Sequences of points on a circle

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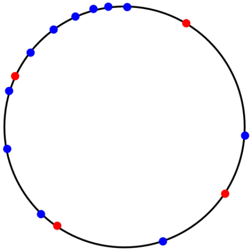
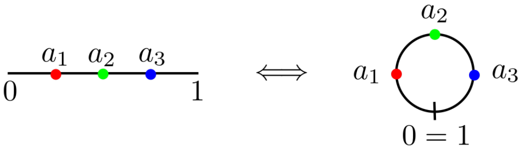


Sequences of points on a circle

N. G. de Brujin, P. Erdős, 1948

Consider a sequence $a = a_1, a_2, \ldots$ of real numbers modulo 1.

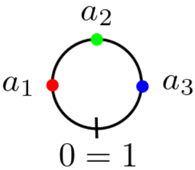


Points $a_1, \ldots, a_n$ divide the circle into n intervals.

Interval lengths

Let $M_n^1(a)$ and $m_n^1(a)$ denote the largest and the smallest interval length. Clearly:

$$n \cdot M_n^1(a) \geq 1 \geq n \cdot m_n^1(a)$$

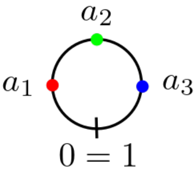


Here $M_3^1(a) = 0.5$ and $m_3^1(a) = 0.25$.

Interval lengths

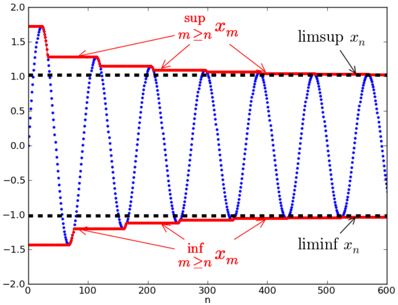
Let $M_n^r(a)$ and $m_n^r(a)$ denote the largest and the smallest total length of r consecutive intervals. Clearly:

$$n \cdot M_n^r(a) \geq r \geq n \cdot m_n^r(a)$$



Here $M_3^2(a) = 0.75$ and $m_3^2(a) = 0.5$.

Limit superior and limit inferior



source: Wikipedia

Interval lengths in the limit

Define:

$$\Lambda_r(a) = \limsup_{n\to\infty} nM_n^r(a)$$
$$\lambda_r(a) = \liminf_{n\to\infty} nm_n^r(a)$$

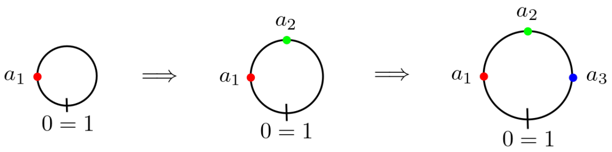
and:

$$\Lambda_r = \inf_a \Lambda_r(a)$$
$$\lambda_r = \sup_a \lambda_r(a)$$

Interval lengths in the limit

Intuition on $nM_n^r(a)$ and $nm_n^r(a)$:

1. We start with a_1 on a circle of circumference 1.
2. Each time we add a new point we increase the circumference by 1.
3. The circle's semantics never change - it always represents the mod 1 additive group.



The goal

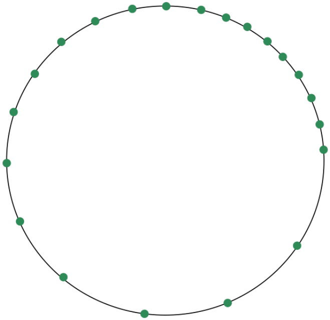
We will determine:

$$\Lambda_1 = 1/\ln 2$$
$$\lambda_1 = 1/\ln 4$$

And provide bounds on Λ_r and λ_r.

An important sequence

Let $a_k = \lg(2k-1)$.



Lemma

The numbers $a_1, ..., a_n$ are distinct and the sets $\{a_1, ..., a_n\}$ and $\{\lg(n), \lg(n+1), ..., \lg(2n-1)\}$ are equal (everything modulo 1). Moreover, the numbers $\lg(n), \lg(n+1), ..., \lg(2n-1)$ appear in this order on a circle.

