A not 3-choosable planar graph without 3-cycles

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Optymalizacja Kombinatoryczna 2022/23L

Introduction

$G = (V, E)$ - simple undirected graph

L - a map from V to finite subsets of $\mathbb{N}$

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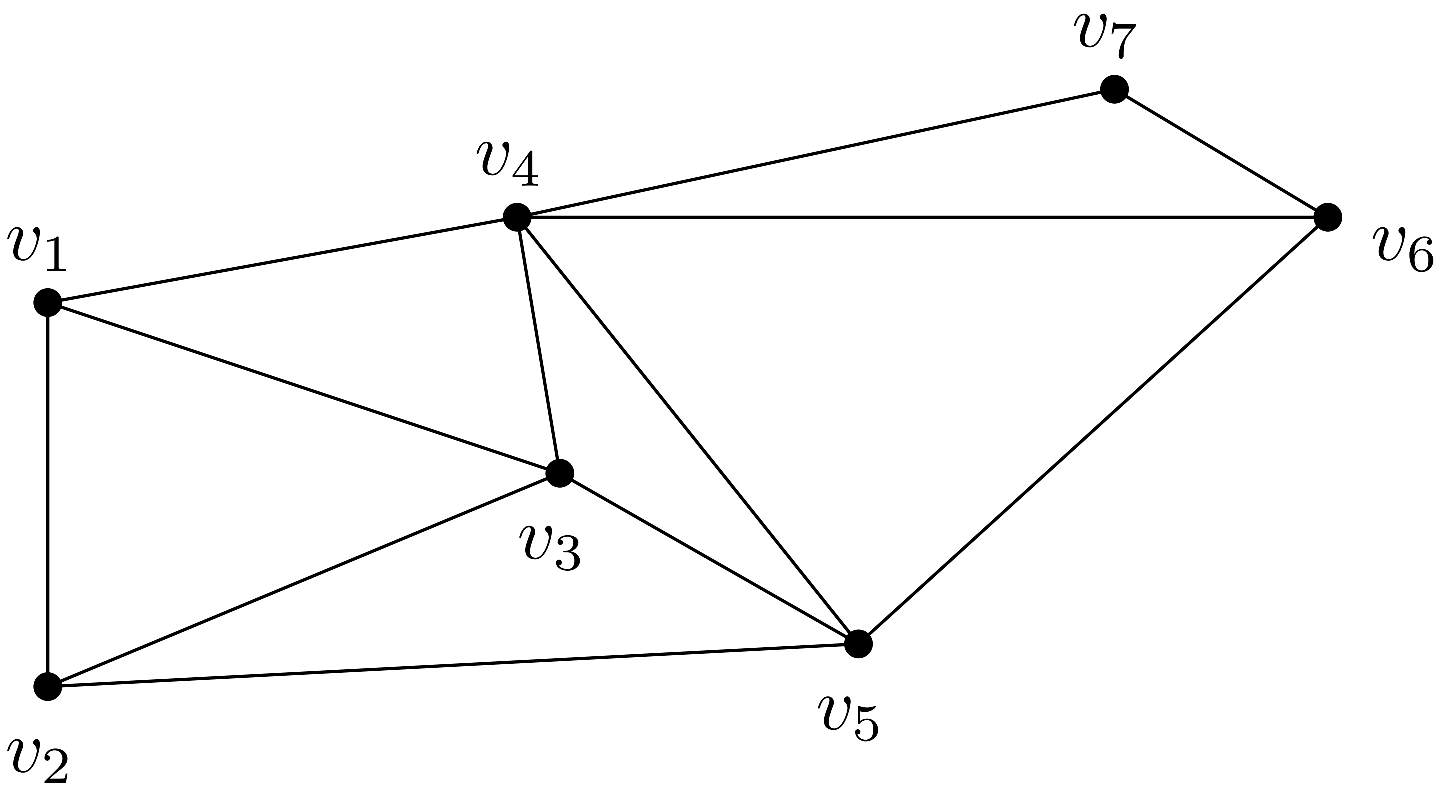
$$L = \{\ v_1 : [2],\ v_2 : [1,3,7],\ v_3 : [3,4],\ v_4 : [2,4,5],\ v_5 : [4,6],\ v_6 : [5,6],\ v_7 : [2,5]\ \}$$



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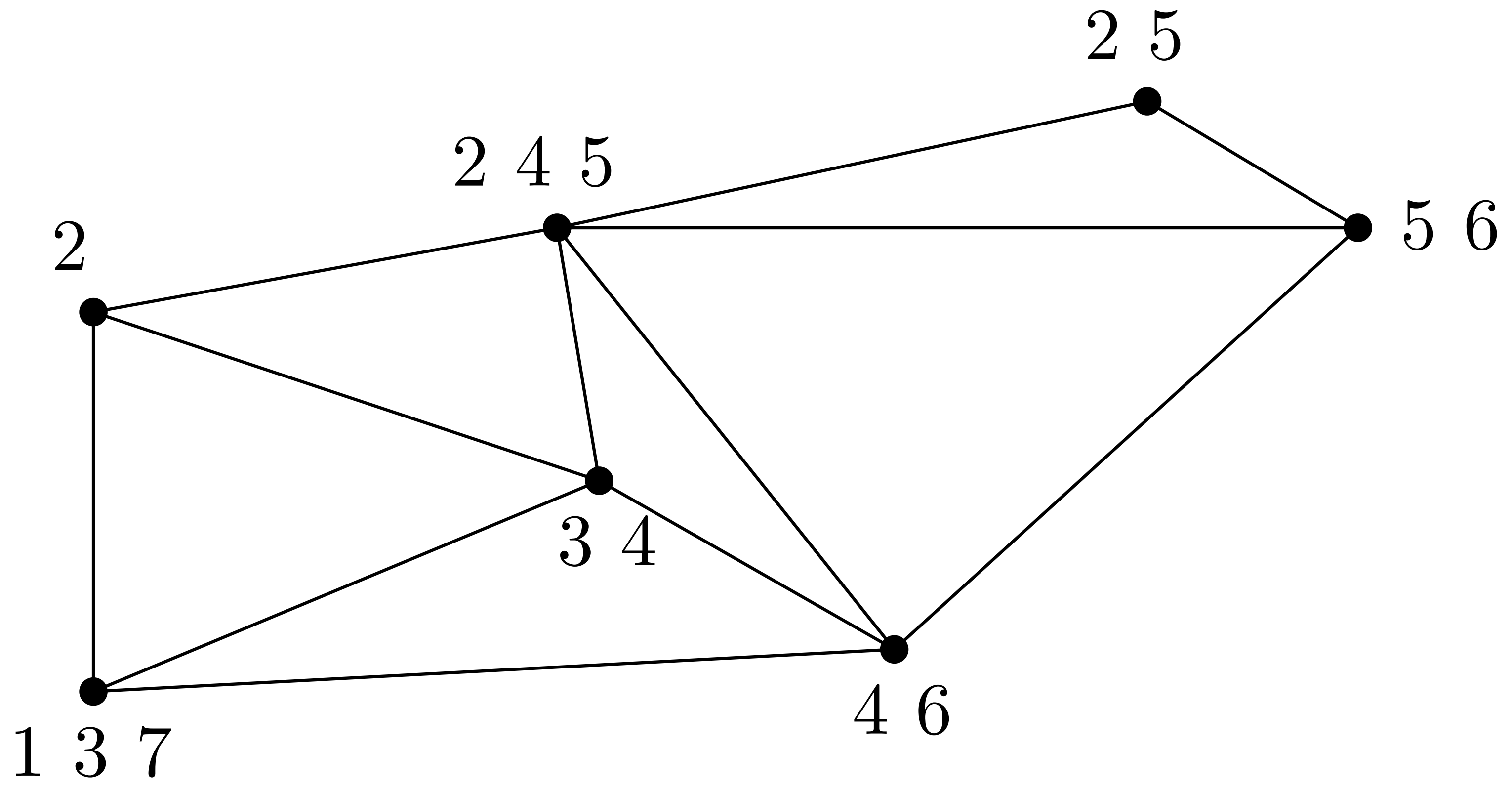

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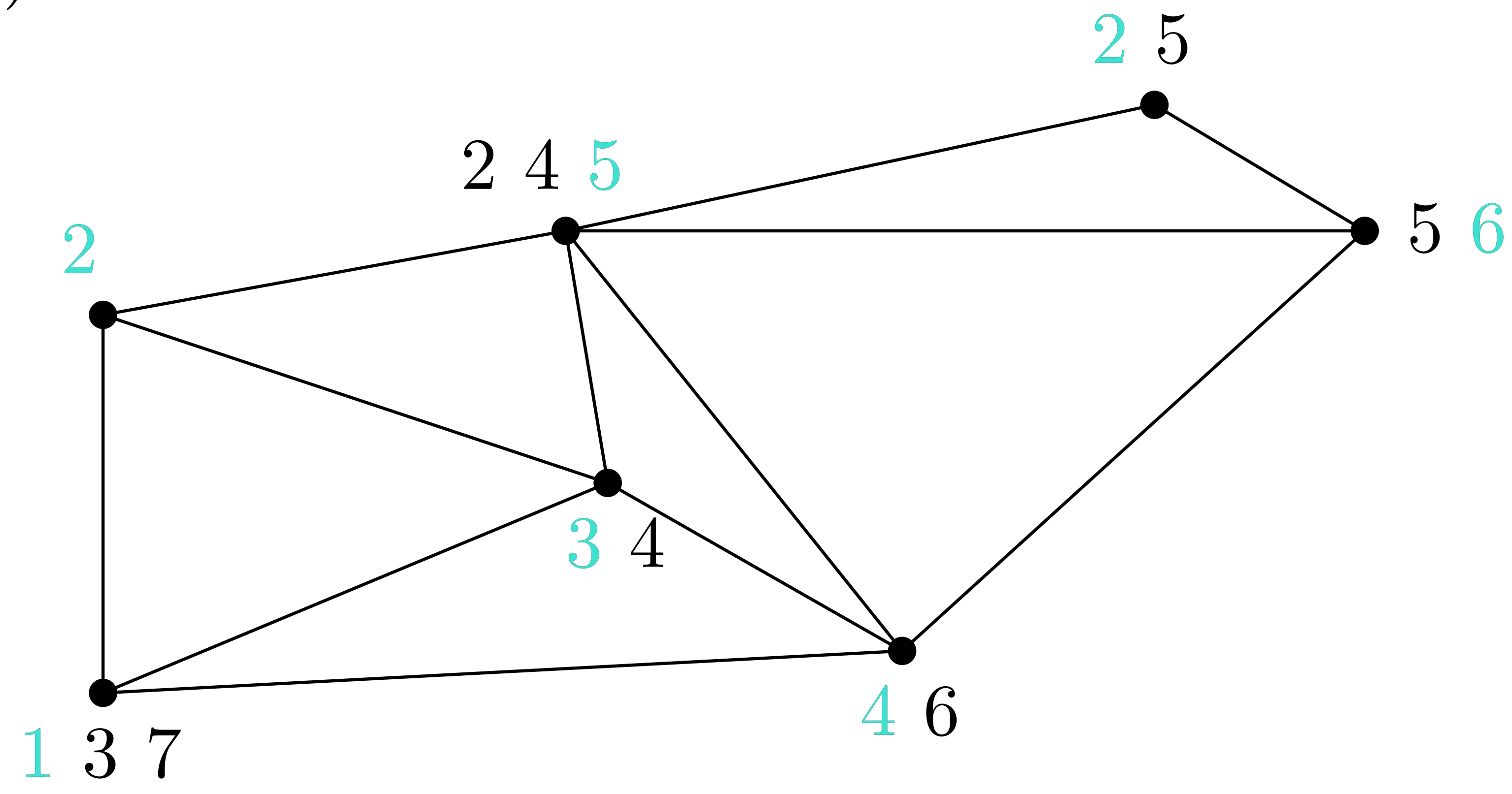
- $\forall_{uv \in E} f(u) \neq f(v)$



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Graph G is *k-choosable* if G is L-colorable for every L s.t.:

