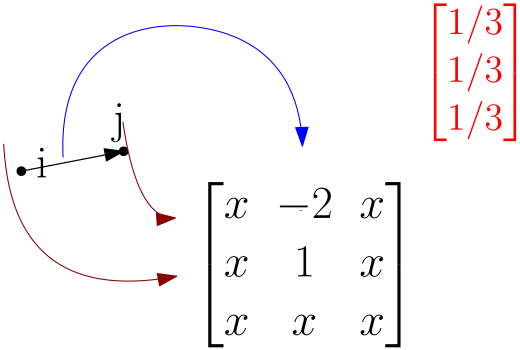
- Using the main Theorem we get
  $deg_G^L(v) - 2deg_H^L(v) + deg_H^R(v) \geq \lfloor \frac{1}{3} deg(v) \rfloor - 1.$



$$x_v = \frac{2}{3} deg_G^L(v) + \frac{1}{3} deg_G^R(v) = \frac{1}{3} deg_G(v) - deg_G^L(v)$$

Proof sketch

- Using the main Theorem we get $deg_G^L(v) - 2deg_H^L(v) + deg_H^R(v) \geq \lfloor \frac{1}{3}deg(v) \rfloor - 1.$



$$x_v = \frac{2}{3}deg_G^L(v) + \frac{1}{3}deg_G^R(v) = \frac{1}{3}deg_G(v) - deg_G^L(v)$$

$$x'_v \geq \frac{1}{3}deg_G(v) - deg_G^L(v) - 2$$

Eulerian orientations

Definition

- A Graph polynomial coefficients can be determined solely by counting Eulerian orientations of a graph. These are defined as follows: An orientation D of the graph G if $deg_D^+(v) = deg_D^-(v)$ for every $v \in V(G)$.

Eulerian orientations

Definition

- A Graph polynomial coefficients can be determined solely by counting Eulerian orientations of a graph. These are defined as follows: An orientation D of the graph G if $deg_D^+(v) = deg_D^-(v)$ for every $v \in V(G)$.
- A subgraph H of G is called even if $|E(H)|$ is even and is called odd otherwise.

Eulerian orientations

Definition

- A Graph polynomial coefficients can be determined solely by counting Eulerian orientations of a graph. These are defined as follows: An orientation D of the graph G if $deg_D^+(v) = deg_D^-(v)$ for every $v \in V(G)$.
- A subgraph H of G is called even if $|E(H)|$ is even and is called odd otherwise.
- $EE(D)$ and $EO(D)$ are respectively the number of even and odd Eulerian orientations

Eulerian orientations

Definition

- A Graph polynomial coefficients can be determined solely by counting Eulerian orientations of a graph. These are defined as follows: An orientation D of the graph G if $deg_D^+(v) = deg_D^-(v)$ for every $v \in V(G)$.
- A subgraph H of G is called even if $|E(H)|$ is even and is called odd otherwise.
- $EE(D)$ and $EO(D)$ are respectively the number of even and odd Eulerian orientations

Alon-Tarsi [1992]

If G has orientation D s.t $EE(D) \neq EO(D)$ then D is an Alon-Tarsi orientation. If D is an Alon-Tarsi orientation of G, and if L is a list assignment on G for which $|L(v)| > deg_D^+(v)$ at each vertex $v \in V(G)$, then G is L-choosable.

Alon Tarsi number

Definition

Alon Tarsi number is defined as: $AT(G)$ is the minimum value k such that G has an Alon-Tarsi orientation of maximum out-degree less than k. In particular $AT(G) \geq ch(G)$

Alon Tarsi number

Definition

Alon Tarsi number is defined as: $AT(G)$ is the minimum value k such that G has an Alon-Tarsi orientation of maximum out-degree less than k. In particular $AT(G) \geq ch(G)$

S. Akbari, M. Dalirrooyfard, K. Ehsani, K. Ozeki, and R. Sherkati [2020]

Let G be a graph, let H be a spanning subgraph of G, and let $F : V(G) \to 2^{\mathbb{N}}$ be a map. If there exists an Alon-Tarsi orientation D of H such that $|F(v)| \leq deg_D^+(v)$ for every vertex $v \in V(G)$, then G has an F-avoiding orientation.:

Alon Tarsi number

Definition

Alon Tarsi number is defined as: $AT(G)$ is the minimum value k such that G has an Alon-Tarsi orientation of maximum out-degree less than k. In particular $AT(G) \geq ch(G)$

S. Akbari, M. Dalirrooyfard, K. Ehsani, K. Ozeki, and R. Sherkati [2020]

Let G be a graph, let H be a spanning subgraph of G, and let $F : V(G) \to 2^{\mathbb{N}}$ be a map. If there exists an Alon-Tarsi orientation D of H such that $|F(v)| \leq deg_D^+(v)$ for every vertex $v \in V(G)$, then G has an F-avoiding orientation.:

Despite the upper theorem proof is based on an original graph polynomial, it can be proved using the polynomial defined in the main theorem.

