On the discrepancy of circular sequences of reals

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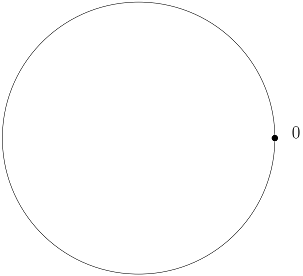
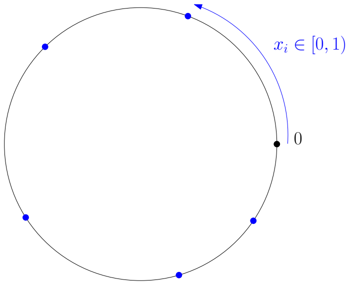
April 20, 2023

Circular sequences



- Circle C of circumference 1. Zero is on the right.

Circular sequences



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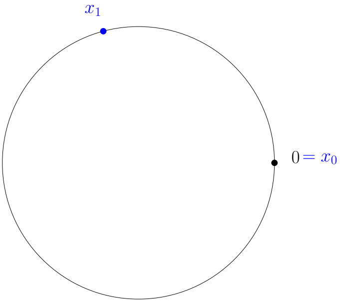
Circular sequences



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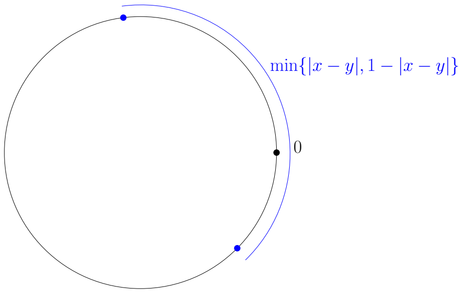
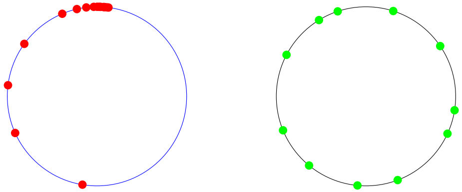
Circular sequences



- Circle C of circumference 1. Zero is on the right.
- Sequence of real numbers from $[0, 1)$, each number represents the distance from zero in a counterclockwise direction.
- Wlog $x_0 = 0$ and $x_1 \leq 1/2$.
- We define the distance between two points as $||x - y|| = \min\{|x - y|, 1 - |x - y|\}$.

Uniformly distributed sequences



Important in many branches of mathematics: ergodic theory, diophantine approximation, numerical integration, mathematical statistics, ...

How to measure the irregularity of distribution of sequences of reals?

Function ω

De Bruijn and Erdős proposed the following function:

$$\omega(x) = \liminf_{n\to\infty} \inf_{1\le i<j\le n} n||x_i - x_j||$$

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Theorem

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and this bound is tight — it is achieved by a sequence $x_n = \{\log_2(2n-1)\}$ *(braces denote the fractional part)*

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and this bound is tight — it is achieved by a sequence $x_n = \{\log_2(2n-1)\}$ *(braces denote the fractional part)*

This sequence is not well-distributed — consecutive points are very close together.

Function D

$$D(x) = \inf_n \inf_m n||x_m - x_{m+n}||$$

For this measure of discrepancy, any run of n consecutive terms of x must be just as well dispersed as the first n terms of x.

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Theorem

$$D(x) \leq \alpha_0 = \frac{3 - \sqrt{5}}{2} \approx 0.381966$$

and this bound is tight.

A sequence that achieves α_0

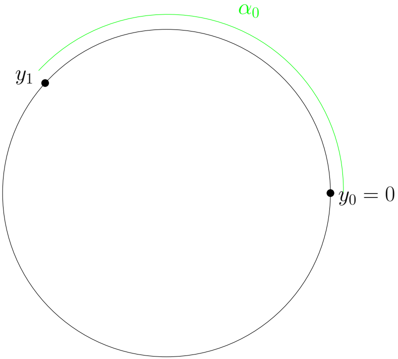
Let $y_n = \{n\alpha_0\}$.

Claim

$D(y) = \alpha_0$

Note that this sequence is quite well dispersed.