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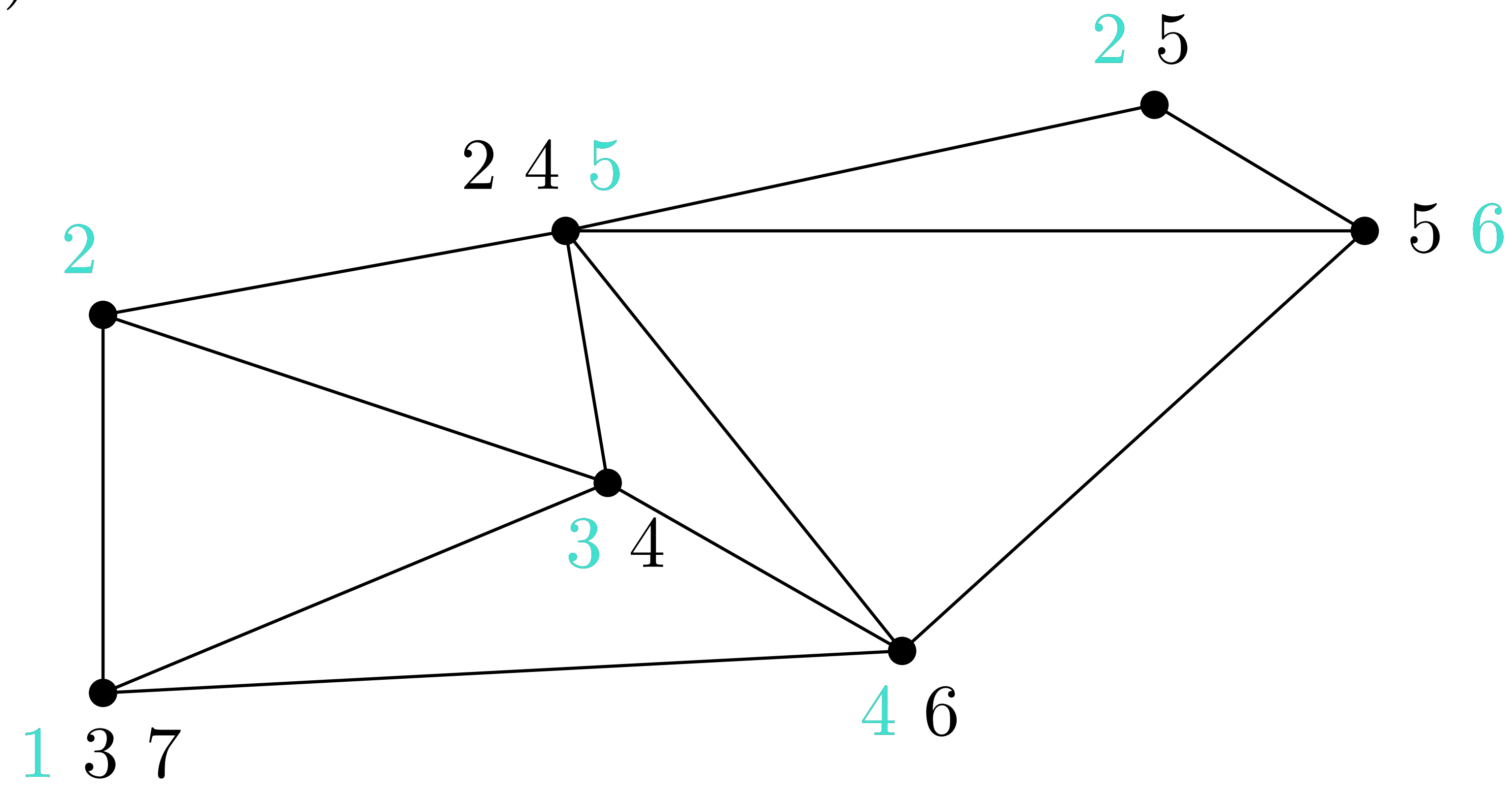

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...because, G is L-colorable for $L \equiv [1..k]$

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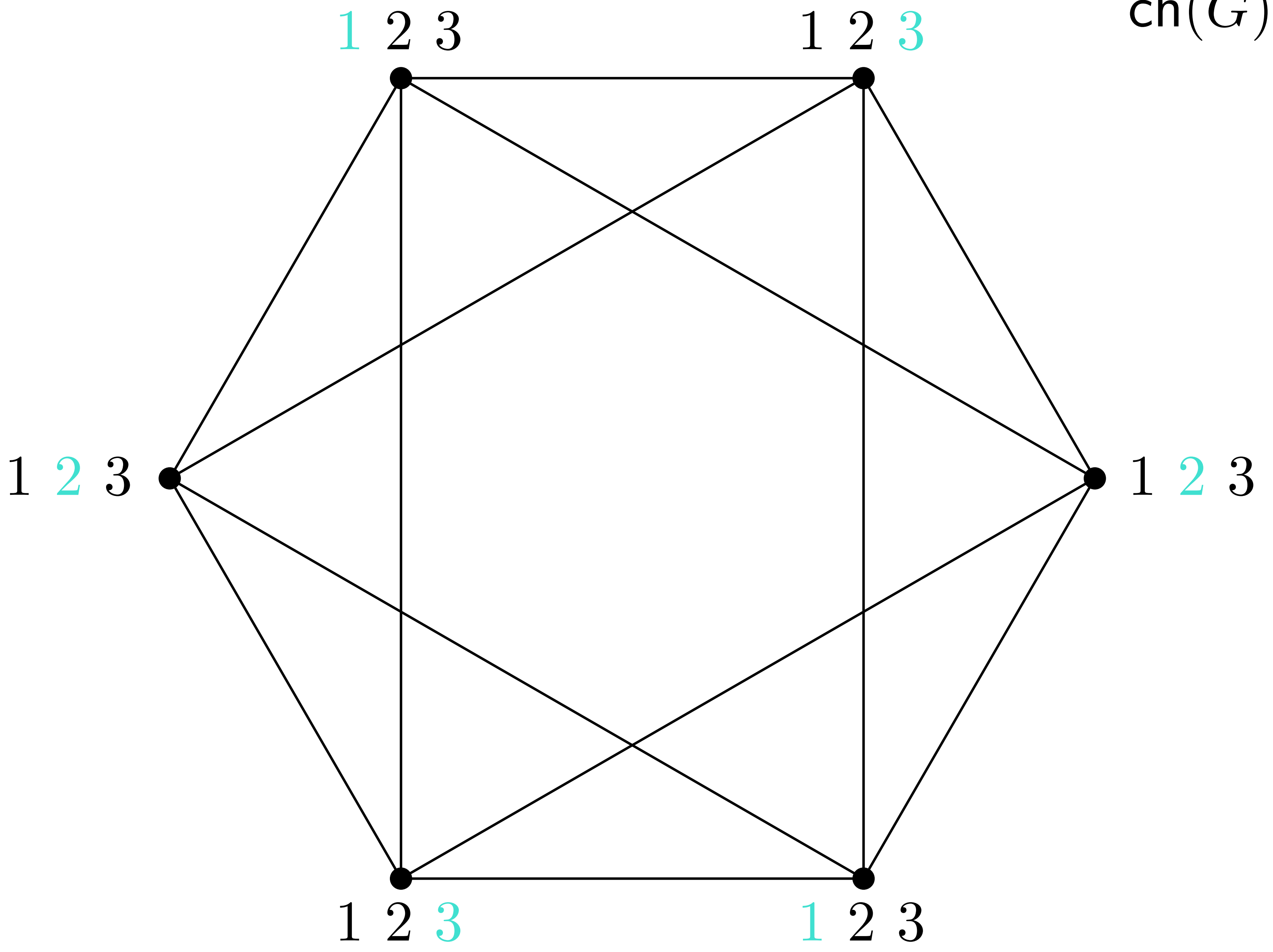

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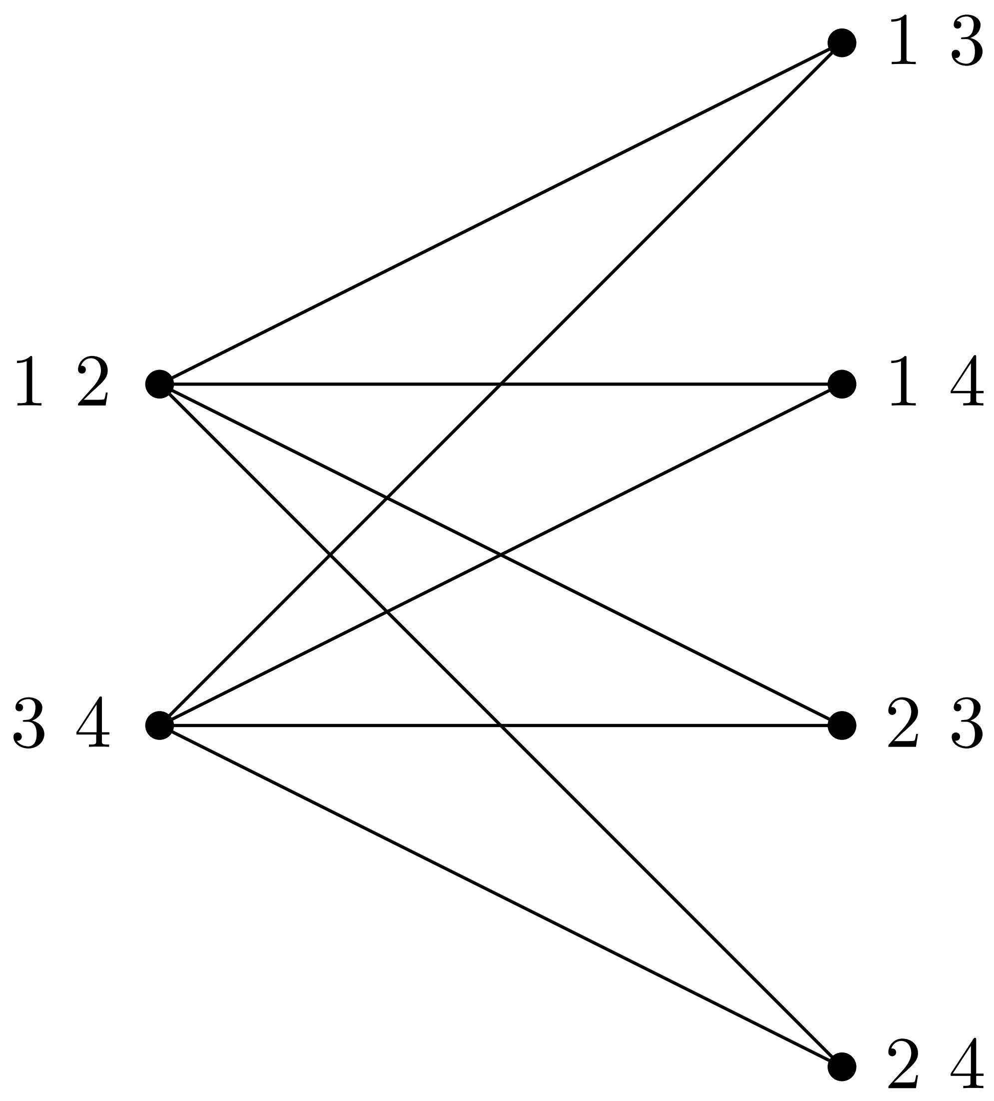

