Edge Lower Bounds for List Critical Graphs, via Discharging

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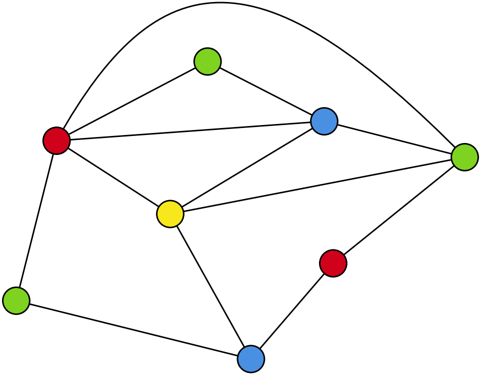
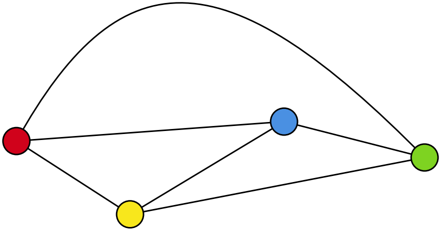
Introduction

A *k-coloring* of a graph G assigns a color from $\{1, \ldots, k\}$ to each vertex of G such that adjacent vertices get distinct colors. $\chi(G)$ is the least integer t such that G is t-colorable.



Introduction

G is *k-critical* when $\chi(G) = k$ and every proper subgraph H of G has $\chi(H) < k$. For a graph G with $\chi(G) = k$, every minimal subgraph H such that $\chi(H) = k$ must be k-critical.



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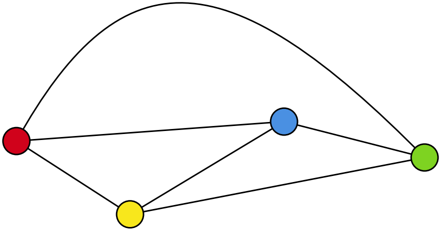
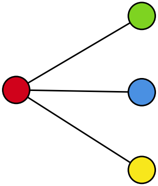


As a result, many questions about the chromatic number of a graph can be reduced to corresponding question about k-critical graphs.

Introduction

One natural question is how few edges an n-vertex k-critical graph G can have?

A k-critical graph G must have $\delta(G) \geq k-1$, which implies that $2||G|| \geq (k-1)|G|$.



$$d(v) = k-2 \implies \chi(G) = k-1$$

Introduction

Kostochka, Yancey (2014)

Every k-critical graph satisfies

$$||G|| \geq \left(\frac{k}{2} - \frac{1}{k-1}\right)|G| - \frac{k(k-3)}{2(k-1)}$$

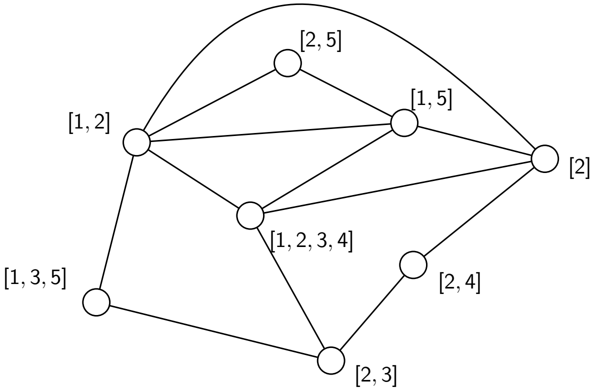
The bound is tight for $k = 4$ and $|G| \geq 6$. Also, for each $k \geq 5$, it is tight for infinitely many values of $|G|$.

The result has numerous applications to coloring problems. Thus, it is natural to consider the same question for more general types of coloring.