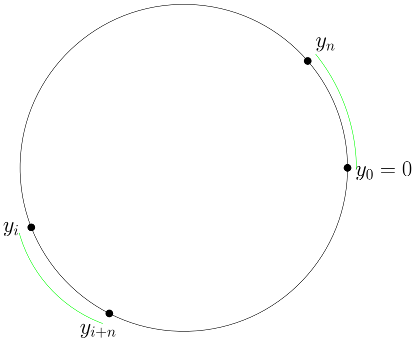
Observations



Observation

$||y_1 - y_0|| = \alpha_0 \implies D(y) \leq \alpha_0$

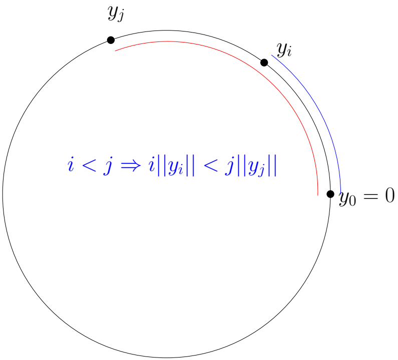
Observations



Observation

$n||y_{i+n} - y_i|| = n||y_n - y_0|| \implies$ *it is sufficient to focus on terms* $n||y_n||$

Observations



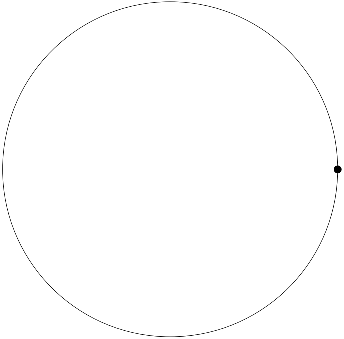
Observation

If $i < j$ and $||y_i|| < ||y_j||$, then $i||y_i|| < j||y_j||$. Therefore, it is sufficient to focus on terms y_n, such that $||y_s|| > ||y_n||$ for all $s < n$.

Which y_n satisfy the last property?

Leaders

$y_k = \{k\alpha_0\}$



Leaders:

Leaders

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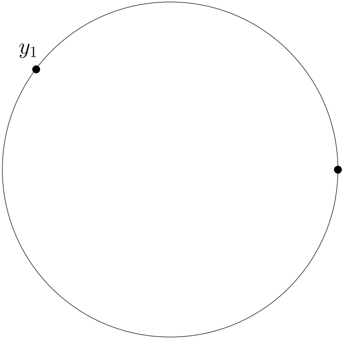
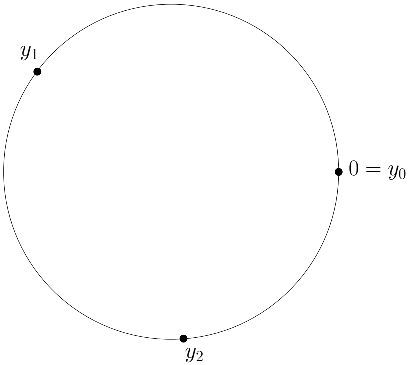


Leaders:

1

Leaders

$y_k = \{k\alpha_0\}$

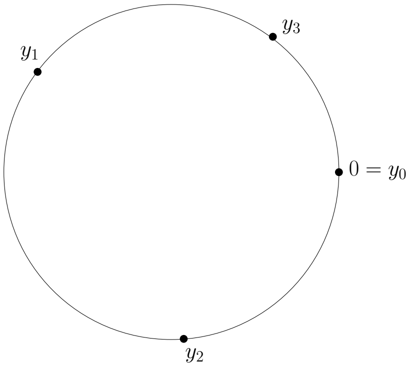


Leaders:

1 2

Leaders

$y_k = \{k\alpha_0\}$



Leaders:

1 2 3

Leaders

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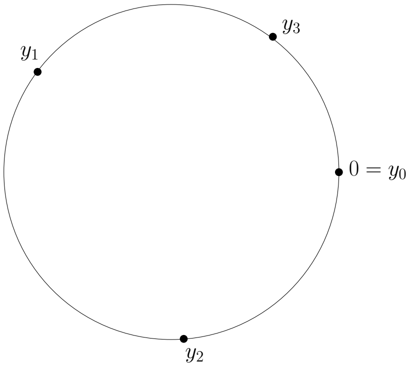

Leaders:

1 2 3

y_3 lies between y_0 and y_1 ...