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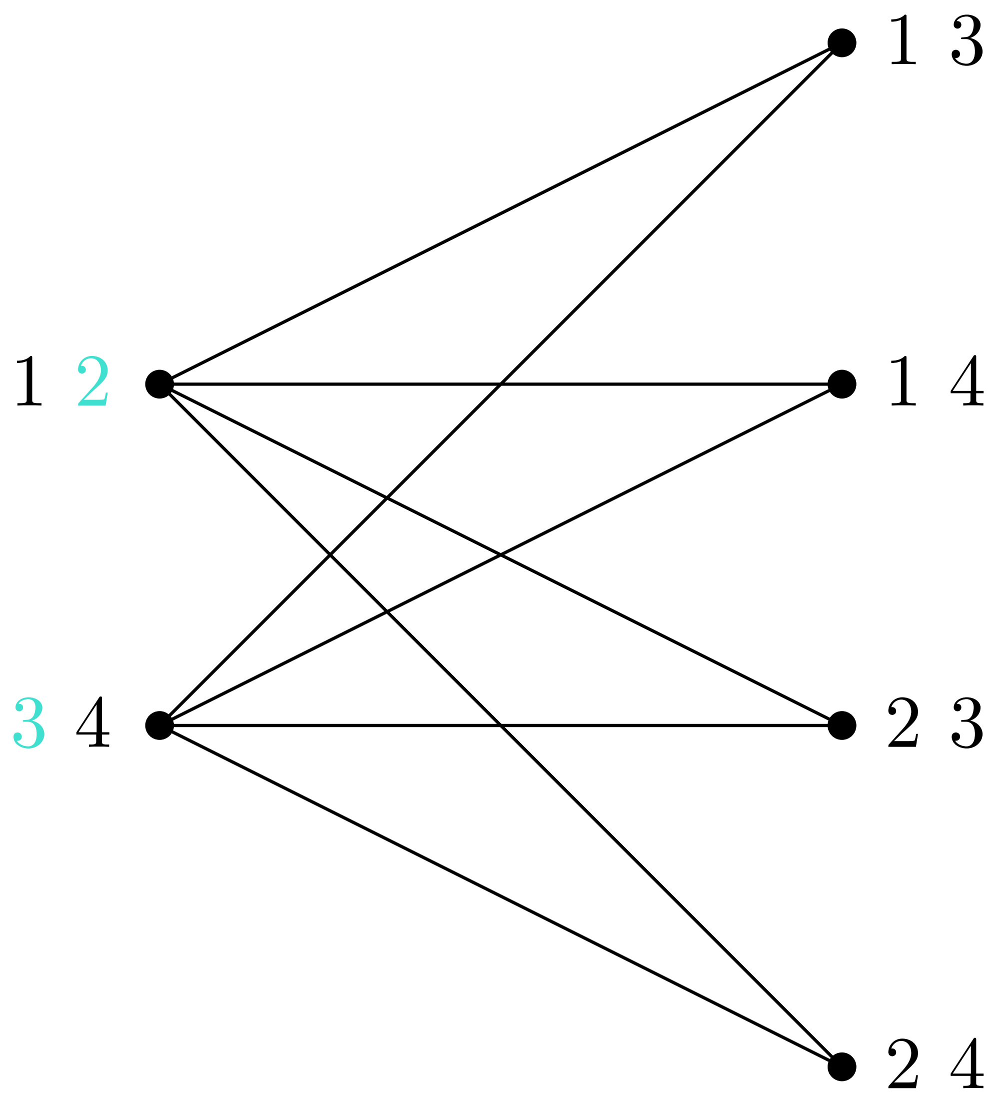

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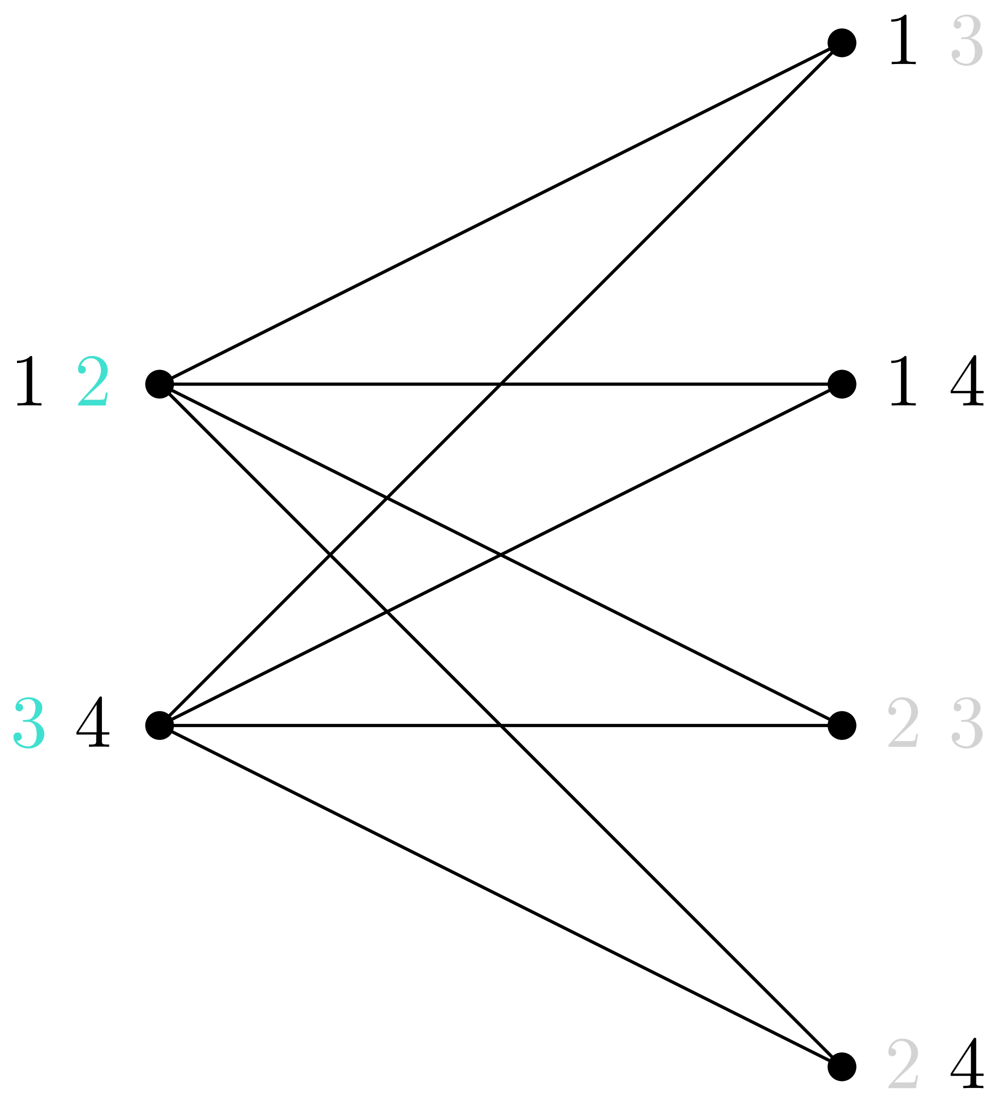

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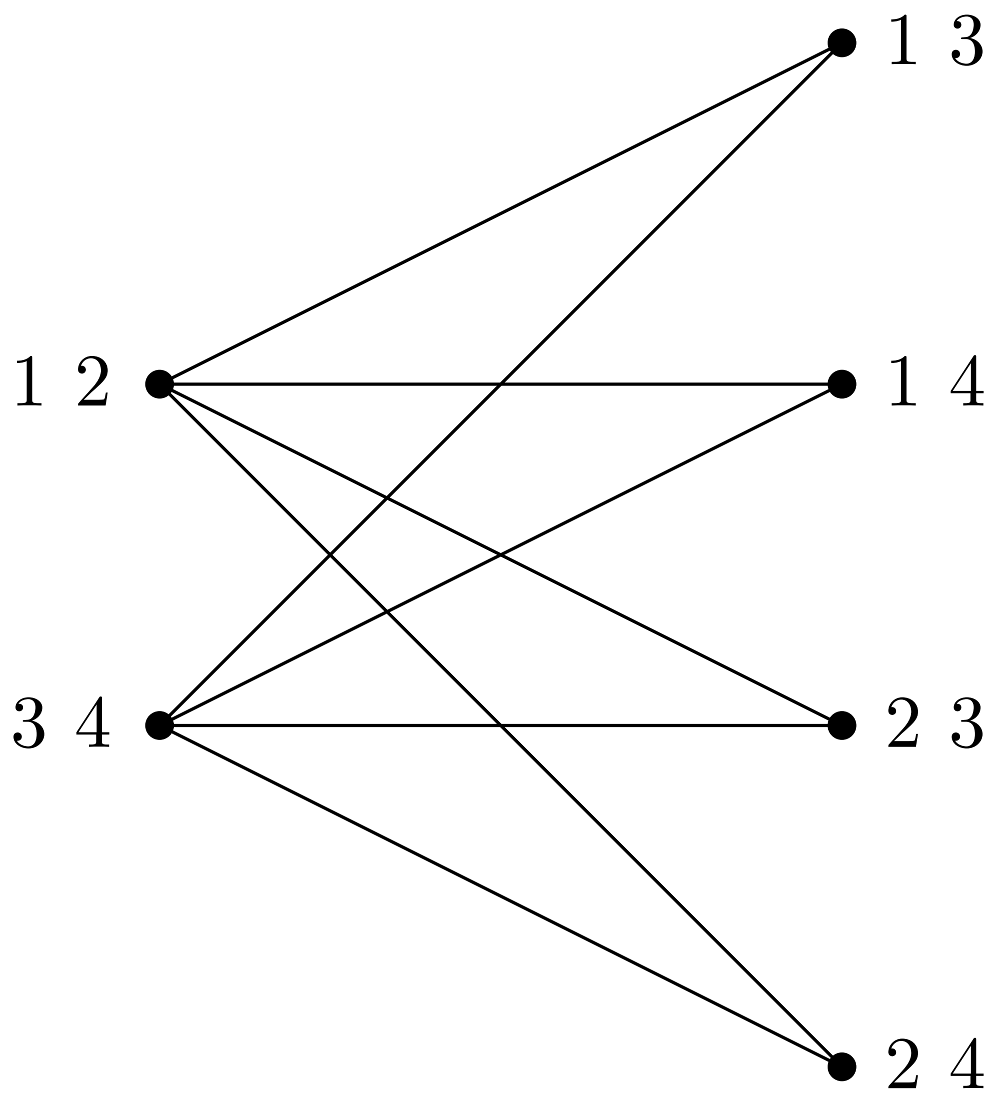

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This can be generalized to K_{q,q^q}

- $\chi(K_{q,q^q}) = 2$
- $\mathsf{ch}(K_{q,q^q}) \geq q$

State of the art

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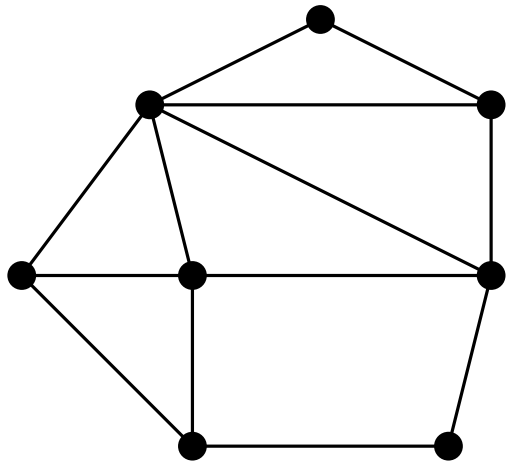
Not every planar graph of girth 4 is 3-choosable.

Girth of G is the length of its smallest cycle.



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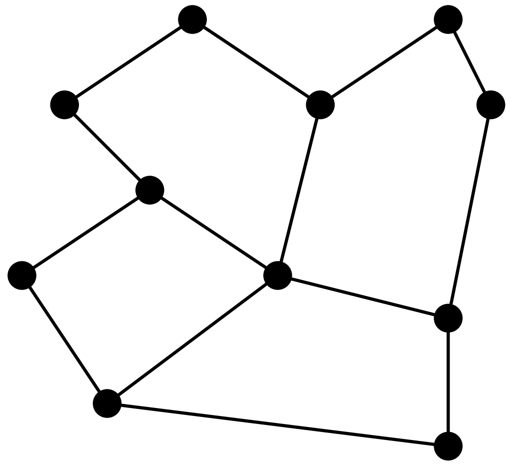