Introduction

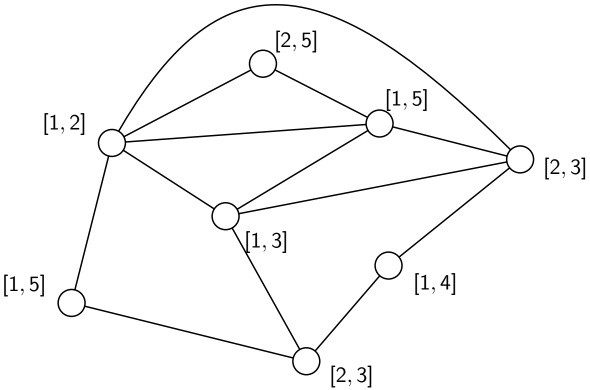
A *list assignment* L assigns to each vertex v of a graph G a set of allowable colors.

An *f-assignment* is a list assignment L such that $|L(v)| = f(v)$ for all v.



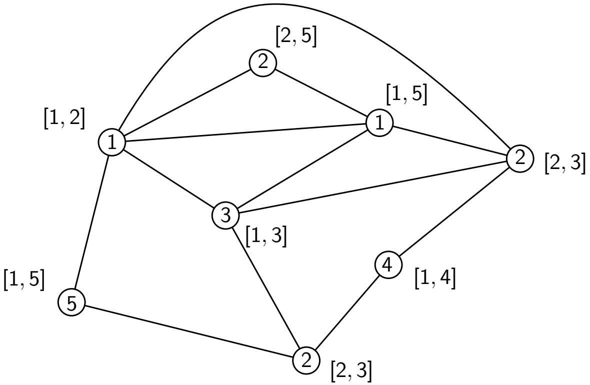
Introduction

A *k-assignment* is an f-assignment such that $f(v) = k$ for all v.



Introduction

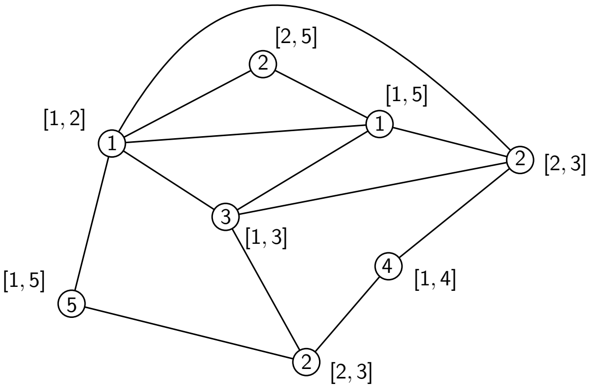
An *L-coloring* is a proper coloring φ of G such that $\varphi(v) \in L(v)$ for all v.



Introduction

A graph G is *k-choosable* if G is L-colorable for all k-assignments L.

The *list chromatic number* $\chi_L(G)$ of G is the least integer t such that G is t-choosable.



Introduction

From the definitions it is easy to check that always

$$\chi(G) \leq \chi_L(G) \leq \Delta(G) + 1$$

This implies that a lower bound for edges in a k-list critical also applies for k-critical graphs.

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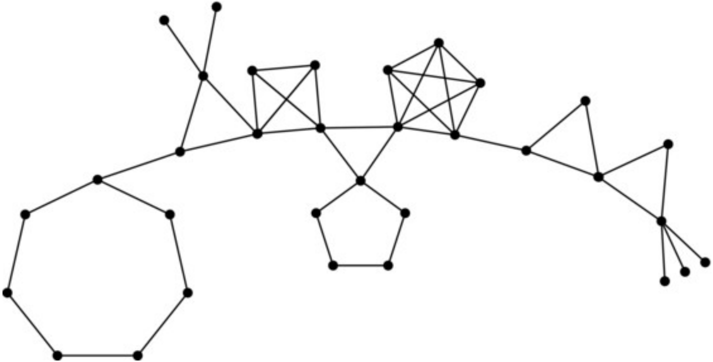
Gallai tree

A *Gallai tree* is a connected graph in which each block is a clique or an odd cycle.