Leaders

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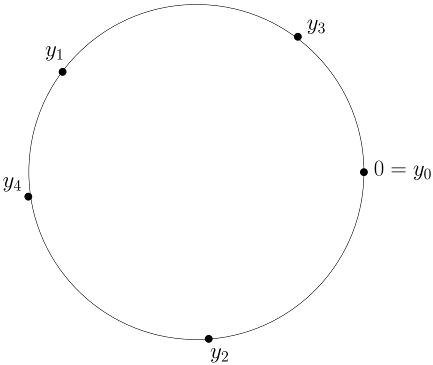


Leaders:

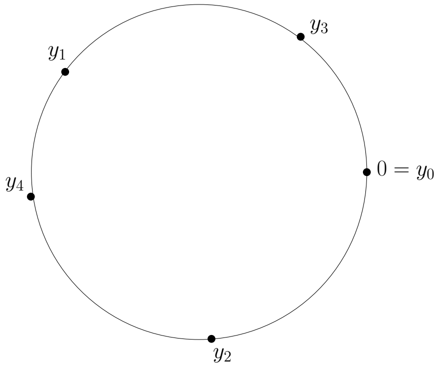
1 2 3

y_3 lies between y_0 and y_1 ...

... so y_4 lies between y_1 and y_2.

Leaders

$y_k = \{k\alpha_0\}$



Leaders:

1 2 3

$y_3 \in (y_0, y_1)$

$y_4 \in (y_1, y_2)$

Leaders

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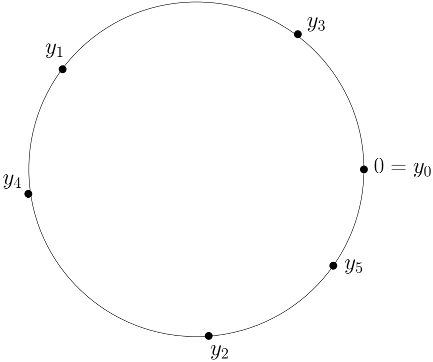


Leaders:

1 2 3

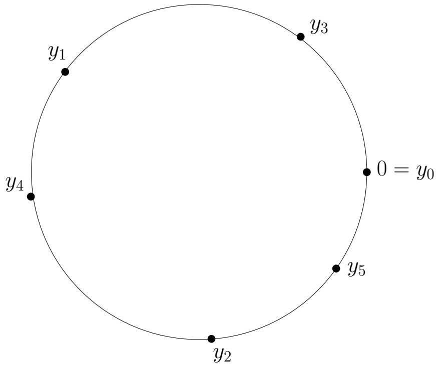
$y_3 \in (y_0, y_1)$

$y_4 \in (y_1, y_2)$

$y_5 \in (y_2, y_3) \ni 0$

Leaders

$y_k = \{k\alpha_0\}$



Leaders:

1 2 3 5

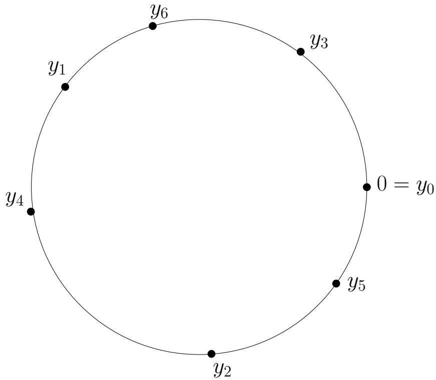
$y_3 \in (y_0, y_1)$

$y_4 \in (y_1, y_2)$

$y_5 \in (y_2, y_0)$

Leaders

$y_k = \{k\alpha_0\}$



Leaders:

1 2 3 5

$y_3 \in (y_0, y_1)$

$y_4 \in (y_1, y_2)$

$y_5 \in (y_2, y_0)$

$y_6 \in (y_3, y_1)$

Leaders

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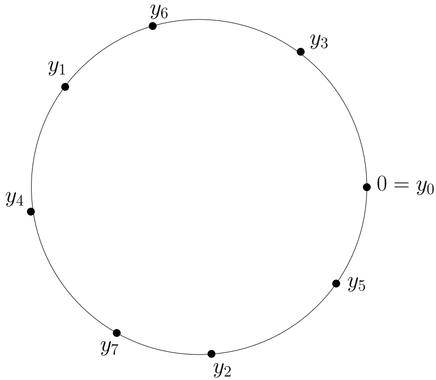

Leaders:

1 2 3 5

$y_3 \in (y_0, y_1)$

$y_4 \in (y_1, y_2)$

$y_5 \in (y_2, y_0)$

$y_6 \in (y_3, y_1)$

$y_7 \in (y_4, y_2)$

Leaders

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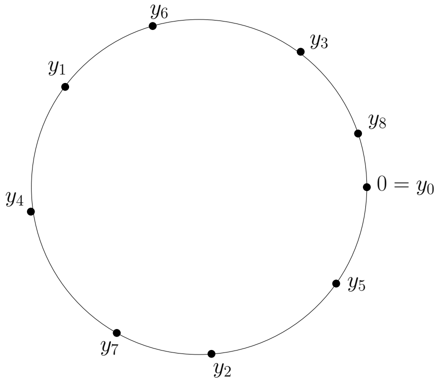


Leaders:

1 2 3 5

$y_3 \in (y_0, y_1)$

$y_4 \in (y_1, y_2)$

$y_5 \in (y_2, y_0)$

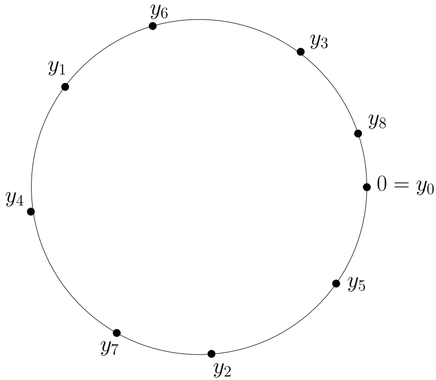
$y_6 \in (y_3, y_1)$

$y_7 \in (y_4, y_2)$

$y_8 \in (y_5, y_3) \ni 0$

Leaders

$y_k = \{k\alpha_0\}$



Leaders:

1 2 3 5 8

$y_3 \in (y_0, y_1)$

$y_4 \in (y_1, y_2)$

$y_5 \in (y_2, y_0)$

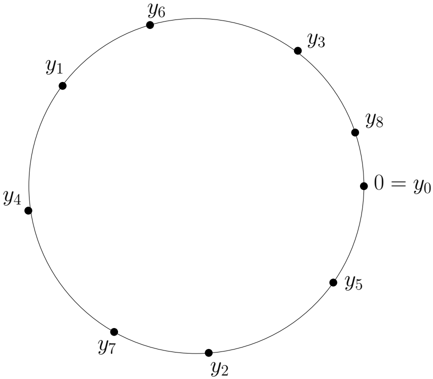
$y_6 \in (y_3, y_1)$

$y_7 \in (y_4, y_2)$

$y_8 \in (y_0, y_3)$

Leaders

$y_k = \{k\alpha_0\}$



Leaders:

1 2 3 5 8

$y_3 \in (y_0, y_1)$

$y_4 \in (y_1, y_2)$

$y_5 \in (y_2, y_0)$

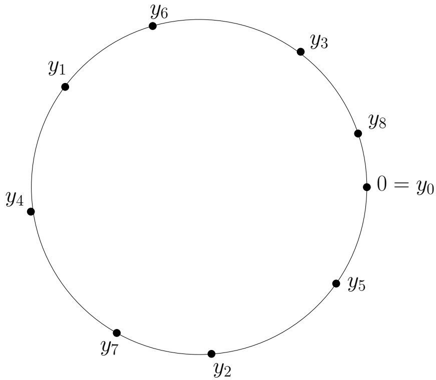
$y_6 \in (y_3, y_1)$

$y_7 \in (y_4, y_2)$

$y_8 \in (y_0, y_3)$

Leaders

$y_k = \{k\alpha_0\}$



Leaders:

1 2 3 5 8

$y_3 \in (y_0, y_1)$ from above

$y_4 \in (y_1, y_2)$

$y_5 \in (y_2, y_0)$ from below

$y_6 \in (y_3, y_1)$

$y_7 \in (y_4, y_2)$

$y_8 \in (y_0, y_3)$ from above

Facts about Fibonacci numbers

Folklore

$$F_n = \frac{1}{\sqrt{5}}(\tau^n - \sigma^n)$$

where $\tau = (1+\sqrt{5})/2$ and $\sigma = (1-\sqrt{5})/2$. Thus $\tau + \sigma = 1$ and $\tau\sigma = -1$.