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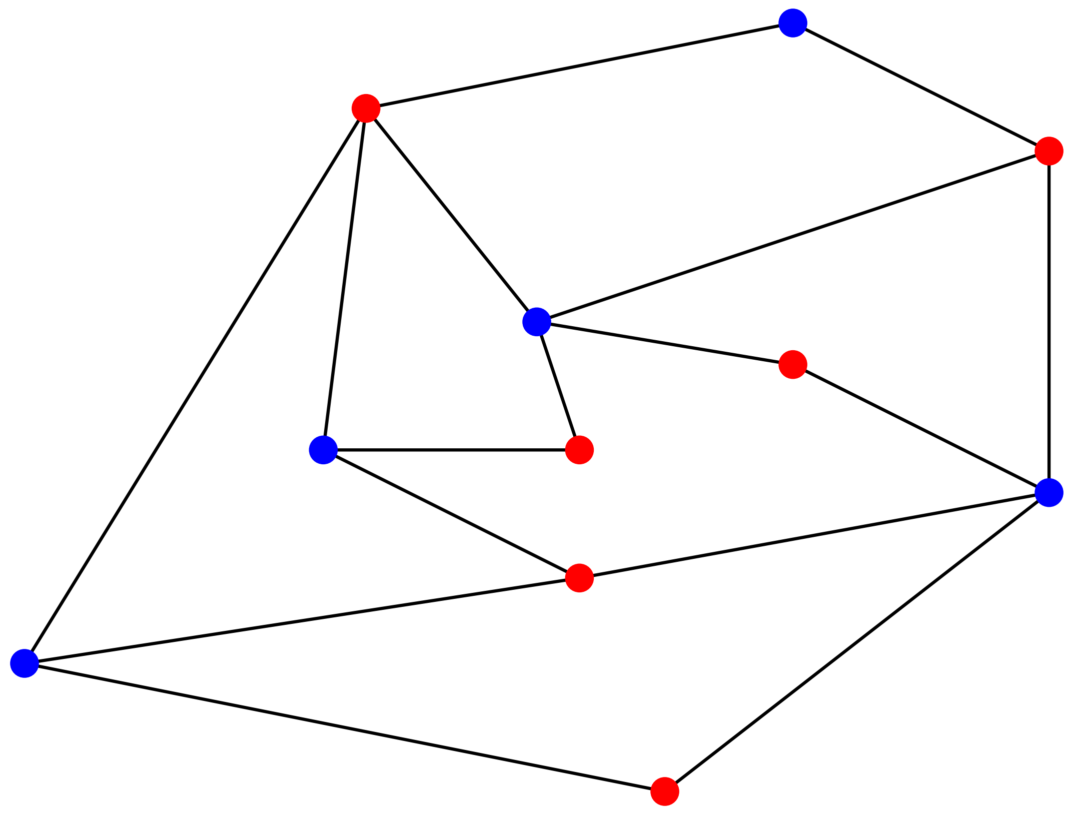

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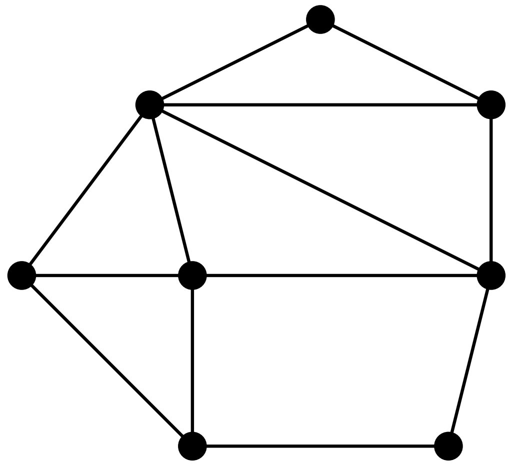

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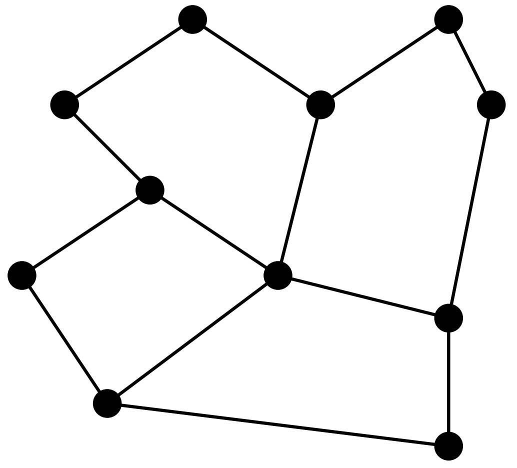

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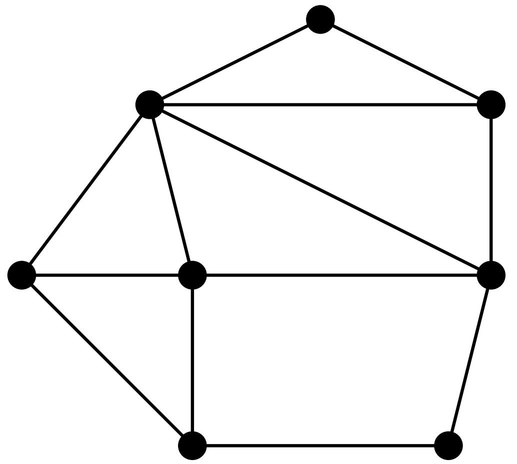

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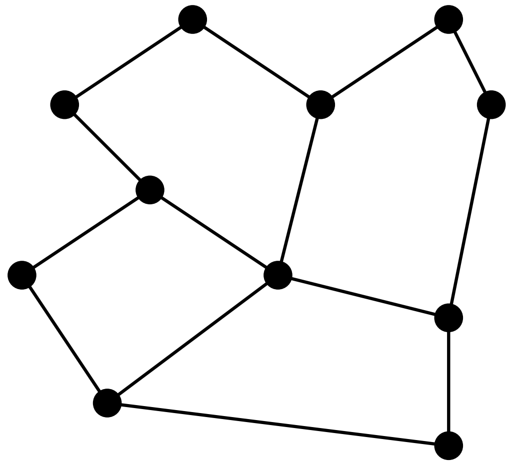

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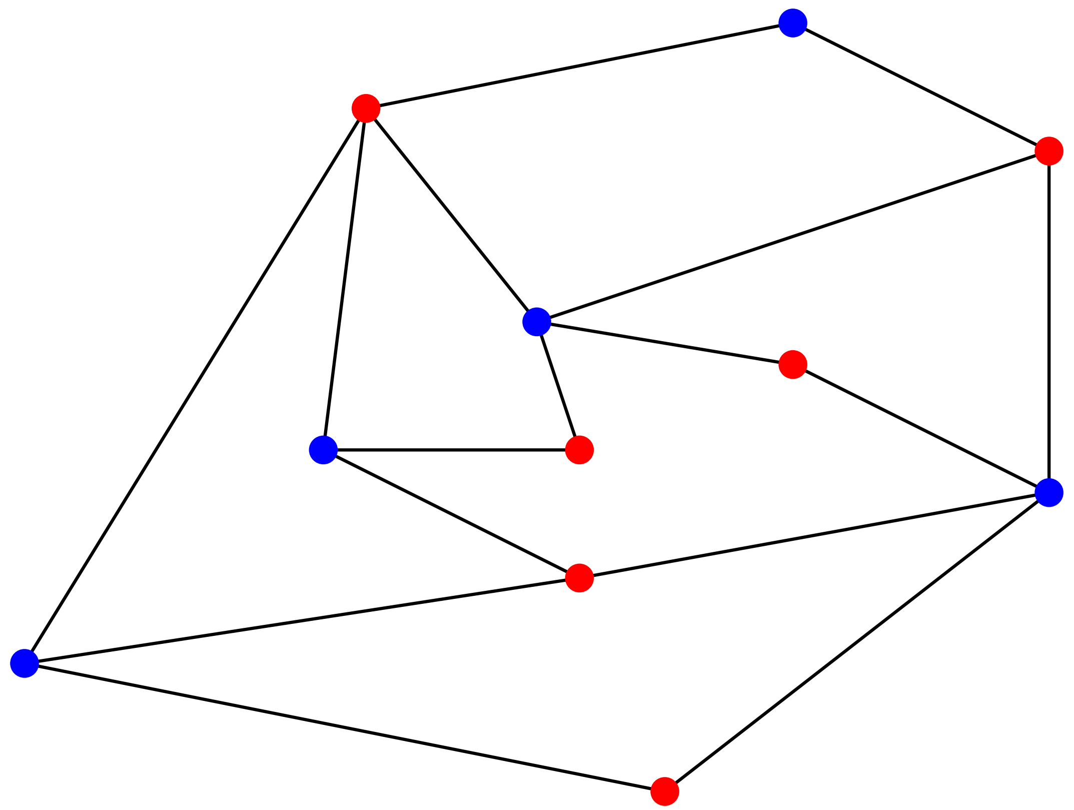

mad(G) - maximum average degree of G over all its subgraphs

Proof:

- For planar bipartite graphs,
  $e \leq 2v - 4 \Rightarrow$ mad$(G) < 4$
- Bipartite graphs are $(\lceil \frac{\text{mad}(G)}{2} \rceil + 1)$-choosable

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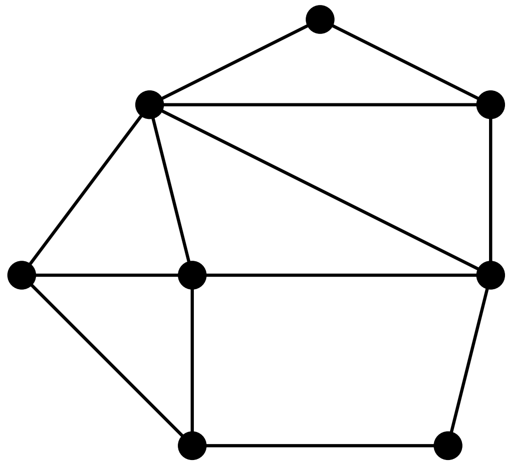

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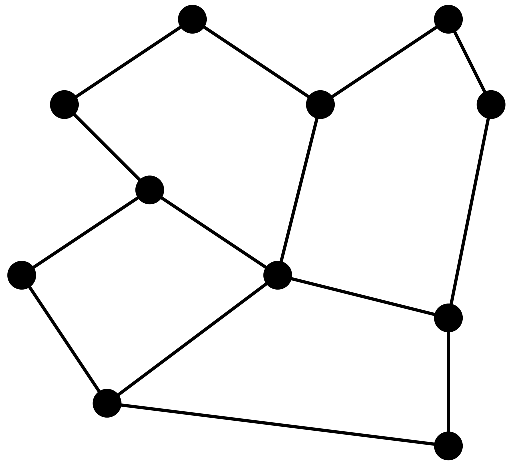

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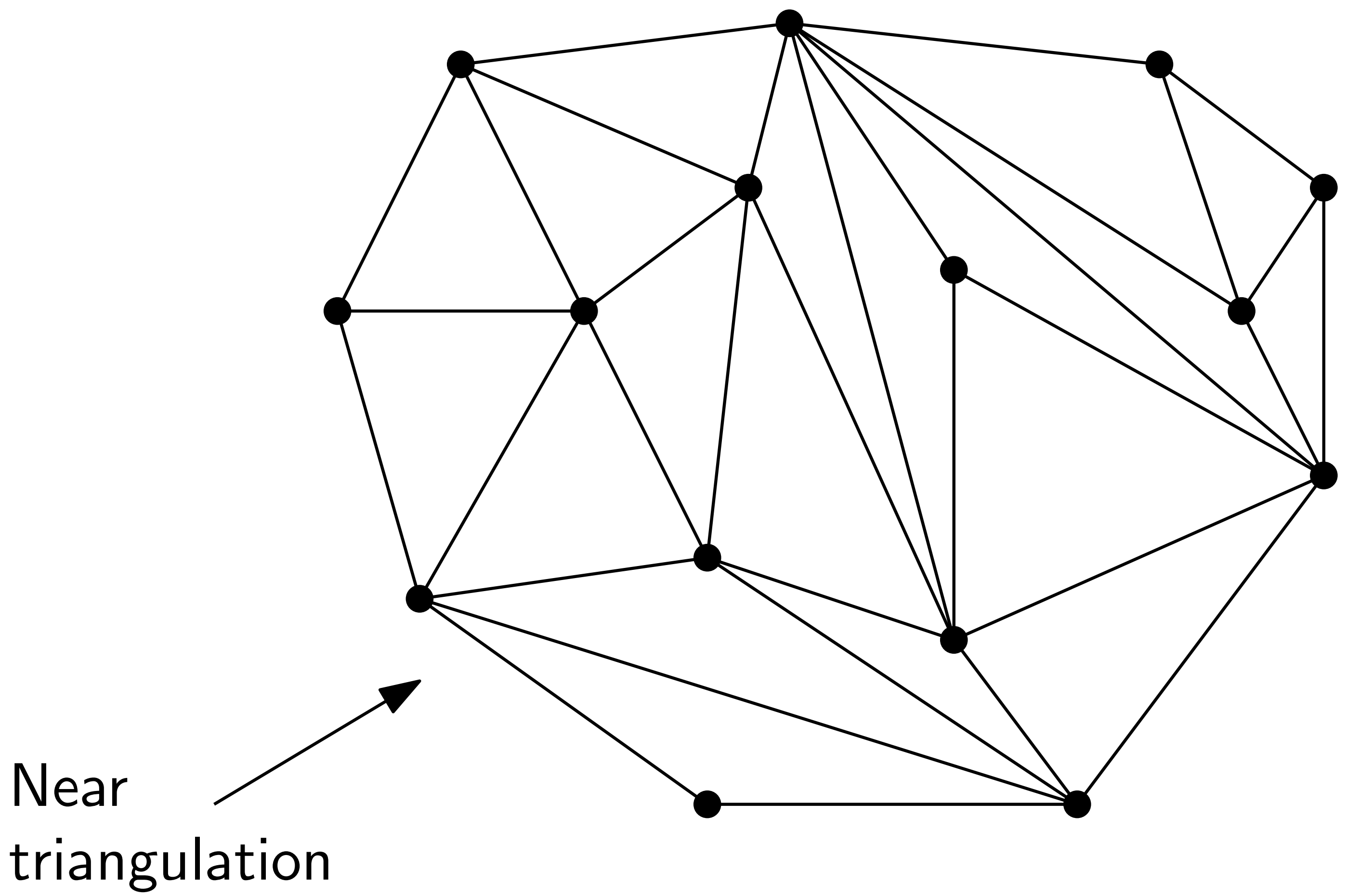