Gallai tree with 15 blocks

Gallai tree

Let $\mathcal{T}_k$ denote the set of all Gallai trees of degree at most $k-1$, excluding K_k. For convenience, we write $d(G)$ to denote the average degree of G.

Lemma 2.1 (Gallai)

For $k \geq 4$ and $T \in \mathcal{T}_k$, we have

$$d(T) < k - 2 + \frac{2}{k-1}$$

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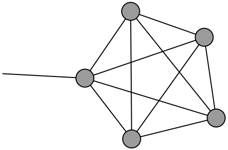
Suppose the lemma is false and choose a minimal counterexample T. Now T has at least two blocks. Let B be an endblock of T.

Lemma 2.1

If $B = K_{k-1}$, remove all vertices of B from T to get T'.



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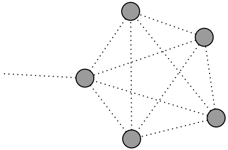


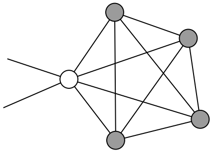
By the minimality of $|T|$, we have

$$\begin{aligned} 2||T|| - (k-1)(k-2) - 2 = 2||T'|| \\ &< \left(k - 2 + \frac{2}{k-1}\right)|T'| \\ &= \left(k - 2 + \frac{2}{k-1}\right)(|T| - (k-1)) \end{aligned}$$

Hence, $2||T|| < (k - 2 + \frac{2}{k-1})|T|$, a contradiction.

Lemma 2.1

If $B = K_t$ for some $t \in \{2, \dots, k-2\}$, remove the non-cut vertices of B.



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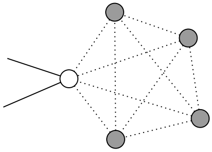

By the minimality of $|T|$, we have

$$2||T|| - t(t-1) = 2||T'|| < \left(k - 2 + \frac{2}{k-1}\right)|T'|$$
$$= \left(k - 2 + \frac{2}{k-1}\right)(|T| - (t-1))$$

Hence a contradiction

$$2||T|| < \left(k - 2 + \frac{2}{k-1}\right)|T|$$

Lemma 2.1

The case when B is an odd cycle is similar - less edges just make the inequality stronger. $\square$

Theorem 2.2

Theorem (Gallai, 1963)

Let $k \geq 4$ and G be a k-list critical graph, with $G \neq K_k$. We have

$$d(G) > k - 1 + \frac{k-3}{k^2-3}$$

We use the discharging method with initial charges $\mu(v) = d(v)$.

- each k^+-vertex gives charge $\frac{k-1}{k^2-3}$ to each of its $(k-1)$-neighbors
- vertices in each component of the subgraph induced by $(k-1)$-vertices share their charges equally.

Theorem 2.2

If v is a k^+-vertex, then

$$\begin{aligned}\mu^*(v) &\geq d(v) - \frac{k-1}{k^2-3}d(v)\\ &= \left(1 - \frac{k-1}{k^2-3}\right) d(v)\\ &\geq \left(1 - \frac{k-1}{k^2-3}\right) k\\ &= k - 1 + \frac{k-3}{k^2-3}\end{aligned}$$

Theorem 2.2

Let T be a component of the subgraph induced by $(k-1)$-vertices.

Lemma

T is a Gallai tree.

The vertices in T receive total charge

$$\frac{k-1}{k^2-3}\sum_{v\in V(T)}(k-1-d_T(v)) = \frac{k-1}{k^2-3}((k-1)|T|-2||T||)$$

After distributing the charges, each vertex in T receives charge

$$\frac{k-1}{k^2-3}((k-1)-d(T))$$

By Lemma 2.1 this is greater than

$$\frac{k-1}{k^2-3}\left((k-1)-\left(k-2+\frac{2}{k-1}\right)\right) = \frac{k-1}{k^2-3}\cdot\frac{k-3}{k-1} = \frac{k-3}{k^2-3}$$

