Flip distance to a non-crossing perfect matching

Demian Banakh
Department of Theoretical Computer Science
Jagiellonian University

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Non-crossing perfect matching

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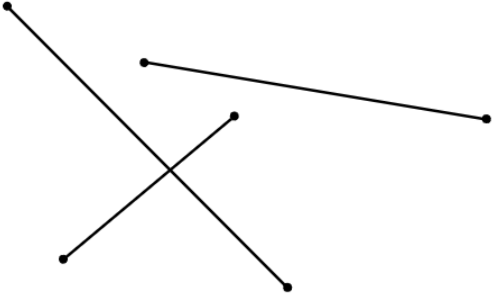

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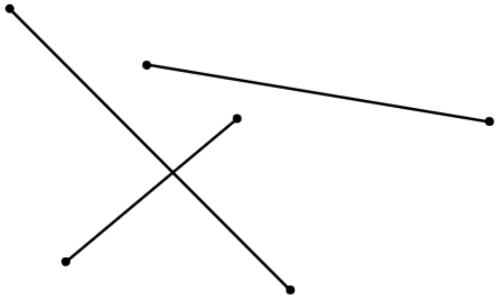

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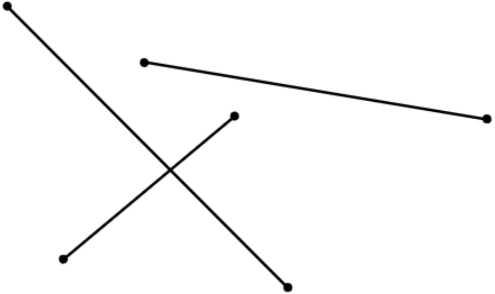
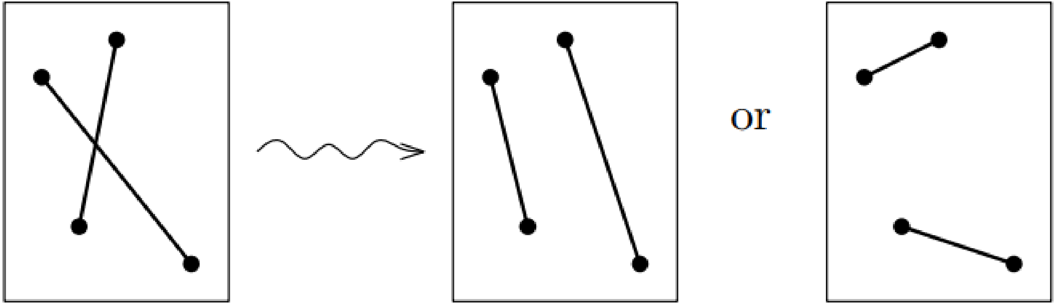
- Note: P always has a perfect non-crossing matching



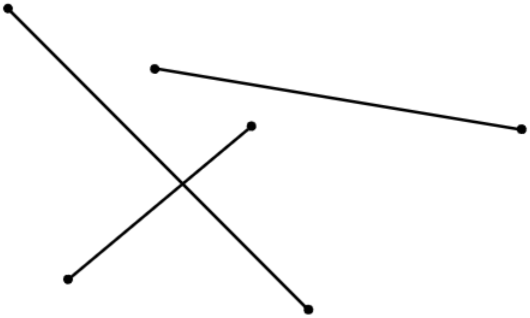
Flip operation

A flip operation replaces 2 crossing segments with 2 non-crossing ones on the same set of points.



Flip operation

- To prove that any P has a non-crossing matching, we start with any matching M



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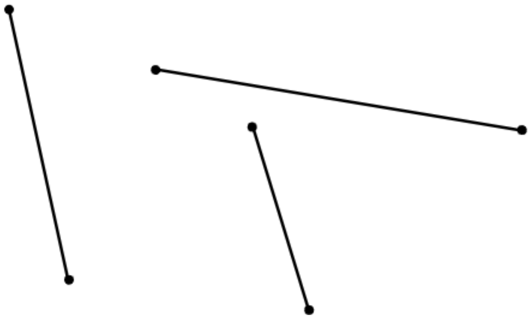
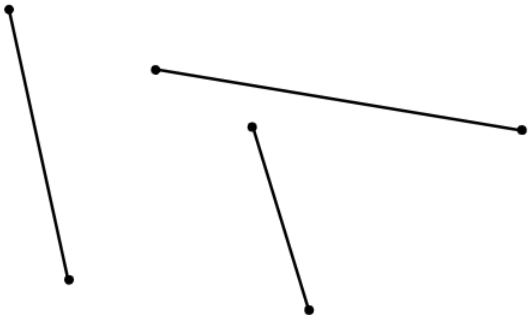


