On the structure of k-connected graphs without K_k minor

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k-connected graph

Definition

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Expansion lemma

If G is a k-connected graph, and G' is obtained from G by adding a new vertex y with at least k neighbors in G, then G' is k-connected.

Hadwiger's conjecture

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Proven for $k \leq 6$

Motivation

Question

What do K_k-minor-free graphs look like?

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Conjecture

A minimal counterexample to Hadwiger's conjecture is k-connected.

Proven for $k \leq 7$. A minimal counterexample is k-edge-connected.

Results

Theorem (Robersten et al.)

Let G be a simple 6-connected non-apex graph. If G contains three 4-cliques, say, L_1, L_2, L_3, such that $|L_i \cap L_j| \leq 2$, then G contains a K_6 as a minor.

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Theorem (Kawarabayashi, Toft)

Let G be a 7-connected graph with at least 19 vertices. Suppose G contains three 5-cliques, say, L_1, L_2, L_3, such that $|L_1 \cup L_2 \cup L_3| \geq 12$, then G contains a K_7-minor.

Results

Main theorem

Let G be a $(k+2)$-connected graph where $k \geq 5$. If G contains three k-cliques, say, L_1, L_2, L_3, such that $|L_1 \cup L_2 \cup L_3| \geq 3k - 3$, then G contains a K_{k+2}-minor.

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Main theorem

Let G be a $(k+2)$-connected graph where $k \geq 5$. If G contains three k-cliques, say, L_1, L_2, L_3, such that $|L_1 \cup L_2 \cup L_3| \geq 3k-3$, then G contains a K_{k+2}-minor.

Proof by contradiction. Assume G does not contain K_{k+2} as a minor.

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Let $P_1, \ldots, P_{k+2}$ be disjoint good paths. Contracting each path to a vertex results in a clique K_{k+2} corresponding to a K_{k+2} minor.

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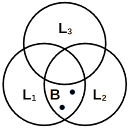
Lemma 1.

There do not exist $(k+2)$ mutually disjoint good paths in G.

Lemma 2.

$|L_i \cap L_j - L_k| \leq 1$

Otherwise $|L_i \cap L_j - L_k| \geq 2$, let $B \subseteq L_i \cap L_j - L_k$, $|B| = 2$. $G - B$ is k-connected.
There are k disjoint good paths from L_3 to $L_1 \cup L_2 - B$ in $G - B$, which imply $k + 2$ disjoint good paths in G.



Structure of the graph

Lemma (Mader, Robertson)

Let G be a graph, let Z_1, Z_2, ... , Z_h be subsets of V, and let $k \geq 1$ be an integer. Then exactly one of the following two statements holds:

1. There are k mutually disjoint good paths in G.
2. There exist $W \subseteq V$, a partition $Y_1, \ldots, Y_n$ of $V - W$ and subsets $X_i \subseteq Y_i$ such that:
   2.1 $|W| + \sum_{i=1}^{n} \lfloor \frac{1}{2} |X_i| \rfloor < k$
   2.2 no vertex in $Y_i - X_i$ has a neighbor in $V - (W \cup Y_i)$ and $Y_i \cap (L_1 \cup L_2 \cup L_3) \subseteq X_i$
   2.3 every good path in $G - W$ has an edge contained in some Y_i

Lemma 3.

There exist $W \subseteq V$, a partition $Y_1, \ldots, Y_n$ of $V - W$ and subsets $X_i \subseteq Y_i$ such that:

1. $|W| + \sum_{i=1}^{n} \lfloor \frac{1}{2}|X_i| \rfloor \leq k + 1$
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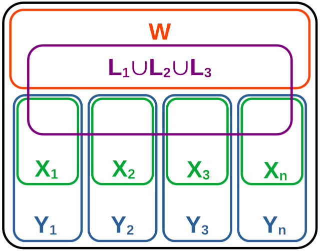
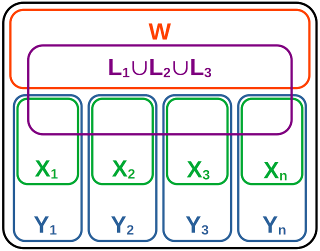
3. every good path in $G - W$ has an edge contained in some Y_i



Lemma 4.