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Refined bound on $||T||$

There are two places where the bound in Theorem 2.2 could be improved. Either:

1. the bound in Lemma 2.1 is loose
2. many k^+-vertices finish with extra charge

Refined bound on $||T||$

There are two places where the bound in Theorem 2.2 could be improved. Either:

1. the bound in Lemma 2.1 is loose ($q(T)$ is small)
2. many k^+-vertices finish with extra charge ($q(T)$ is large)

A good way to quantify this slackness is with the parameter $q(T)$, which denotes the number of non-cut vertices in T that appear in copies of K_{k-1}.

Refined bound on $||T||$

Without more reducible configurations we can't hope to prove $d(T) < k - 3$, since T could be a copy of K_{k-2}. This is why the next bound has the form

$$2||T|| \leq (k - 3 + p(k))|T|$$

It is helpful to handle separately the cases $K_{k-1} \not\subseteq T$ and $K_{k-1} \subseteq T$. The former is simpler, since it implies $q(T) = 0$.

Lemma 3.1

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Let $p : \mathbb{N} \to \mathbb{R}, f : \mathbb{N} \to \mathbb{R}$. For all $k \geq 5$ and $T \in \mathcal{T}_k$ with $K_{k-1} \nsubseteq T$, we have

$$2||T|| \leq (k - 3 + p(k))|T| + f(k)$$

whenever p and f satisfy all of the following conditions:

1. $p(k) \geq \frac{-f(k)}{k-2}$
2. $p(k) \geq \frac{-f(k)}{5} + 5 - k$
3. $0 \geq f(k) \geq -k + 2$
4. $p(k) \geq \frac{3}{k-2}$

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Suppose the lemma is false and choose a minimal counterexample T.

Lemma 3.1

- T is K_t for some $t \in \{2, \ldots, k-2\}$ - contradiction with (3) after substituting (1)
- T is C_{2r+1} - contradiction with (2)

Lemma 3.1

Let D be an induced subgraph such that $T' = T \backslash D$ is connected. By the minimality of T we have

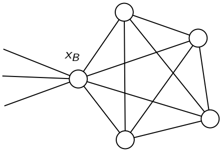
$$2||T'|| \leq (k - 3 + p(k))|T'| + f(k)$$

Since T is a counterexample, subtracting this inequality gives

$$2||T|| - 2||T'|| > (k - 3 + p(k))|D|$$

Lemma 3.1

Suppose T has an endblock B that is K_t for some $t \in \{3, \ldots, k-3\}$; let x_B be a cut vertex of B and let $D = B - x_B$.



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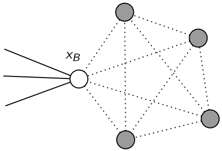


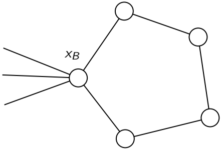
We get

$$2||T|| - 2||T'|| = t(t-1) > (k - 3 + p(k))(t-1)$$

which is a contradiction, since $t \leq k-3$ and $p(k) > 0$.

Lemma 3.1

Suppose that B is an odd cycle. Again, let $D = B - x_B$.



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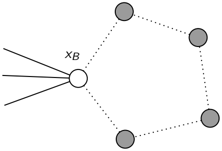


Now we get

$$2||T|| - 2||T'|| = 2|B| > (k - 3 + p(k))(|B| - 1)$$

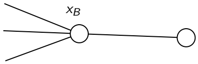
This simplifies to

$$|B| < 1 + \frac{2}{k - 5 + p(k)}$$

which is a contradiction, since the denominator is at least 1 (using (4)).

Lemma 3.1

Suppose that B is K_2.



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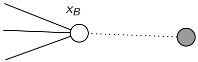


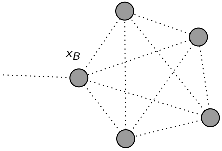
Now we get

$$2||T|| - 2||T'|| = 2 > k - 3 + p(k)$$

which is again a contradiction, since $k \geq 5$ and $p(k) > 0$.