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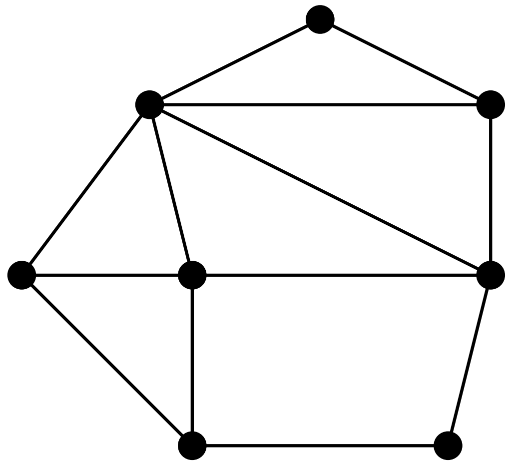

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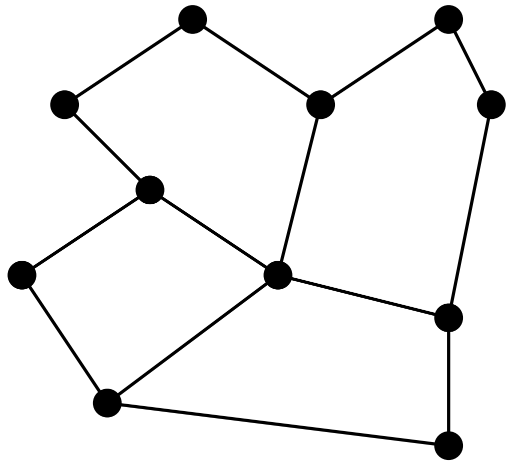

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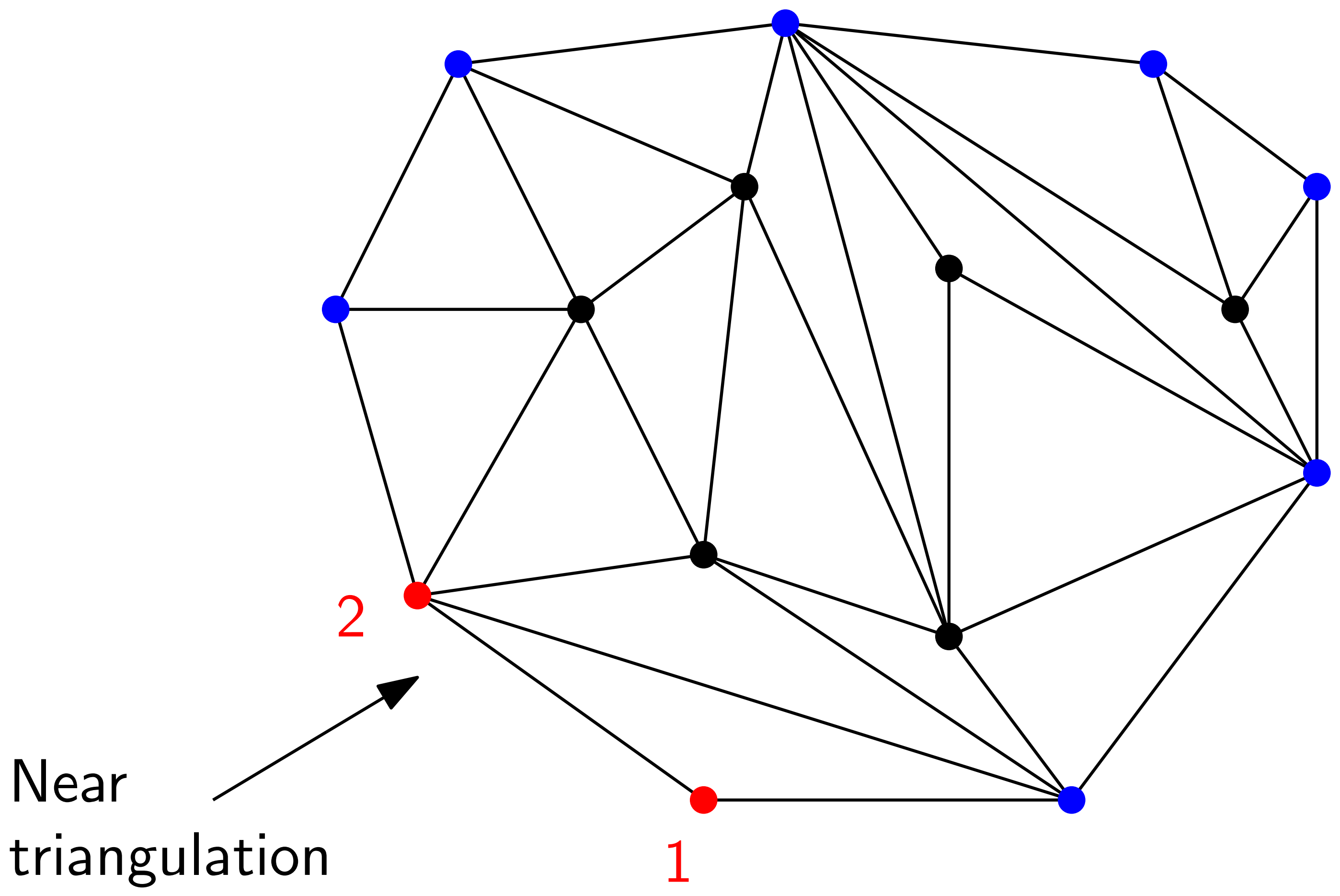

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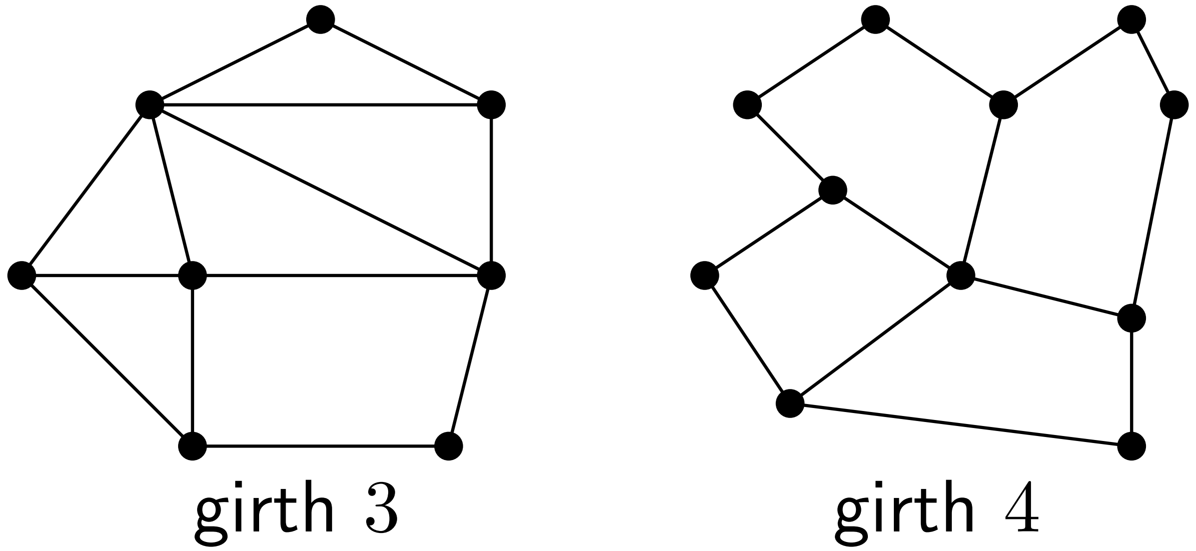

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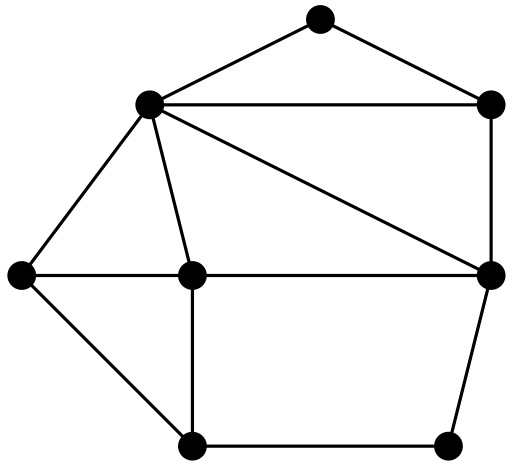

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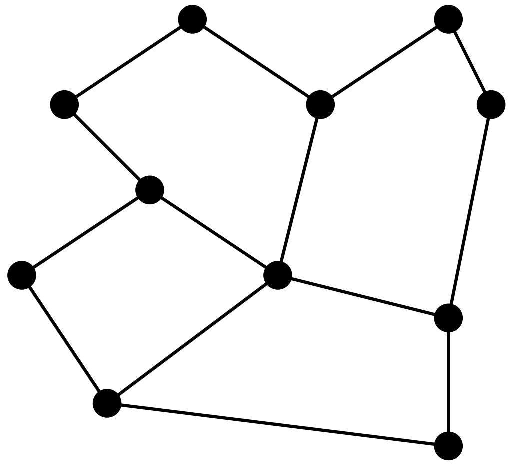

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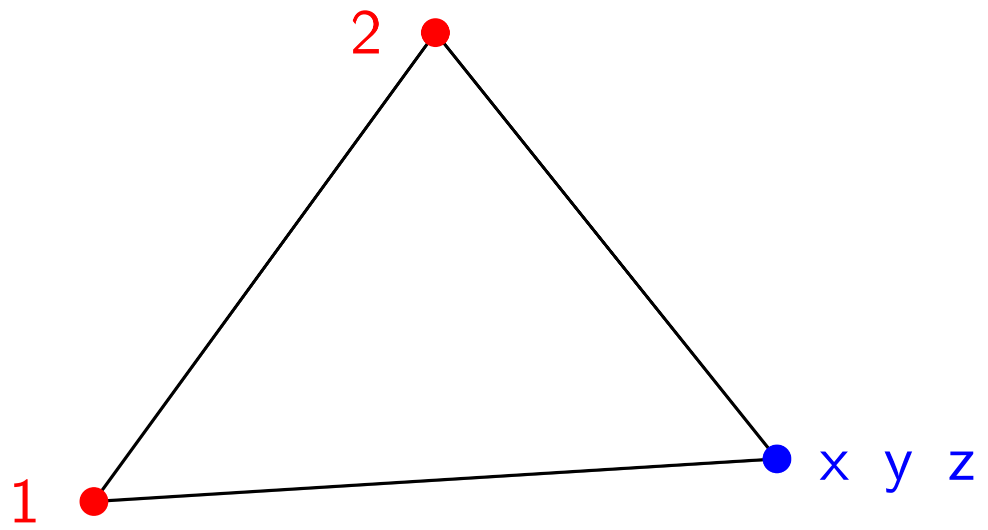