Observation

$$\begin{aligned} F_t\,\alpha_0 - F_{t-2} &= \frac{1}{\sqrt{5}}\left(\frac{1}{\tau^2}(\tau^t - \sigma^t) - (\tau^{t-2} - \sigma^{t-2})\right) \\ &= \frac{1}{\sqrt{5}}(\sigma^{t-2} - \sigma^{t+2}) \\ &= \frac{1}{\sqrt{5}}(1-\sigma^4)\,\sigma^{t-2} = \sigma^t. \end{aligned}$$

Conclusions

$$
\begin{aligned}
F_t \, ||F_t \alpha_0|| &= |F_t \, \sigma^t| \\
&= \frac{1}{\sqrt{5}}(\tau^t \, \sigma^t - \sigma^{2t}) \\
&= \frac{1}{\sqrt{5}}((-1)^t - \sigma^{2t}) \\
&\leq \frac{1}{\sqrt{5}}(1 - \sigma^4) = \alpha_0.
\end{aligned}
$$

Proof that $D(x) \leq \alpha_0$

Suppose $x = (0, x_1, x_2, \ldots)$ has $D(x) = \alpha \geq \alpha_0 \approx 0.381966$

Observation

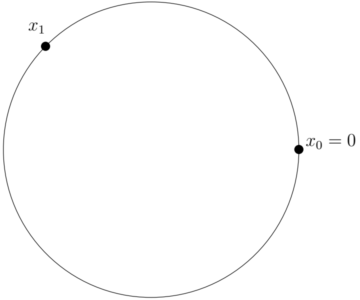
$$||x_m - x_{m+n}|| \geq \frac{\alpha}{n} \ \left(\geq \frac{\alpha_0}{n}\right)$$

for all m, n

We will show that this sequence follows the same order of terms around C as the sequence y.

The order of points

$D(x) = \alpha \geq \alpha_0 \Rightarrow ||x_i - x_{i+k}|| \geq \frac{\alpha}{k} \geq \frac{\alpha_0}{k}$



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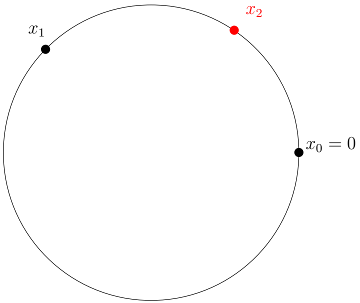


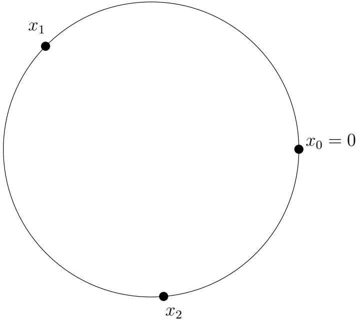
If x_2 went between x_0 and x_1, then

$$\frac{1}{2} \geq ||x_1|| = ||x_2|| + ||x_1 - x_2|| \geq \alpha(\frac{1}{2} + 1)$$

$$\Rightarrow \alpha \leq \frac{1}{3}$$

The order of points

$D(x) = \alpha \geq \alpha_0 \Rightarrow ||x_i - x_{i+k}|| \geq \frac{\alpha}{k} \geq \frac{\alpha_0}{k}$



– All 3 consecutive points have order $\langle 0, 1, 2 \rangle$

The order of points

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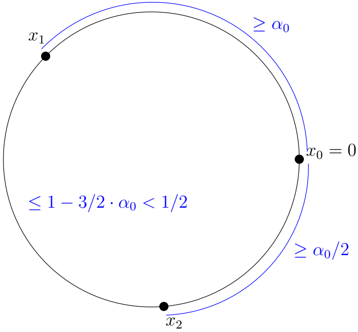

– For all i, x_{i+1} lies on the right side of x_i.
– All 3 consecutive points have order $\langle 0, 1, 2 \rangle$

The order of points

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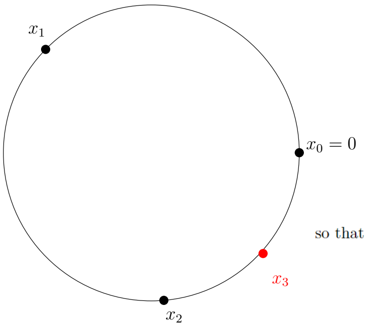