Flip operation

- To prove that any P has a non-crossing matching, we start with any matching M
- Perform flip operation while possible
- Since the total length of the segments decreases, this process has to end at some point



How many flips is necessary / sufficient in this algorithm, for fixed M?

Flip distance

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- Applications: e.g. improving approximate Euclidean TSP

Main results

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Theorem (Bonnet and Miltzow; 2016)

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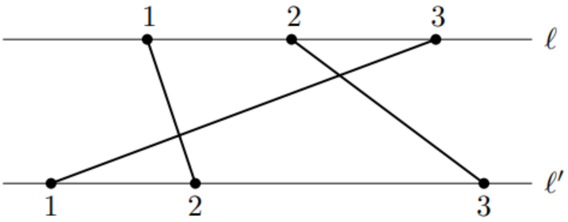

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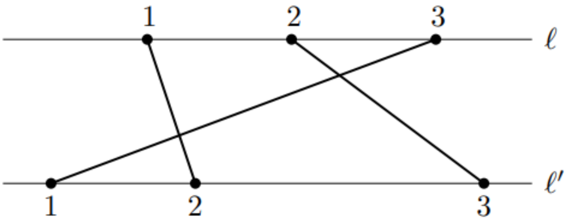

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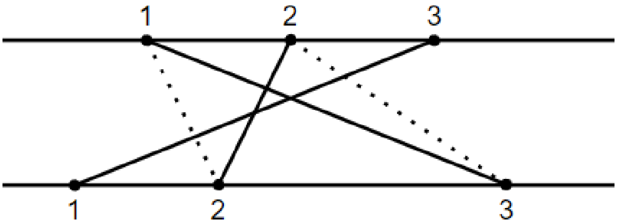

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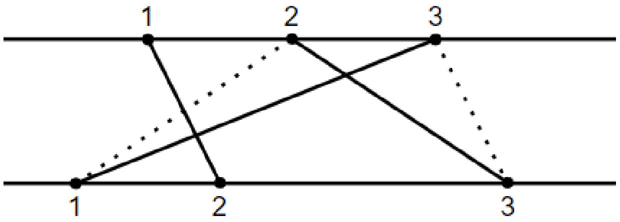

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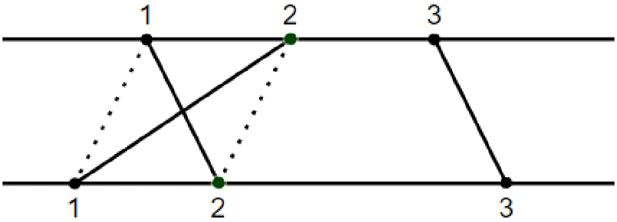

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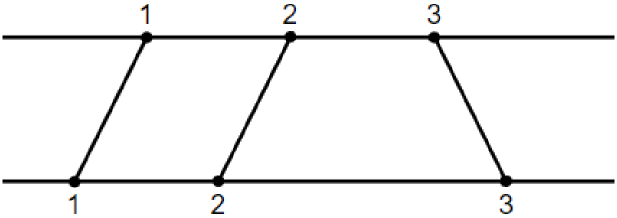

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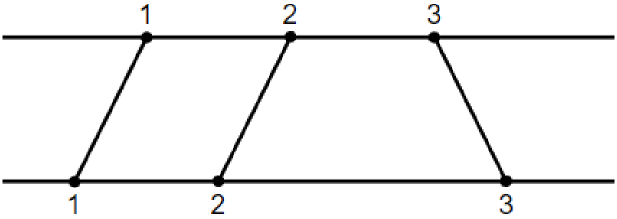