Let $M = (L_1 \cap L_2) \cup (L_2 \cap L_3) \cup (L_3 \cap L_1)$



Following hold:

1. $M \subseteq W$

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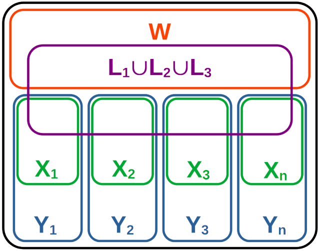

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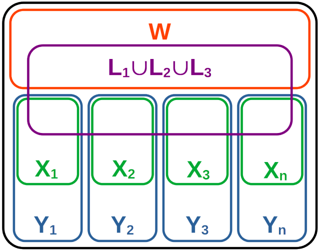

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4. $|L_1 \cap L_2 \cap L_3| \leq 1$
5. $|L_i \cup L_j| \geq k + 2$

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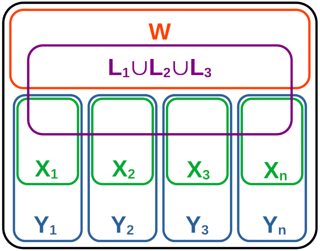

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6. $W \cup X_1 \cup \cdots \cup X_n \supseteq L_1 \cup L_2 \cup L_3$

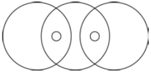


Fig. 1. (2, 0).

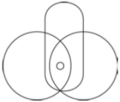


Fig. 2. (1, 1).

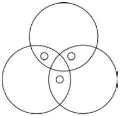
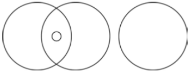


Fig. 3. (3, 0).



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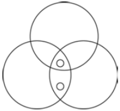


Fig. 5. (2, 1).

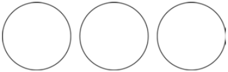
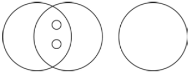


Fig. 6. (0, 0).



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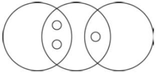

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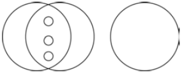

Fig. 9. (3, 0).

Legend for Figs. 1–9. $(i, j) : i = |M|, \; j = |L_1 \cap L_2 \cap L_3|$.

Since $|M| + |L_1 \cap L_2 \cap L_3| \leq 3$ and $|L_1 \cap L_2 \cap L_3| \leq 1$

Lemma 5.

$n \geq k-3$ and equality holds when $W = M$, $|L_1 \cap L_2 \cap L_3| = 1$ and $L_1 \cup L_2 \cup L_3 = W \cup X_1 \cup \cdots \cup X_n$

$$\begin{aligned} 2(k+1) &\geq 2\left(|W| + \sum_{i=1}^{n} \lfloor \frac{1}{2}|X_i| \rfloor \right) \geq 2|W| + \sum_{i=1}^{n} |X_i| - n \\ &\geq |W| + |L_1 \cup L_2 \cup L_3| - n \geq |M| + |L_1 \cup L_2 \cup L_3| - n \\ &= |L_1| + |L_2| + |L_3| - |L_1 \cap L_2 \cap L_3| - n \geq 3k - 1 - n \end{aligned}$$

Thus $n \geq k - 3$

Lemma 6.

$|X_i|$ is odd for each $1 \leq i \leq n$

If $X_i = \emptyset$, then W is a cutset of size at most $k+1$, so $X_i \neq \emptyset$.

Suppose $|X_i|$ is even, let $v \in X_i$.

Let $W^* = W \cup \{v\}$, $X_i^* = X_i - \{v\}$, $Y_i^* = Y_i - \{v\}$ and $X_j^* = X_j$, $Y_j^* = Y_j$ for $j \neq i$.

Sets $W^*, Y_1^*, \ldots, Y_n^*, X_1^*, \ldots, X_n^*$ satisfy lemma 3. which contradicts maximality of $|W|$.

Lemma 7.

Definition

Let G' be the subgraph obtained from GW by deleting all edges contained in any Y_j.
Let A_i be the union of the vertex subsets of all components of G' containing some vertex of L_i.