Lemma 3.1

In the case where $B = K_{k-2}$ we will remove x_B from T as well, so we let $D = B$.



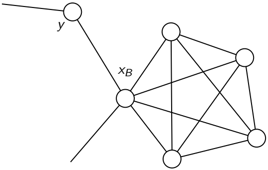
We have 2 subcases. When $d_T(x_B) = k - 2$, we have

$$(k-2)(k-3) + 2 > (k - 3 + p(k))(k-2)$$

contradicting (4).

Lemma 3.1

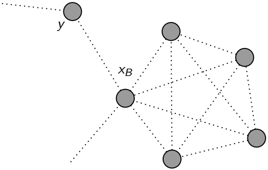
The only remaining case is when B is K_{k-2} and $d_T(x_B) = k - 1$. We may assume that every endblock of T is a copy of K_{k-2} that shares a vertex with an odd cycle.



Choose an endblock B that is the end of a longest path in the block-tree of T. Consider a neighbor y of x_B on the odd cycle.

Lemma 3.1

If y lies only on the cycle, let $D = B \cup \{y\}$.



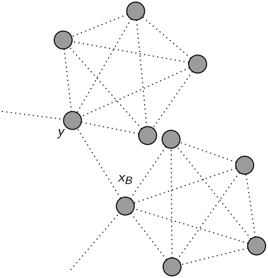
We get

$$(k-2)(k-3) + 2 \cdot 3 > (k - 3 + p(k))(k-1)$$

This eventually, by (4), leads to $6 > k + \frac{3}{k-2}$, which is a contradiction.

Lemma 3.1

If y lies also in an endblock A that is a copy of K_{k-2} (because B is at the end of a longest path), let $D = B \cup A \cup \{y\}$.



We get

$$2(k-2)(k-3) + 2 \cdot 3 > (k-3+p(k)) \cdot 2(k-2)$$

which simplifies to $3 > (k-2)p(k)$, again contradicting (4). □

Lemma 3.2

Lemma 3.2

Let $p : \mathbb{N} \to \mathbb{R}, f : \mathbb{N} \to \mathbb{R}, h : \mathbb{N} \to \mathbb{R}$. For all $k \geq 5$ and $T \in \mathcal{T}_k$ with $K_{k-1} \subseteq T$, we have

$$2||T|| \leq (k - 3 + p(k))|T| + f(k) + h(k)q(T)$$

whenever p, f and h satisfy all of the following conditions:

1. $f(k) \geq (k-1)(1 - p(k) - h(k))$
2. $p(k) \geq \frac{3}{k-2}$
3. $p(k) \geq h(k) + 5 - k$
4. $p(k) \geq \frac{2+h(k)}{k-2}$
5. $(k-1)p(k) + (k-3)h(k) \geq k + 1$

The proof is similar to that of Lemma 3.1.

Refined bound on $||T||$

Lemmas 3.1 and 3.2 give the tightest bounds for different values of $p(k)$. For the discharging part, it will be convenient to use optimal values that work in both lemmas:

- $p(k) = \frac{3k-5}{k^2-4k+5}$
- $f(k) = -\frac{2(k-1)(2k-5)}{k^2-4k+5}$
- $h(k) = \frac{k(k-3)}{k^2-4k+5}$

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Corollary 3.3

For $k \geq 5$ and $T \in \mathcal{T}_k$ with $K_{k-1} \subseteq T$, we have

$$2||T|| \leq \left(k - 3 + \frac{3k-5}{k^2-4k+5}\right)|T| - \frac{2(k-1)(2k-5)}{k^2-4k+5} + \frac{k(k-3)}{k^2-4k+5}q(T)$$

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Discharging overview

Let $\mathcal{T}(G)$ be a graph induced on $(k-1)$-vertices from G and $Q(T)$ be a set of vertices contained in some K_{k-1} in $\mathcal{T}(G)$.