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- We can do the flips corresponding to single swaps in bubble-sort
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- Add random small offset to points positions to ensure no collinearity

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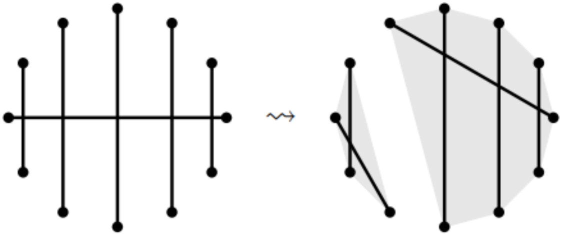

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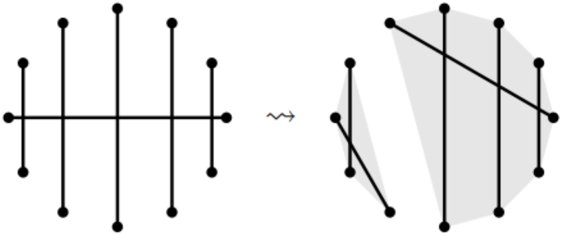

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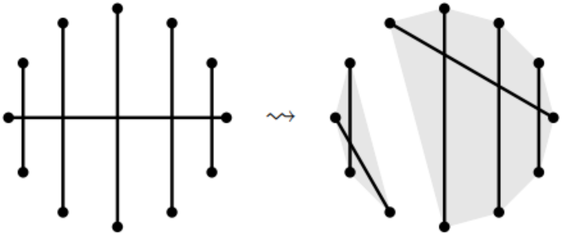

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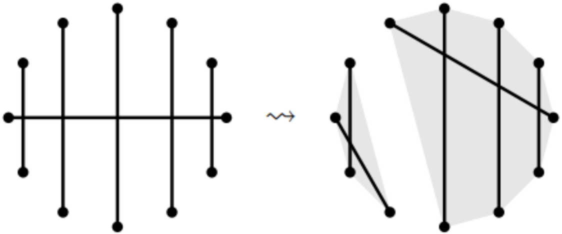

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- $[H(n) = 1 + H(k) + H(n-k)$ and $H(1) = 0] \implies H(n) = n - 1$

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Idea: Define some non-negative potential function $\Phi(M)$ bounded by function of n, so that any flip decreases the potential of the matching.

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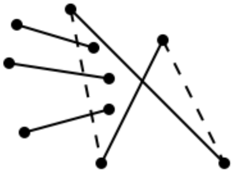

Figure 5: After the depicted flip, the number of crossings goes from 1 to 3.

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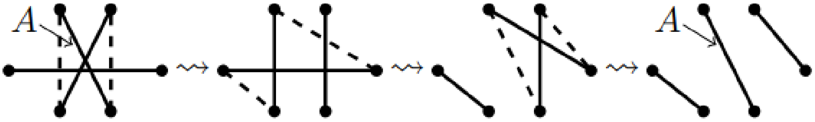

Figure 6: Segment A disappears and reappears.

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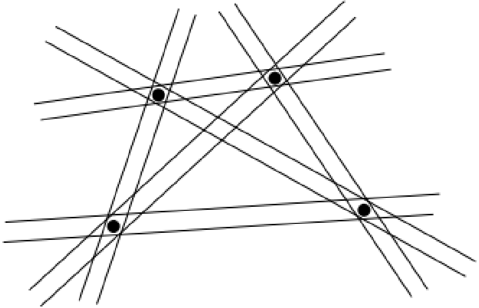
- Set of lines L: $\forall p, q \in P$ add 2 lines slightly above and below line pq



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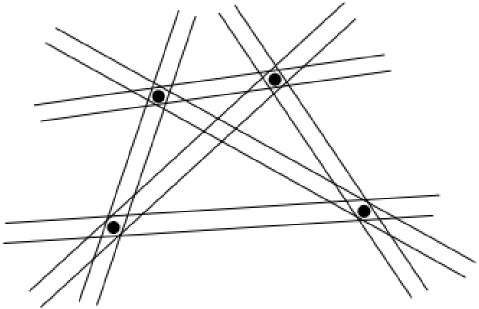