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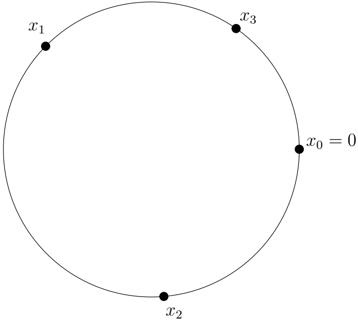
$$1 = ||x_1|| + ||x_1 - x_2|| + ||x_2 - x_3|| + ||x_3|| \geq \alpha(1 + 1 + 1 + \frac{1}{3}) = \alpha(\frac{10}{3})$$

so that

$$\alpha \leq \frac{3}{10} < \alpha_0$$

The order of points

$D(x) = \alpha \geq \alpha_0 \Rightarrow ||x_i - x_{i+k}|| \geq \frac{\alpha}{k} \geq \frac{\alpha_0}{k}$



– For all i, x_{i+1} lies on the right side of x_i.
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– All 4 consecutive points have order $\langle 0, 3, 1, 2 \rangle$

The order of points

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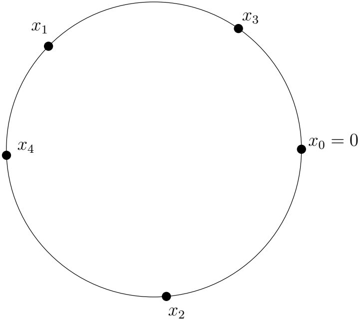

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Thus, x_4 lies between x_1 and x_2

The order of points

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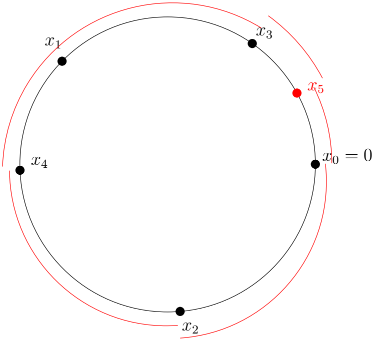

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- All 3 consecutive points have order $\langle 0, 1, 2 \rangle$
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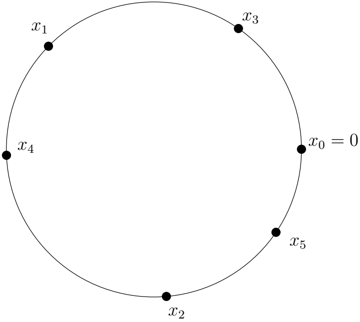
$$
\begin{aligned}
1 &= ||x_5|| + ||x_5 - x_3|| + ||x_3 - x_4|| + ||x_4 - x_2|| + ||x_2|| \\
&\geq \alpha\left(\tfrac{1}{5} + \tfrac{1}{2} + 1 + \tfrac{1}{2} + \tfrac{1}{2}\right) = \alpha\left(\tfrac{27}{10}\right)
\end{aligned}
$$

which implies

$$\alpha \leq \frac{10}{27} < 0.3704 < \alpha_0,$$

The order of points

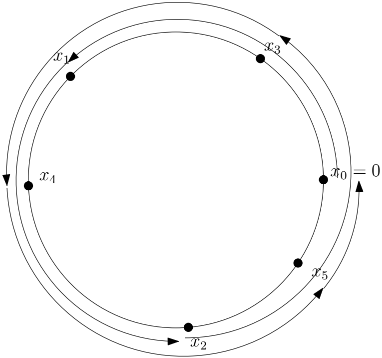
$D(x) = \alpha \geq \alpha_0 \Rightarrow ||x_i - x_{i+k}|| \geq \frac{\alpha}{k} \geq \frac{\alpha_0}{k}$



– For all i, x_{i+1} lies on the right side of x_i.
– All 3 consecutive points have order $\langle 0, 1, 2 \rangle$
– All 4 consecutive points have order $\langle 0, 3, 1, 2 \rangle$
– All 6 consecutive points have order $\langle 0, 3, 1, 4, 2, 5 \rangle$

The order of points

$D(x) = \alpha \geq \alpha_0 \Rightarrow ||x_i - x_{i+k}|| \geq \frac{\alpha}{k} \geq \frac{\alpha_0}{k}$