An important sequence

Induced interval lengths are:

$$\lg((n+1)/n), \ldots, \lg(2n/(2n-1))$$

The largest and the smallest (scaled by n):

$$nM_n^1(a) = \frac{n \ln(1+1/n)}{\ln 2} \longrightarrow \frac{1}{\ln 2}$$
$$nm_n^1(a) = \frac{n \ln(1-1/2n)^{-1}}{\ln 2} \longrightarrow \frac{1}{\ln 4}$$

Consequently:

$$\Lambda_1(a) = 1/\ln 2$$
$$\lambda_1(a) = 1/\ln 4$$

Deriving Λ_1

We derive a lower bound for $\Lambda_1(a)$.

Let a be a sequence, $n \in N$ and g such that for all $n \leq k < 2n$:

$$g > kM_k^1(a) \tag{1}$$

Let $d_1, ..., d_n$ denote the interval lengths induced by the sequence $a_1, ..., a_n$ in descending order:

$$d_1 \geq d_2 \geq ... \geq d_n$$

Also:

$$d_1 + ... + d_n = 1 \tag{2}$$

Deriving Λ_1

Now incrementally insert points $a_{n+1}, ..., a_{2n-1}$. Since any of these points "destroys" at most one interval, we're going to have:

$$\begin{aligned} M_n^1(a) &\geq d_1 \\ M_{n+1}^1(a) &\geq d_2 \\ &\dots \\ M_{2n-1}^1(a) &\geq d_n \end{aligned} \tag{3}$$

Deriving Λ_1

From (1): $g > kM^1_k(a)$ we have:

$$g\left(\frac{1}{n} + ... + \frac{1}{2n-1}\right) > M^1_n(a) + ... + M^1_{2n-1}(a)$$

From (3): $M^1_{n+k-1}(a) \geq d_k$ we have

$$g\left(\frac{1}{n} + ... + \frac{1}{2n-1}\right) > d_1 + ... + d_n$$

And from (2): $d_1 + ... + d_n = 1$:

$$g\left(\frac{1}{n} + ... + \frac{1}{2n-1}\right) > 1$$

Deriving Λ_1

Finally:

$$g > \left(\frac{1}{n} + ... + \frac{1}{2n-1}\right)^{-1}$$

And since it works for any $g > kM_k^1(a)$, there must exist $n \leq k < 2n - 1$ such that:

$$kM_k^1(a) \geq \left(\frac{1}{n} + ... + \frac{1}{2n-1}\right)^{-1} = \sigma_n$$

Considering the properties of the harmonic sequence we can prove that $\sigma_n < 1/\ln 2$ and that $\sigma_n \longrightarrow 1/\ln 2$.

Deriving Λ_1

σ_n makes a lower bound on $nM_n^1(a)$, and since it converges to $1/\ln 2$, we have $\Lambda_1(a) \geq 1/\ln 2$.

Since it holds for any sequence a, we have:

$$\Lambda_1 = \inf_a \Lambda_1(a) \geq 1/\ln 2$$

Finally, because we have shown a sequence $a_k = \lg(2k-1)$ for which $\Lambda_1(a) = 1/\ln 2$, the bound must be tight:

$$\Lambda_1 = 1/\ln 2$$

$\square$

Bounding Λ_r

Using similar reasoning the authors show that:

$$kM_k^r(a) \geq \left(\frac{1}{rn} + \ldots + \frac{1}{r(n+1)-1}\right)^{-1}$$

Which is translated into a bound on Λ_r in the same manner:

$$\Lambda_r \geq 1/\ln(1+1/r)$$

But we don't know if it's tight because we didn't see a sequence which attains it.

Deriving λ_1

The authors use reasoning similar to what we have already seen:

1. For some range of k, bound $km_k^1(a)$ by a value which converges to $1/\ln 4$.
2. Deduce that $\lambda_1 \leq 1/\ln 4$.
3. Use the sequence $a_k = \lg(2k-1)$ to argue that $\lambda_1 = 1/\ln 4$

Bounding λ_r

The authors also argue that in general:

$$\lambda_r \leq \left(\frac{r}{r+1}\right) / \ln(1 + 1/r)$$

But we don't know if it's tight because we didn't see a sequence which attains it.

What's more?

Along with Λ_r and λ_r the authors also define μ_r:

$$\mu_r(a) = \limsup_{n\to\infty} M_n^r(a)/m_n^r(a)$$

$$\mu_r = \inf_a \mu_r(a)$$

which bounds from below the ratio of the largest interval to the smallest interval of any sequence in the limit.

They show that:

$$\mu_r \geq 1 + 1/r$$

which is tight when $r = 1$ due to the sequence $a_k = \lg(2k-1)$.

Thank you