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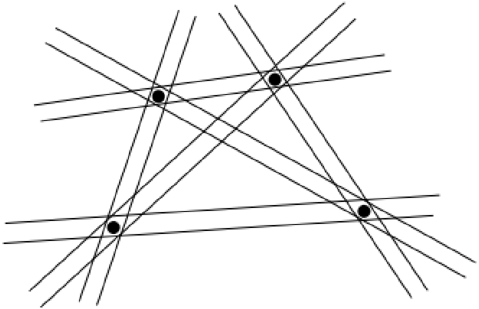

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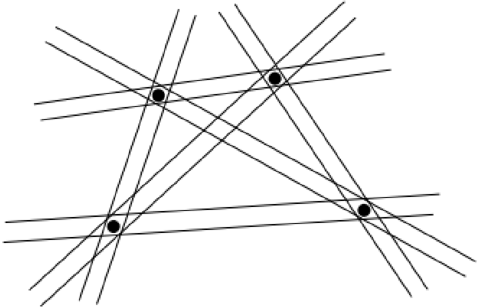

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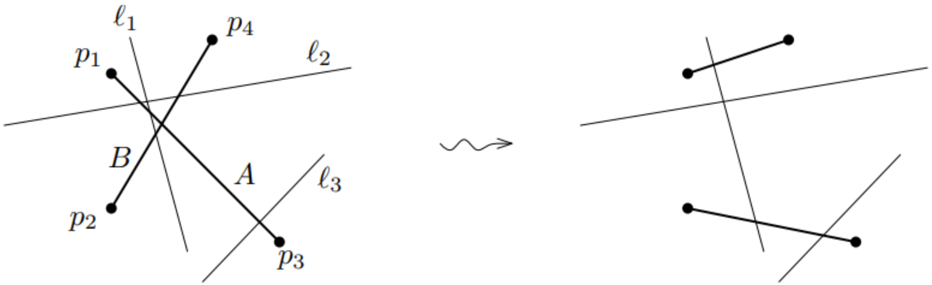

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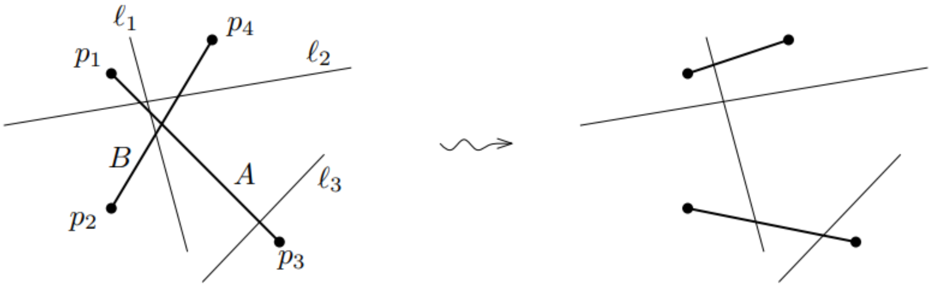

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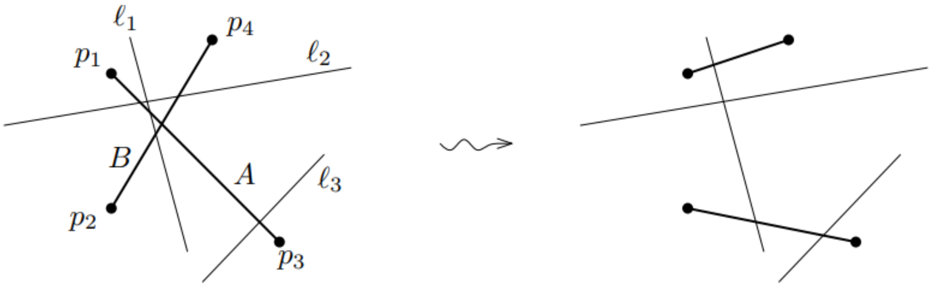