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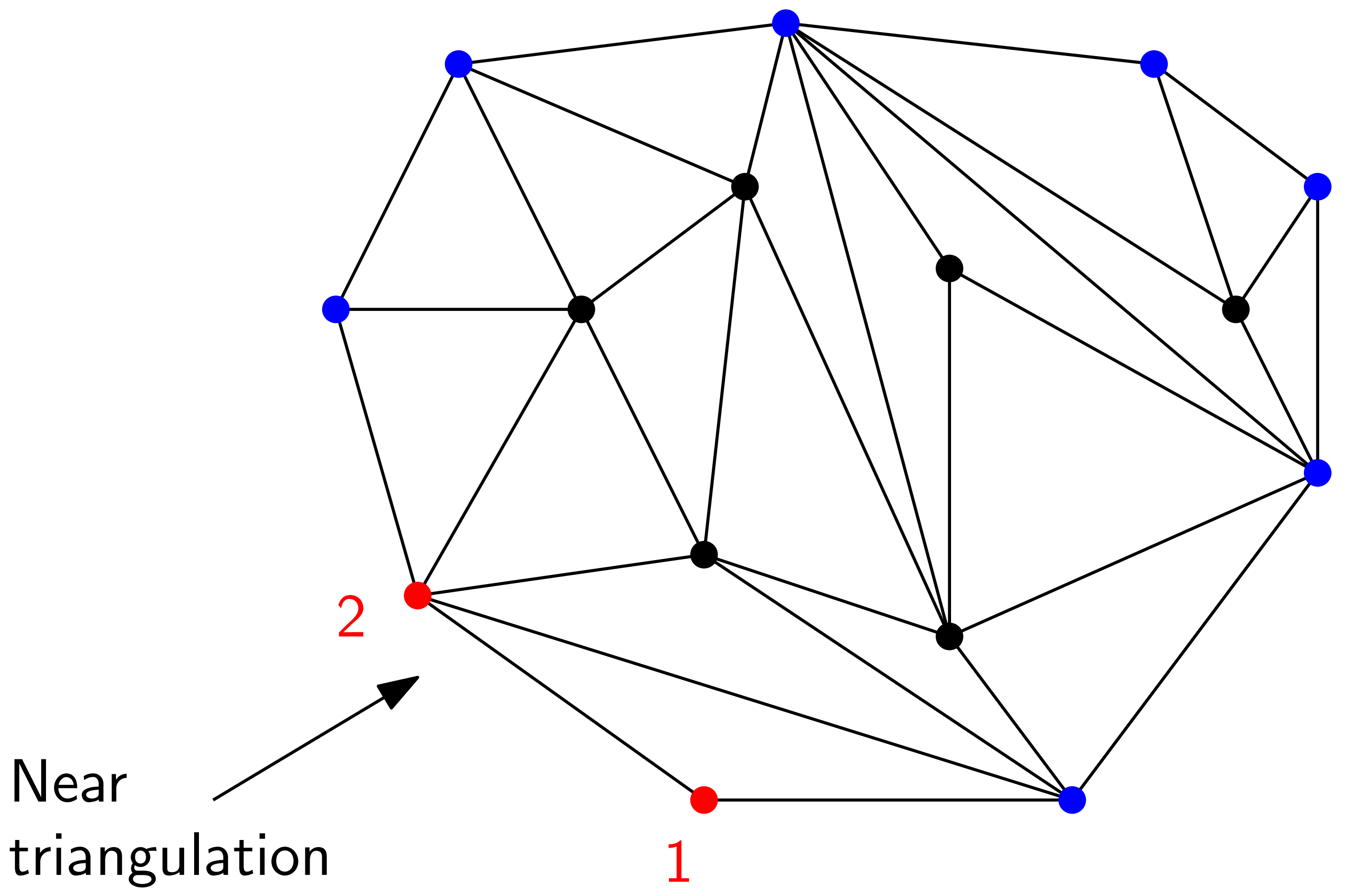

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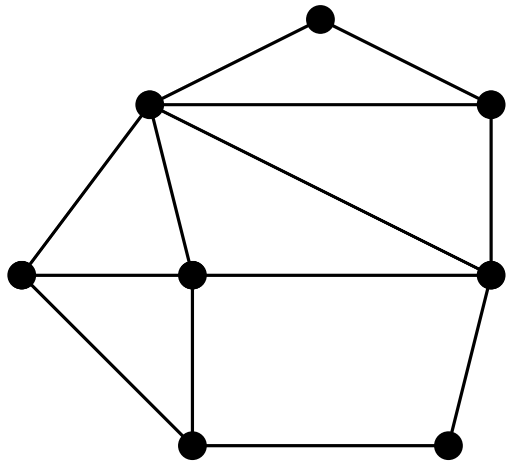

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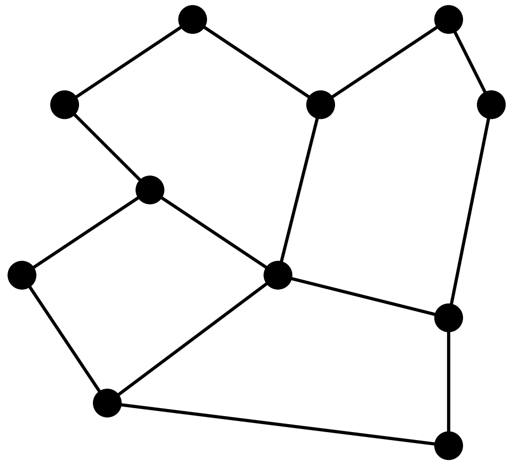

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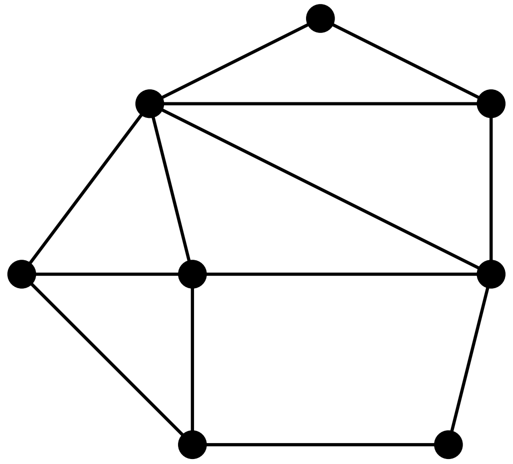

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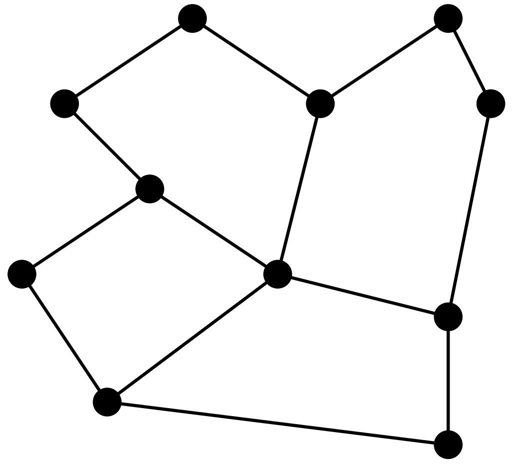

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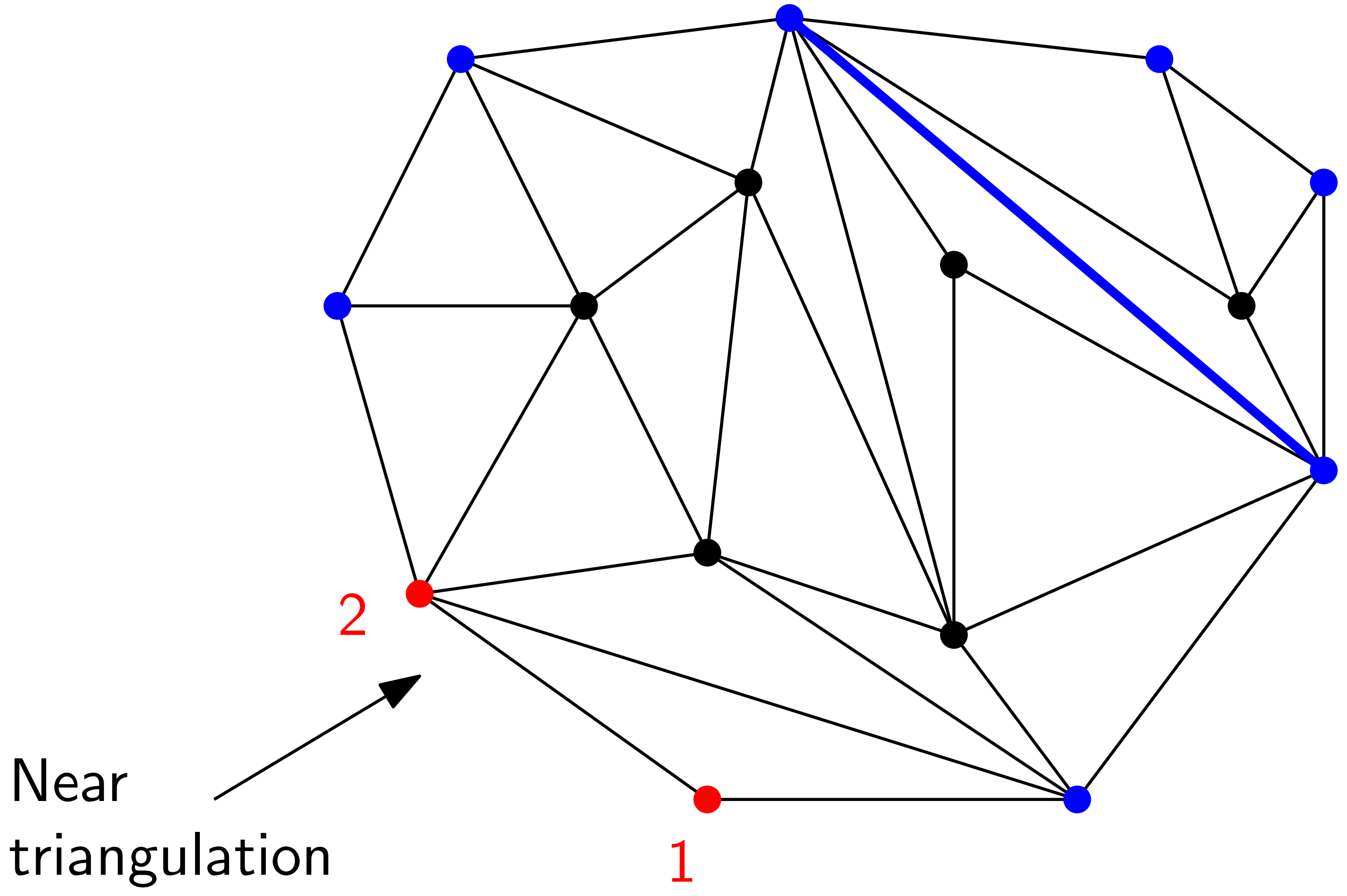