Note that in the proof of Gallai's bound, all $(k+1)^+$-vertices finish with extra charge. The idea is to have the k^+-vertices give slightly less charge ϵ. The additional charge γ needed in $(k-1)$-vertices will be sent via each edge incident to a vertex in $Q(T)$, i.e., one counted by $q(T)$. We will show that k-vertices do not send too much charge in this step.

Discharging overview

Let $\mathcal{B}(G)$ be a bipartite graph with one part being the k-vertices of G and the other part the components of $\mathcal{T}(G)$. Put an edge between v and T if and only of $N(v) \cap Q(T) \neq \emptyset$.

Discharging overview

Let $\mathcal{B}(G)$ be a bipartite graph with one part being the k-vertices of G and the other part the components of $\mathcal{T}(G)$. Put an edge between v and T if and only of $N(v) \cap Q(T) \neq \emptyset$.

Lemma 5.2

$\mathcal{B}(G)$ is 2-degenerate for $k \geq 7$.

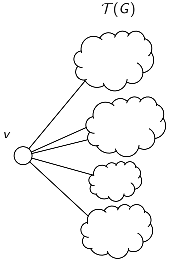
Discharging overview

1. Each k^+-vertex gives charge ϵ to its $(k-1)$-neighbors not in a K_{k-1}.
2. Each $(k+1)^+$-vertex gives charge γ to its $(k-1)$-neighbors in a K_{k-1}.
3. Repeat the following steps until $\mathcal{B}(G)$ is empty.
    1. For each component T with degree at most two:
        1. For each v such that $|N_G(v) \cap Q(T)| = 2$, pick one $x \in N_G(v) \cap Q(T)$ and send charge γ from v to x.
        2. Remove T from $\mathcal{B}(G)$.
    2. For each vertex v with degree at most two:
        1. Send charge γ from v to each $x \in N_G(v) \cap Q(T)$ for each neighbor T in $\mathcal{B}(G)$.
        2. Remove v from $\mathcal{B}(G)$.
4. Have the vertices in each component of $\mathcal{T}(G)$ share their charge equally.

Discharging overview

Lemma 5.1

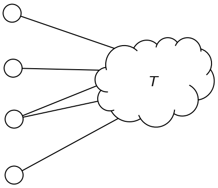
Each k-vertex v has at most 2 neighbors in any component $T \in \mathcal{T}(G)$. Further, v has at most one component in which it has 2 neighbors.



Thus, in step 3, each k-vertex v gives away a total of at most 3γ.

Discharging overview

In step 3.2, T receives charge γ via every edge in $\mathcal{B}(G)$, except possibly two.



Again by lemma 5.1, and combining it with step 2, T receives total charge $\gamma(q(T) - 2)$.

Analyzing the discharging

Theorem 4.1

Let $k \geq 7$ and $p : \mathbb{N} \to \mathbb{R}, f : \mathbb{N} \to \mathbb{R}, h : \mathbb{N} \to \mathbb{R}$. If G is a k-list critical graph, and $G \neq K_k$, then

$$d(G) \geq k - 1 + \frac{2 - p(k)}{k + 2 + 3h(k) - p(k)}$$

whenever p, f, h satisfy all of the following conditions:

1. $f(k) \geq (k-1)(1 - p(k) - h(k))$
2. $p(k) \geq \frac{3}{k-2}$
3. $p(k) \geq h(k) + 5 - k$
4. $p(k) \geq \frac{2+h(k)}{k-2}$
5. $(k-1)p(k) + (k-3)h(k) \geq k + 1$
6. $2(h(k)+1) + f(k) \leq 0$
7. $p(k) + (k-5)h(k) \leq k + 1$