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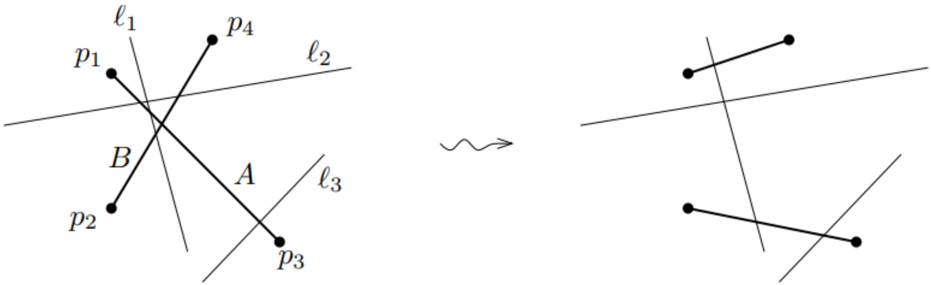

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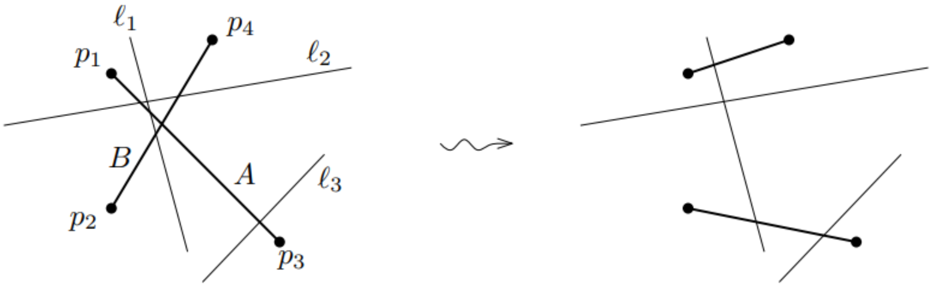
- L always contains at least 2 lines decreasing Φ by 2, for any crossing



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Claim:

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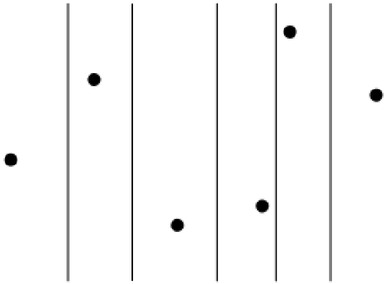

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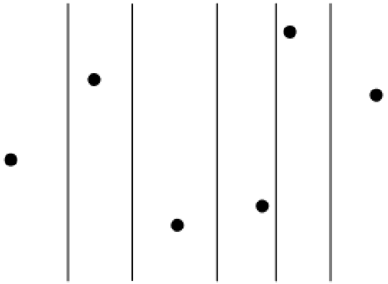

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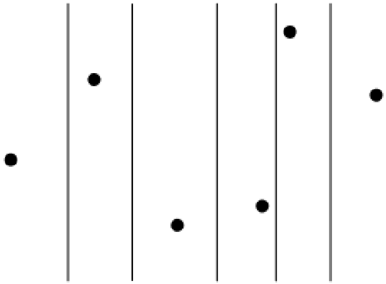

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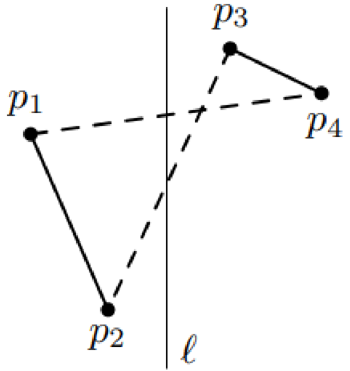
- Consider a crossing on points p_1, p_2, p_3, p_4 (in this order)



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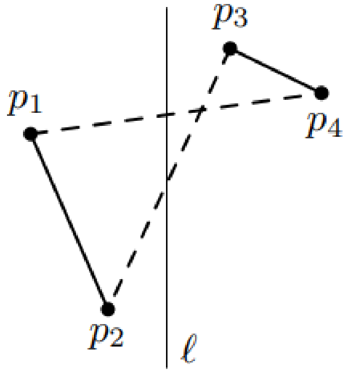
- Consider a crossing on points p_1, p_2, p_3, p_4 (in this order)
- There is at least one line separating p_2 and p_3, which loses 2 intersections after a flip



Conjecture

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- Easy to prove assuming all points P are in convex position
- Possible to show that $(1-\varepsilon)n^2 \leq g(n)$ for large enough n

Distinct flips

Definition

2 flips are *distinct* if the sets of 4 segments involved in the flips are different.