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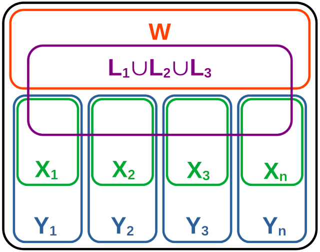

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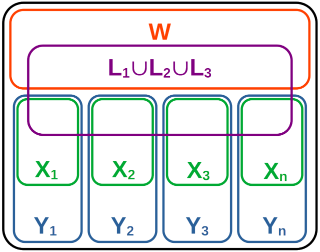

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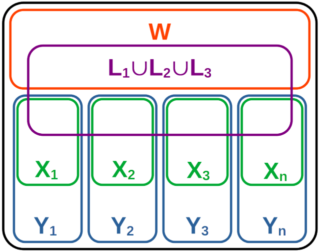

1. $L_i - W \subseteq A - i \subseteq V - W$
2. $A_i \subseteq X_1 \cup \cdots \cup X_n$
3. A_1, A_2 and A_3 are disjoint
4. Any path in $G - W$ between A_i and A_j, $i \neq j$ has at least two vertices in X_k for some k

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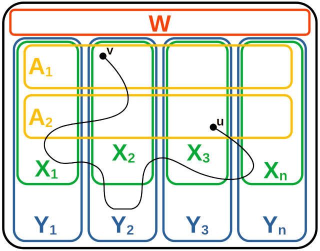

- $v \in A_1$, $u \in A_2$

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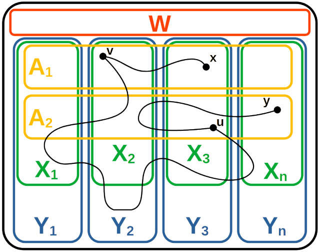

- $v \in A_1$, $u \in A_2$
- Add paths $L_1 \ni x \rightsquigarrow v$ in A_1 and $L_2 \ni y \rightsquigarrow v$ in A_2

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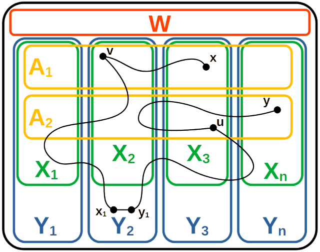

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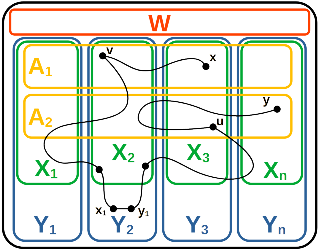

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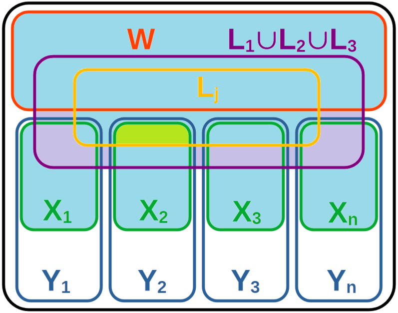
- There has to be a vertex in X_k on path $v \rightsquigarrow x_1$ and $y_1 \rightsquigarrow u$.

Vertex cut

Let $Z = (X_1 \cup \cdots \cup X_n) - (L_1 \cup L_2 \cup L_3)$

Lemma 8.

If $X_i \cap L_j \neq \emptyset$, then $(X_i \Delta L_j) \cup W \cup Z$ is a cutset of G separating $X_i \cap L_j$ from $L_1 \cup L_2 \cup L_3 - W - L_j$.



Lemma 9.

$|A_i \cap X_j| < \frac{1}{2}|X_j|$

Suppose $|A_1 \cap X_1| \geq \frac{1}{2}|X_1|$. There is a vertex $v \in A_1 \cap X_1$.

Since $|L_2 \cup L_3 - W| \geq |L_2 \cup L_3| - |W| \geq k + 2 - |W|$ by lemma 4.5 and $G - W$ is $(k + 2 - |W|)$-connected, there are $(k + 2 - |W|)$ paths of $G - W$ between A_1 and $L_2 \cup L_3 - W$, disjoint except for maybe v with no internal edges in A_1.

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By lemma 6.4, each has at least 2 vertices in X_j for some j. At most $\lfloor \frac{1}{2}|X_j| \rfloor$ have two vertices for each j.

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$$\sum_{j=2}^{n} \frac{1}{2}|X_j|\rfloor \leq k + 1 - |W| - \frac{1}{2}|X_1|\rfloor$$

At least $1 + \lfloor \frac{1}{2}|X_1| \rfloor$ have two vertices in X_1.

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At least $1 + \lfloor \frac{1}{2}|X_1| \rfloor$ have two vertices in X_1.

But each has only one vertex in A_1, so the other vertex is in $X_1 - A_1$.

We have $|X_1 - A_1| \geq 1 + \lfloor \frac{1}{2}|X_1| \rfloor$, which is a contradiction.