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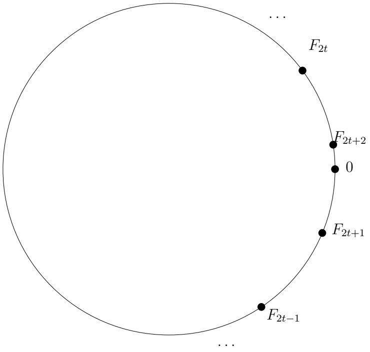
– All 6 consecutive points have order $\langle 0, 3, 1, 4, 2, 5 \rangle$

$$2 \geq \alpha(1 + 1 + 1 + 1 + 1 + \tfrac{1}{5}) = \tfrac{26}{5}\alpha$$

which implies $\alpha \leq \frac{10}{26} = \frac{5}{13}$.

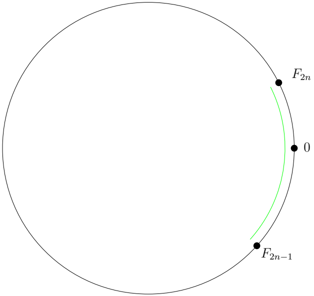
Generalization



We will show that

- The neighbours of zero are always two consecutive Fibonacci numbers. Fibonacci number with an odd index is the one below zero.
- F_{2k+1} is between F_{2k-1} and 0, while F_{2k} is between F_{2k-2} and 0.
- In general, for $F_k \leq N$, N lies between points $N - F_k$ and $N - F_k + F_{k-2}$.

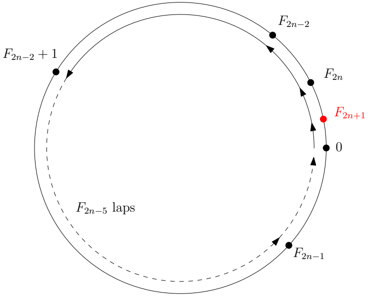
Proof for F_{2n+1}



Inductive assumptions:

- F_{2n-1} and F_{2n} are the neighbours of 0.
- F_{2n+1} lies between F_{2n-1} and F_{2n}.

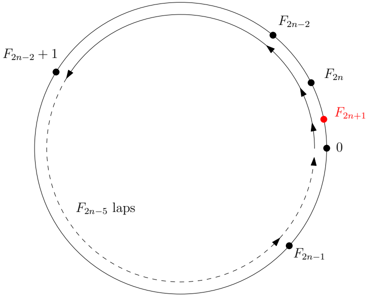
Proof for F_{2n+1}



Suppose that F_{2n+1} lies between 0 and F_{2n}. Then consider the following sequence of terms:

$$\langle 0, F_{2n+1}, F_{2n}, F_{2n-2}, F_{2n-2}+1, F_{2n-2}+2, F_{2n-2}+3, \ldots, F_{2n-1}-1, F_{2n-1}, 0\rangle$$

Proof for F_{2n+1}



It goes around C exactly F_{2n-5} times. Thus, because $||x_m - x_{m+n}|| \geq \frac{\alpha}{n}$, we have

$$\alpha\left(\frac{1}{F_{2n+1}} + \frac{1}{F_{2n-1}} + \frac{1}{F_{2n-1}} + F_{2n-3}\cdot 1 + \frac{1}{F_{2n-1}}\right) \leq F_{2n-5}$$

It turns out this equation does not hold. As a result, F_{2k+1} must lie between 0 and F_{2n-1}.

Proof for F_{2n+1}

In an analogous way we can prove that F_{2n} goes between F_{2n-2} and 0.

$$\langle 0, F_{2n}, F_{2n-1}, F_{2n-1} - 1, F_{2n-1} - 2, F_{2n-1} - 3, \ldots, F_{2n-2} + 1, F_{2n-2}, 0 \rangle$$

Bounds on α

Consider a sequence of points

$$\langle 0, 1, \ldots, F_{2n+1}, 0\rangle$$

It goes around C exactly F_{2n-1} times. Therefore

$$\alpha(F_{2n-1} \cdot 1 + \frac{1}{F_{2n+1}}) \leq F_{2n-1}.$$

which implies

$$\alpha \leq \frac{F_{2n+1}}{F_{2n+3}}$$

However,

$$\lim_{n\to\infty} \frac{F_{2n+1}}{F_{2n+3}} = \frac{1}{\tau^2} = \alpha_0$$

so $\alpha = \alpha_0$.