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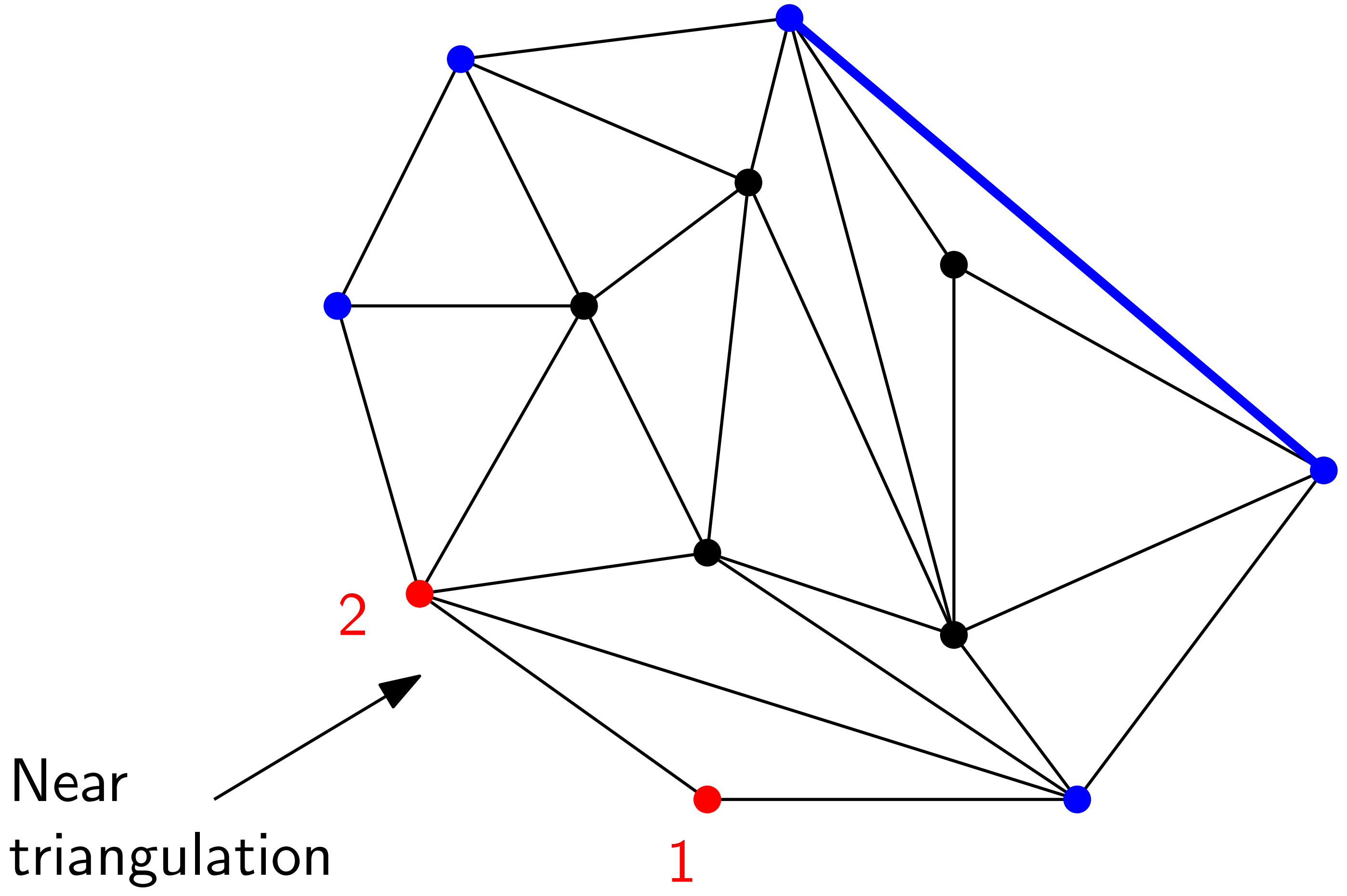

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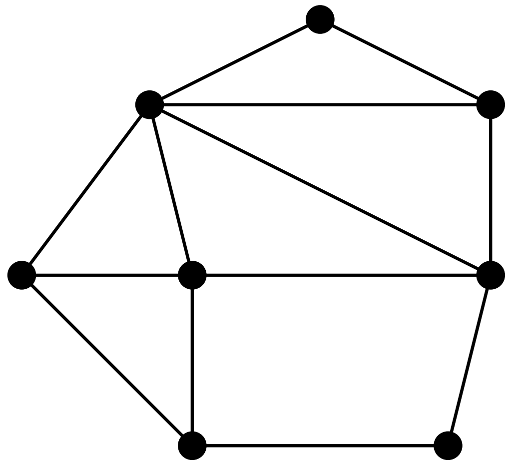

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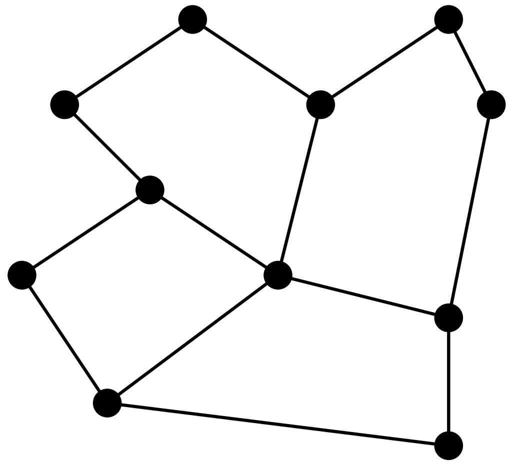

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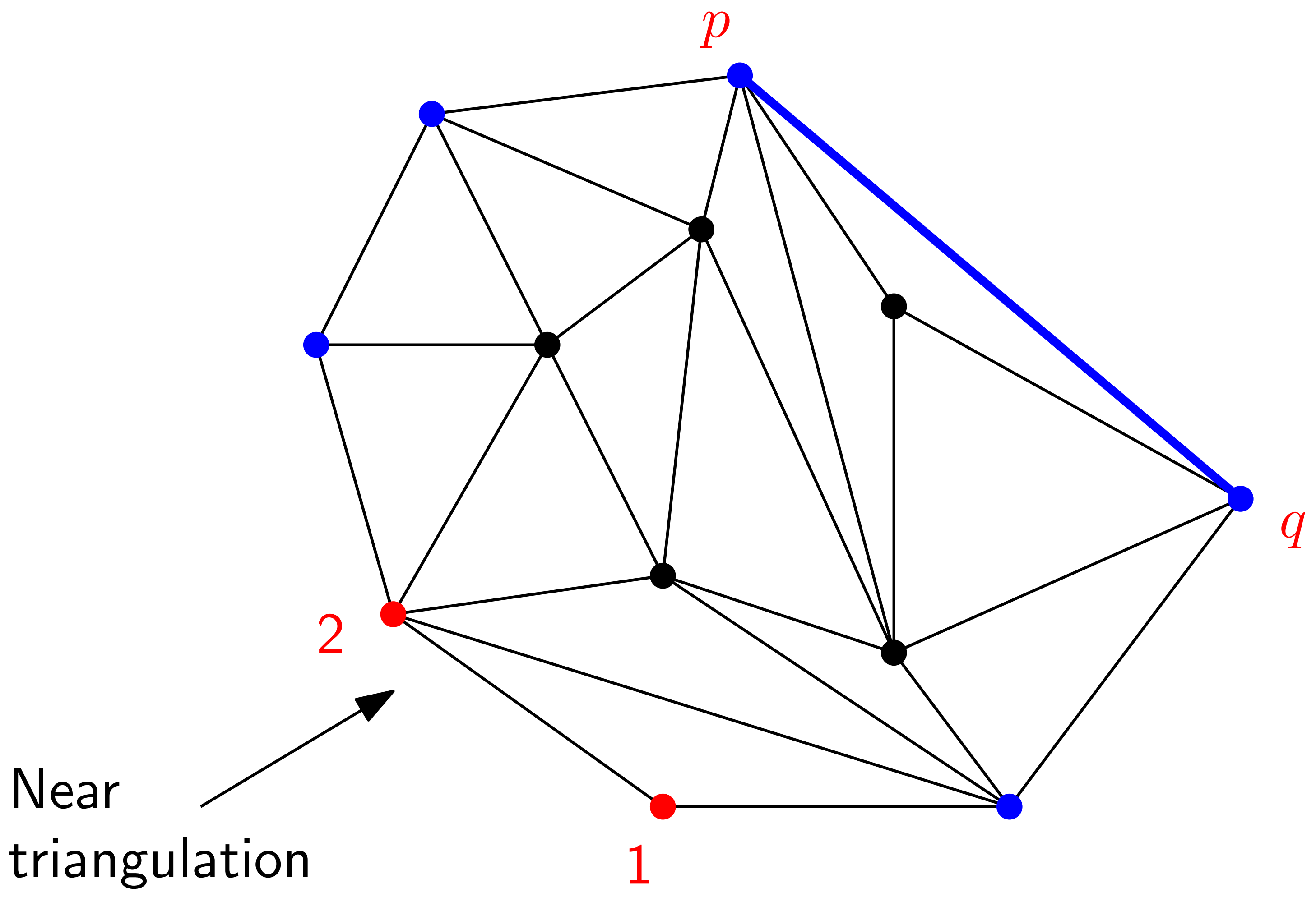

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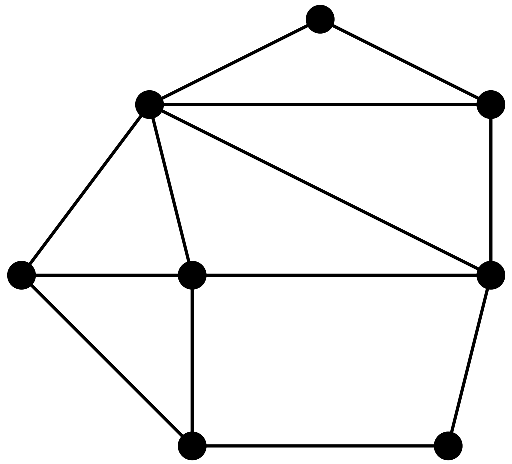

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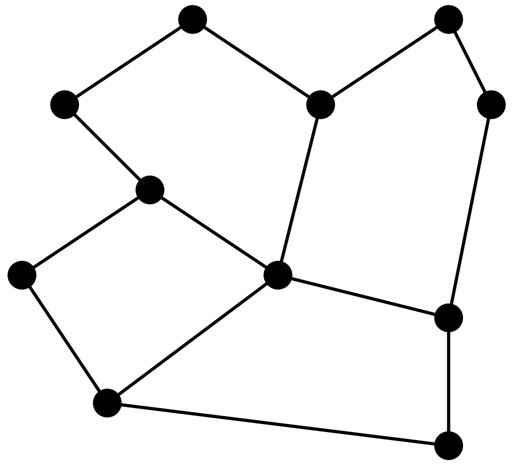

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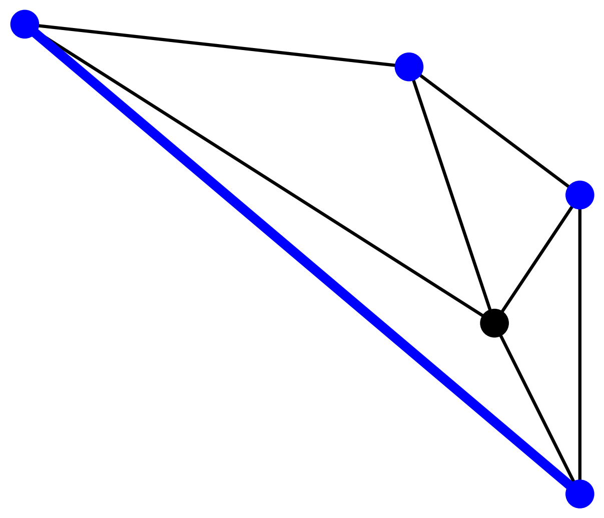

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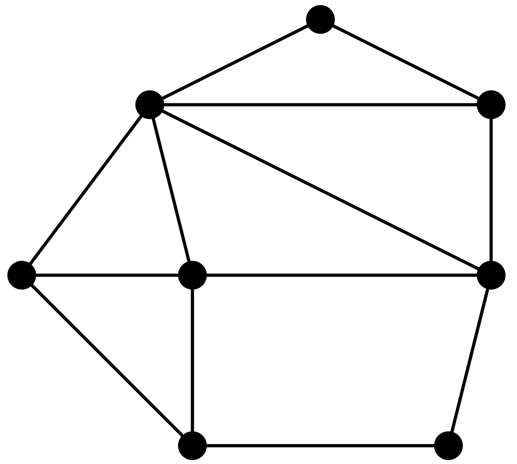

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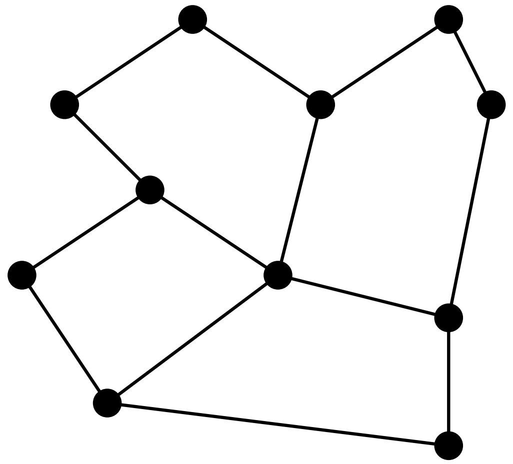

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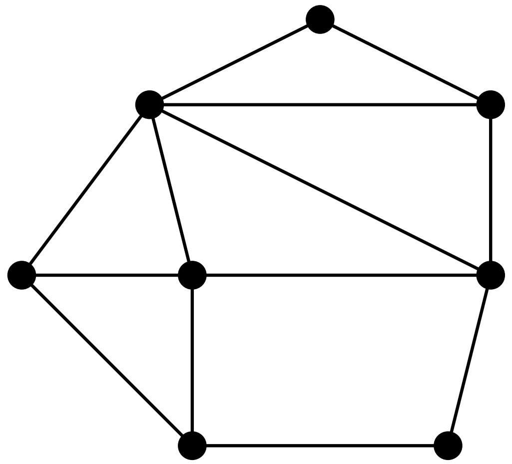