Dual Polynomials

We will consider a $A^{\alpha,\beta}$ transformation of a matrix A

Dual Polynomials

We will consider a $A^{\alpha,\beta}$ transformation of a matrix A

$$\alpha = (1,2)\beta = (2,1)$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} \color{red}{1} & 1 & \color{red}{2} \\ \color{red}{3} & 3 & \color{red}{4} \\ 3 & 3 & 4 \end{bmatrix}$$

Dual Polynomials

We will consider a $A^{\alpha,\beta}$ transformation of a matrix A

$$\alpha = (1,2)\beta = (2,1)$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 3 & 3 & 4 \\ 3 & 3 & 4 \end{bmatrix}$$

And dual polynomials:

$$g = \prod_{i=0}^{n} \sum_{j=0}^{m} a_{ij} y_j$$

$$g* = \prod_{j=0}^{m} \sum_{i=0}^{n} a_{ij} x_i$$

Dual polynomials

Definitions

- $x^\alpha = \prod_{i=0}^{n} x_i^{\alpha_i}$, y^β is defined similarly

Dual polynomials

Definitions

- $x^\alpha = \prod_{i=0}^{n} x_i^{\alpha_i}$, y^β is defined similarly
- $perm(A)$ is a permament of a matrix

Dual polynomials

Definitions

- $x^\alpha = \prod_{i=0}^{n} x_i^{\alpha_i}$, y^β is defined similarly
- $perm(A)$ is a permament of a matrix
- $coeff(y^\beta, g)$ is a coefficient of monomial y^β in g.

Dual polynomials

Definitions

- $x^{\alpha} = \prod_{i=0}^{n} x_i^{\alpha_i}$, y^{β} is defined similarly
- $perm(A)$ is a permament of a matrix
- $coeff(y^{\beta}, g)$ is a coefficient of monomial y^{β} in g.

Theorem

If $||\alpha||_1 = ||\beta||_1$, then
$(\prod_{j=1}^{m} \beta_j!)coeff(y^{\beta}, g) = (\prod_{i=1}^{n} \alpha_i!)coeff(x^{\alpha}, g*) = perm(A^{\alpha,\beta})$

Dual polynomials

Definitions

- $x^\alpha = \prod_{i=0}^{n} x_i^{\alpha_i}$, y^β is defined similarly
- $perm(A)$ is a permament of a matrix
- $coeff(y^\beta, g)$ is a coefficient of monomial y^β in g.

Theorem

If $||\alpha||_1 = ||\beta||_1$, then
$(\prod_{j=1}^{m} \beta_j!)coeff(y^\beta, g) = (\prod_{i=1}^{n} \alpha_i!)coeff(x^\alpha, g*) = perm(A^{\alpha,\beta})$

If $\beta = \{0,1\}^E(G)$, $\alpha = 1^V G$, and M is an incidence matrix (M^β indicates a subgraph) then f satisfies Combinatorial Nullstellensatz and polynomial dual to f is $f* = \prod_{uv \in E'}(x_u - x_v)$, that is a traditional graph polynomial of $G[E']$

Dual polynomials

Theorem: Alon-Tarsi [1992]

if D is an orientation of a graph H satisfying $deg_D^+(v) = t_v$ at each vertex $v \in V(H)$, then

$$|coeff(\prod_{v \in V(G)} x_v^{t_v i}, f^*)| = |EE(D) - EO(D)|$$

Dual polynomials

Theorem: Alon-Tarsi [1992]

if D is an orientation of a graph H satisfying $deg^+_D(v) = t_v$ at each vertex $v \in V(H)$, then

$$|coeff(\prod_{v \in V(G)} x_v^{t_{v\,i}}, f^*)| = |EE(D) - EO(D)|$$

Given that, and a previous theorem, we obtain:

$$(\prod_{j=1}^{m} t_{v\,i}!)coeff(y^\beta, f) = |coeff(\prod_{v \in V(G)} x_v^{t_{v\,i}}, f^*)| = |EE(D) - EO(D)| \neq 0$$