Analyzing the discharging

Step 1 gives charge ϵ to a component T for every incident edge not ending in a K_{k-1}. The number of such edges is exactly

$$A(T) = \sum_{v \in V(T)} (k - 1 - d_T(v)) - q(T) = (k-1)|T| - 2||T|| - q(T)$$

When $K_{k-1} \subseteq T$, Lemma 3.2 gives

$$A(T) \geq (2 - p(k))|T| - f(k) - (h(k) + 1)q(T)$$

Hence, in total T receives charge at least

$$\epsilon A(T) + \gamma(q(T) - 2) \geq \epsilon(2 - p(k))|T| + q(T)(\gamma - \epsilon(h(k)+1)) - (2\gamma + \epsilon f(k))$$

Our goal is to maximize $\epsilon(2 - p(k))$, while ensuring that the final two terms are nonnegative ($\gamma = \epsilon(h(k) + 1)$ and condition (6)).

Analyzing the discharging

Thus T receives charge at least

$$\epsilon(2 - p(k))|T|$$

We also need each k-vertex to end with enough charge. Each of these loses at most $3\gamma + (k-3)\epsilon$, so we take

$$1 - (3\gamma + (k-3)\epsilon) = \epsilon(2 - p(k))$$

which gives exact values of $\epsilon = \frac{1}{k+2+3h(k)-p(k)}$ and $\gamma = \frac{h(k)+1}{k+2+3h(k)-p(k)}$.

Thus after discharging, each k-vertex and $(k-1)$-vertex finishes with charge at least $k - 1 + \epsilon(2 - p(k))$.

Analyzing the discharging

We use lemma 3.1 for the case when $K_{k-1} \nsubseteq T$, the rest follows similarly.

Analyzing the discharging

We use lemma 3.1 for the case when $K_{k-1} \not\subseteq T$, the rest follows similarly.

It remains to check that the $(k+1)^+$-vertices don't give away too much charge. Let v be such vertex, it ends with charge at least

$$d(v) - \gamma d(v) \geq (1-\gamma)(k+1)$$

After inserting the value of γ, the inequality between this and the required charge reduces to condition (7). □

Corollary 4.2

If G is a k-list critical graph, with $k \geq 7$, and $G \neq K_k$, then

$$d(G) \geq k - 1 + \frac{(k-3)(2k-5)}{k^3 + k^2 - 15k + 15}$$

Analyzing the discharging

Corollary 4.2

If G is a k-list critical graph, with $k \geq 7$, and $G \neq K_k$, then

$$d(G) \geq k - 1 + \frac{(k-3)(2k-5)}{k^3 + k^2 - 15k + 15}$$

Corollary 4.4

If G is a k-list critical graph, with $k \in \{5, 6\}$, and $G \neq K_k$, then

$$d(G) \geq k - 1 + \frac{(k-3)(2k-5)}{k^3 + 2k^2 - 18k + 15}$$

History of results

	k-Critical G				k-List Critical G		
k	Gallai [3] $d(G) \geq$	Kriv [11] $d(G) \geq$	KS [8] $d(G) \geq$	KY [10] $d(G) \geq$	KS [8] $d(G) \geq$	KR [6] $d(G) \geq$	Here $d(G) \geq$
4	3.0769	3.1429	—	3.3333	—	—	—
5	4.0909	4.1429	—	4.5000	—	4.0984	**4.1000**
6	5.0909	5.1304	5.0976	5.6000	—	5.1053	**5.1076**
7	6.0870	6.1176	6.0990	6.6667	—	6.1149	**6.1192**
8	7.0820	7.1064	7.0980	7.7143	—	7.1128	**7.1167**
9	8.0769	8.0968	8.0959	8.7500	8.0838	8.1094	**8.1130**
10	9.0722	9.0886	9.0932	9.7778	9.0793	9.1055	**9.1088**
15	14.0541	14.0618	14.0785	14.8571	14.0610	14.0864	**14.0884**
20	19.0428	19.0474	19.0666	19.8947	19.0490	19.0719	**19.0733**