$g'(n)$ = longest sequence of distinct flips over all matchings on $2n$ points

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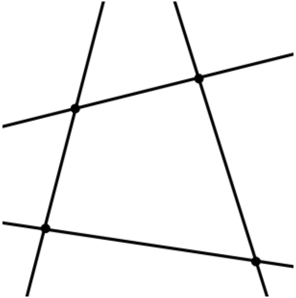
Theorem

$g'(n) = O(n^{\frac{8}{3}})$

Upper bound: $g'(n)$

- Consider a similar potential function as before
- Set of lines L: $\forall p, q \in P$ add line pq



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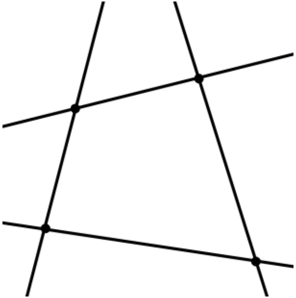

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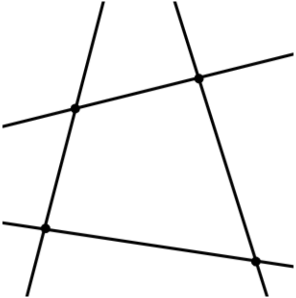
- $\Phi(M) \le n \cdot |L| = n \cdot \binom{2n}{2} \le 2n^3$ for any matching M



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Lemma

For any k, number of flips in any sequence with $|\Delta\Phi| \geq k$ is $O(\frac{n^3}{k})$.

Proof. $\Phi(M) = O(n^3)$.

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Proof. *Next slide.*

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Proof. *Next slide.*

When $k = n^{\frac{1}{3}}$, we get that the total number of distinct flips in any sequence is

$$O(\frac{n^3}{n^{\frac{1}{3}}}) + O(n^2 \cdot n^{\frac{2}{3}}) = O(n^{\frac{8}{3}})$$

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- Fix any $p_1, p_4 \in P$
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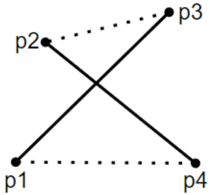

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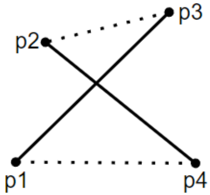

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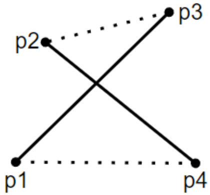
- p_1p_4 can be chosen in $O(n^2)$ ways, so in total $O(n^2k^2)$ flips in any sequence



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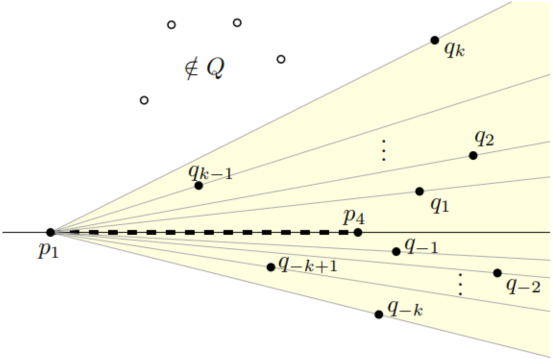

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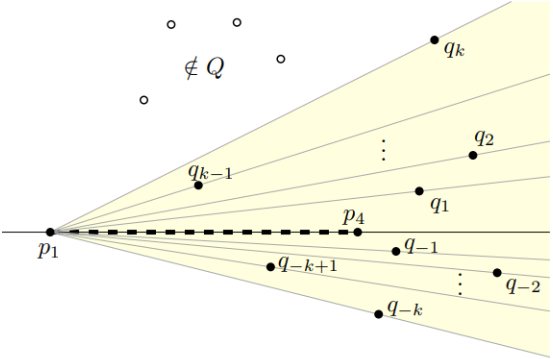
- Let Q be k closest points to the left and k closest points to the right ($2k$ total)