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Lemma 9

$|A_i \cap X_j| < \frac{1}{2}|X_j|$ $|L_i \cap X_j| < \frac{1}{2}|X_j|$

Size of X_i

Lemma 10.

$|X_i| \geq 5$ for every $1 \leq i \leq n$

Proof omitted

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Proof omitted

Using $|W| + \sum_{i=1}^{n} \lfloor \frac{1}{2}|X_i| \rfloor \leq k+1$, we get equations:

$$5n \leq \sum_{j=1}^{n} |X_j| \leq 2(k+1-|W|) + n = 2k+2+n-2|W| \quad (1)$$

Which simplifies to

$$2n \leq k+1-|W| \quad (2)$$

Size of n

Lemma 11.

$n = k - 3$

From lemma 5, $n \geq k - 3$. Assume $n \geq k - 2$. By (2) we have

$$2k - 4 \leq 2n \leq k + 1 - |W| \tag{3}$$

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Thus $k \leq 5 - |W|$, but $k \geq 5$, so $|W| = 0$, $k = 5$ and all equalities of (3) hold. That is $n = k - 2 = 3$.

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From lemma 5, $n \geq k - 3$. Assume $n \geq k - 2$. By (2) we have

$$2k - 4 \leq 2n \leq k + 1 - |W| \tag{3}$$

Thus $k \leq 5 - |W|$, but $k \geq 5$, so $|W| = 0$, $k = 5$ and all equalities of (3) hold. That is $n = k - 2 = 3$. By (1) we have

$$15 \leq \sum_{j=1}^{n} |X_j| \leq 2k + 2 + n - 2|W| = 15$$

Therefore $|X_i| = 5$

Size of n

Lemma 11.

$n = k - 3$

We have $k = 5$, $n = 3$, $|W| = 0$, $|X_i| = 5$.

Note that $|X_1| + |X_2| + |X_3| = |L_1| + |L_2| + |L_3|$.

The fact $|W| = 0$ implies $|L_i \cap L_j| = 0$.

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By lemma 9, $(X_1 L_1) \cup W \cup Z = X_1 L_1$ is a cutset, but $|X_1 \Delta L_1| = 3 + 3 = 6 = k + 1$, which contradicts the fact that G is $(k + 2)$-connected.

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Final step

We have $n = k - 3$, which by lemma 5 implies $W = M$, $|L_1 \cap L_2 \cap L_3| = 1$ and $L_1 \cup L_2 \cup L_3 = W \cup X_1 \cup \cdots \cup X_n$.

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Hence $|W| \geq 1$, $Z = \emptyset$

By (2) we have $2k - 6 = 2n \leq k + 1 - |W|$, that is $k \leq 7 - |W|$

So $|W| \leq 2$, because $k \geq 5$. We have two cases $|W| = 2$ and $|W| = 1$

Final step

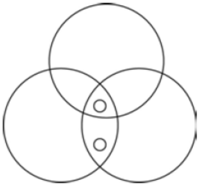
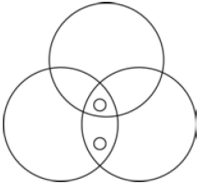


Fig. 5. (2, 1).

When $|W| = 2$, we have $|W| = |M| = 2$, $|L_1 \cap L_2 \cap L_3| = 1$, $k = 5$ and $n = k - 3 = 2$

Final step



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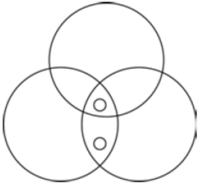
Equality in (2) implies that $|X_1| = |X_2| = 5$

Assume w.l.o.g. that $W \subseteq L_1$ and $|L_1 \cap X_1| = 2$.

Final step



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Equality in (2) implies that $|X_1| = |X_2| = 5$

Assume w.l.o.g. that $W \subseteq L_1$ and $|L_1 \cap X_1| = 2$.

$X_1 \Delta L_1$ is a vertex cut of size at most 6 since $Z = \emptyset$, $W \subseteq L_1$. This contradicts that G is $k + 2 = 7$-connected.

Final step

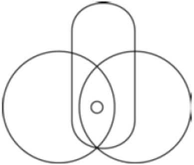

Fig. 2. (1, 1).

When $|W| = 1$, we have $|W| = |M| = |L_1 \cap L_2 \cap L_3| = 1$, and $5 \leq k \leq 6$

Contradiction similar to the previous case.