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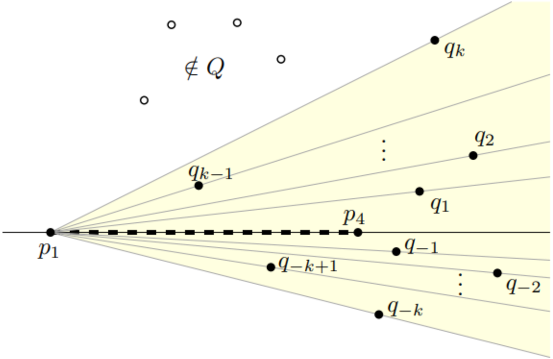

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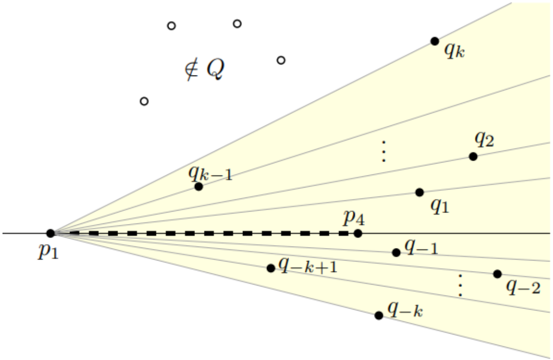

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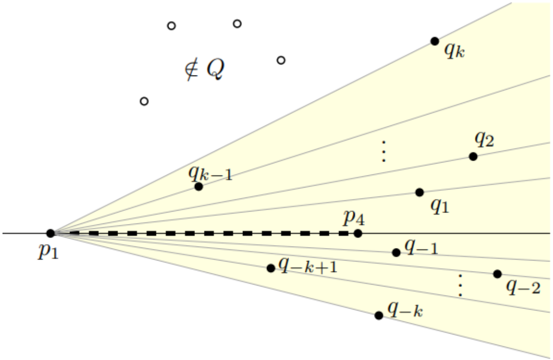

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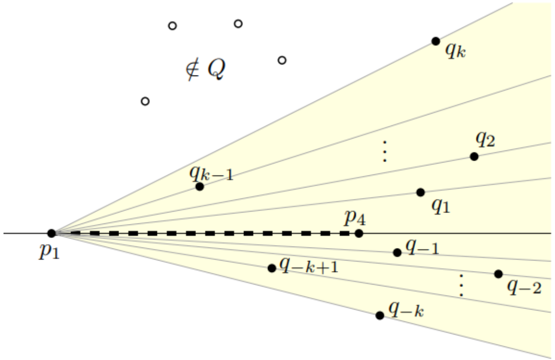

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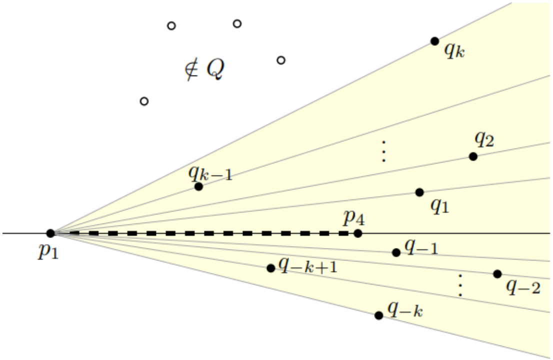
- It means $|\Delta\Phi| \geq k$ - contradiction



Distinct flips: conclusion

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- Repeated flips in a sequence appear to be pretty rare
- So it's believed that $g(n)$ should be $O(g'(n))$

References

Edouard Bonnet and Tillmann Miltzow (2016)
Flip Distance to a Non-crossing Perfect Matching
arXiv

Guilherme D. da Fonseca, Yan Gerard and Bastien Rivier (2023)
On the Longest Flip Sequence to Untangle Segments in the Plane
arXiv

The End