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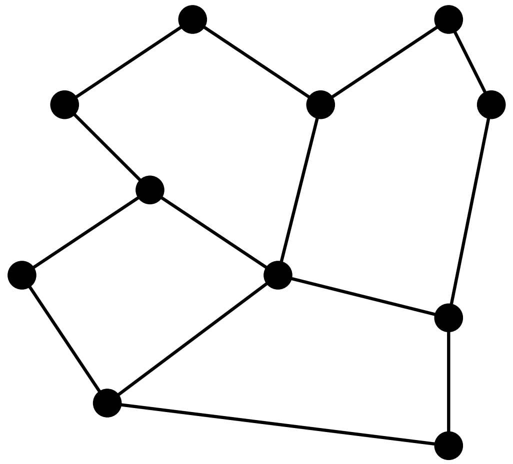

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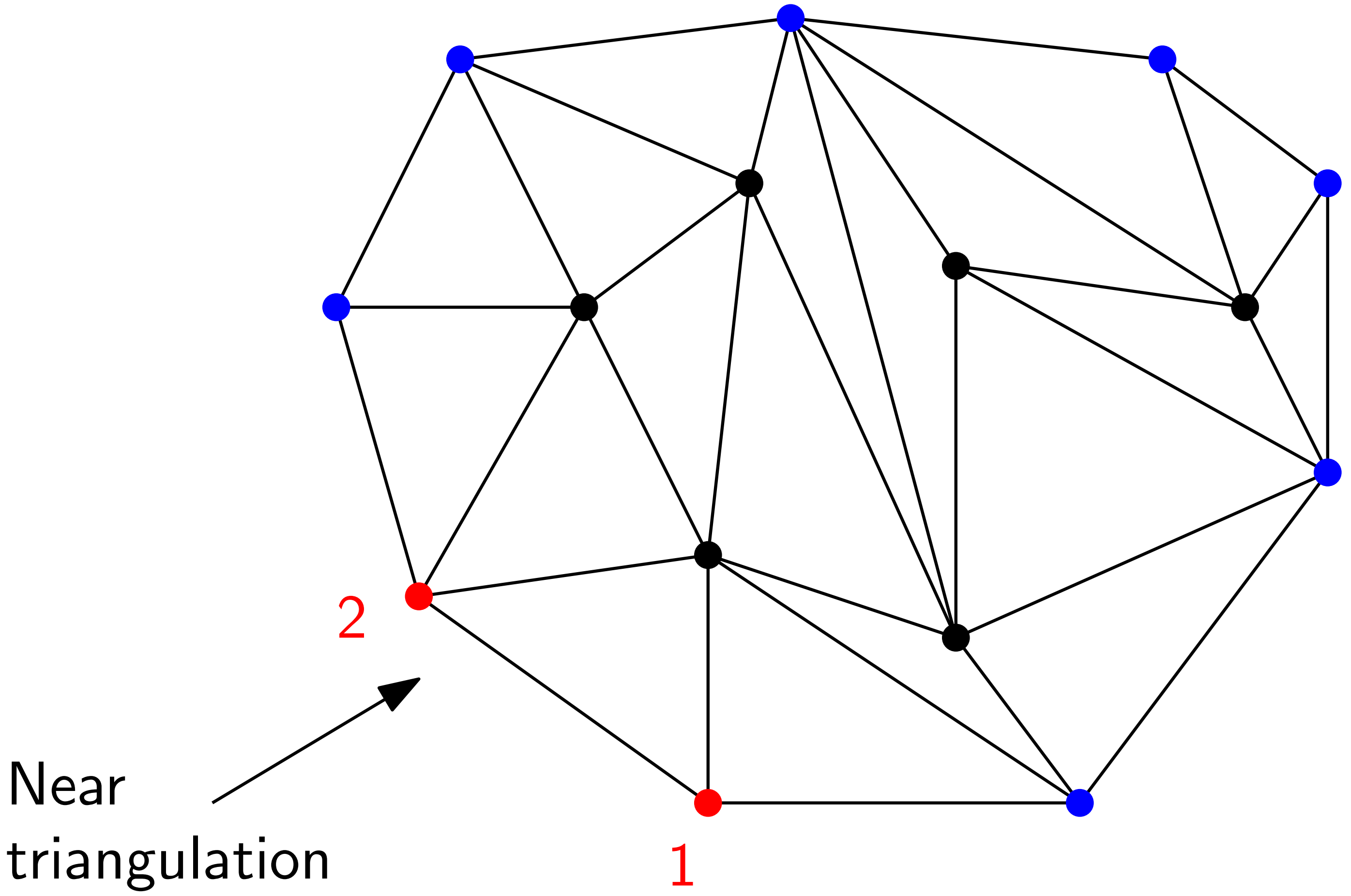

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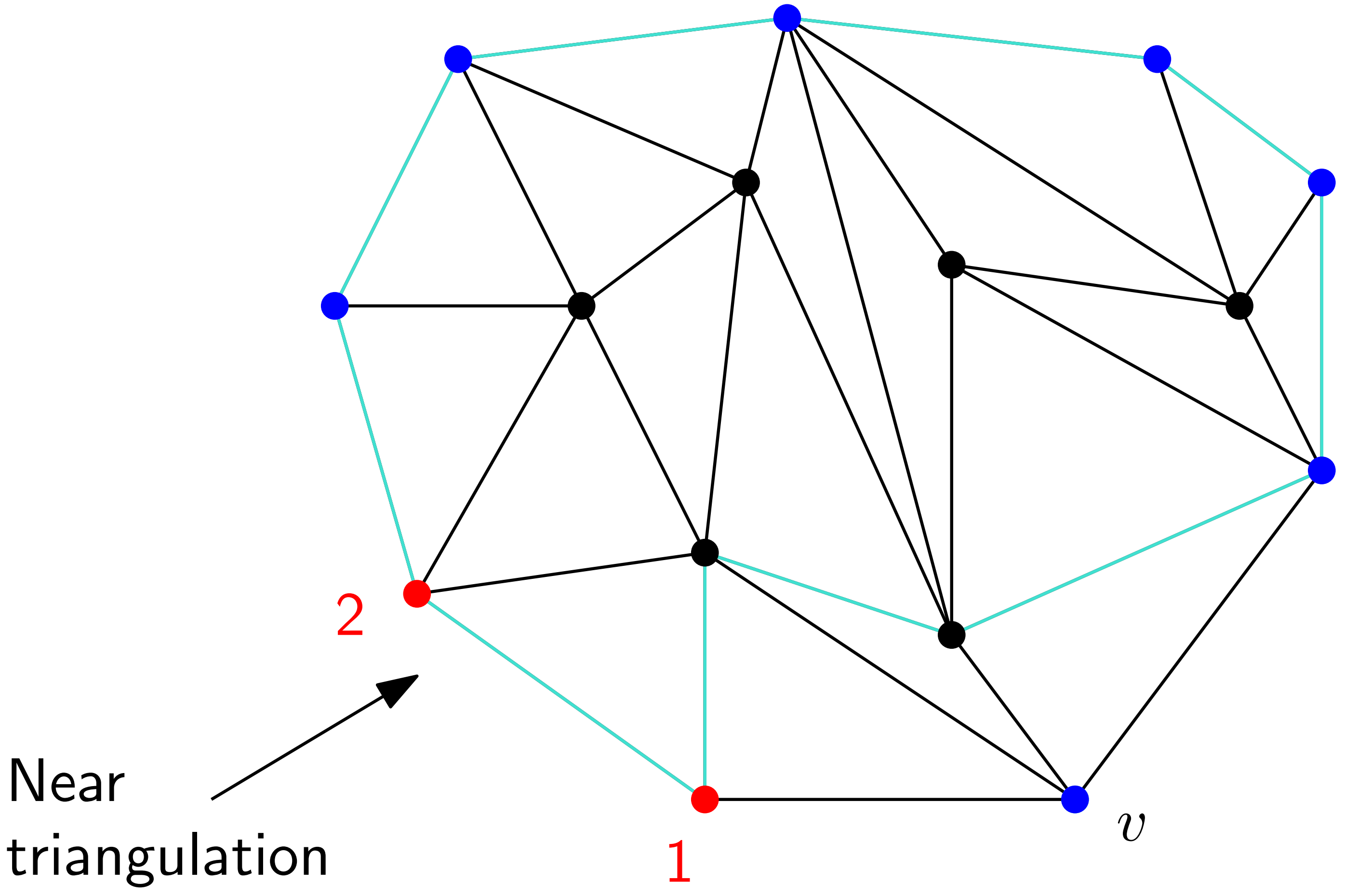

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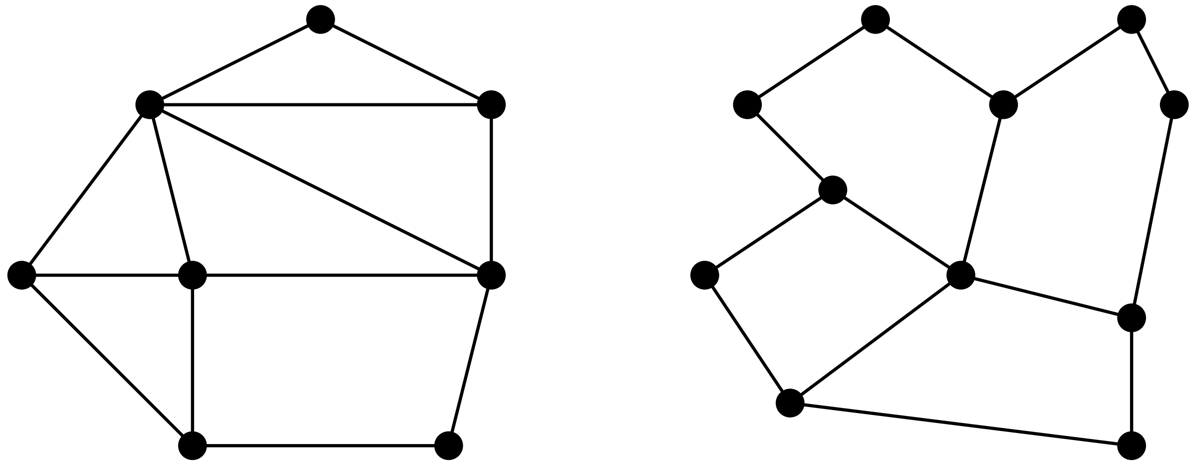

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Let $x, y \in L(v) - \{1\}$

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Not every planar graph of girth 4 is 3-choosable.

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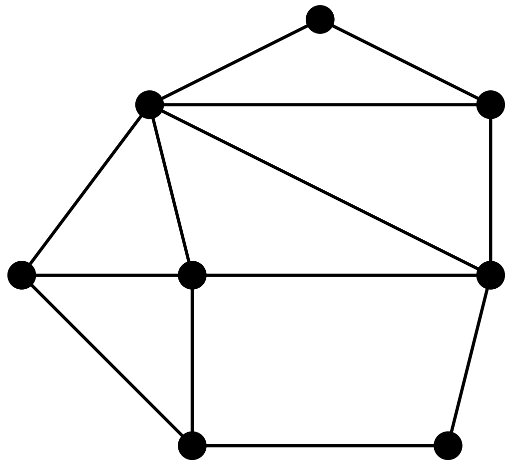
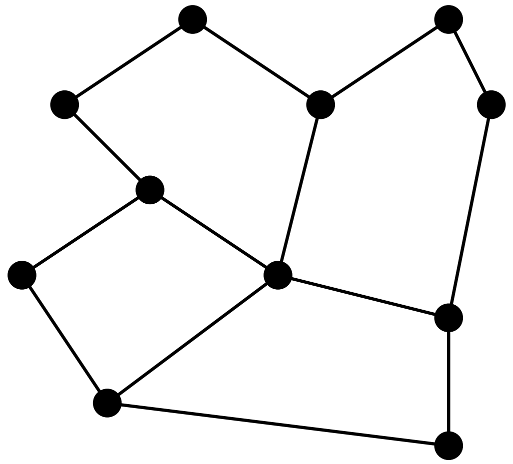
Girth of G is the length of its smallest cycle.



girth 3



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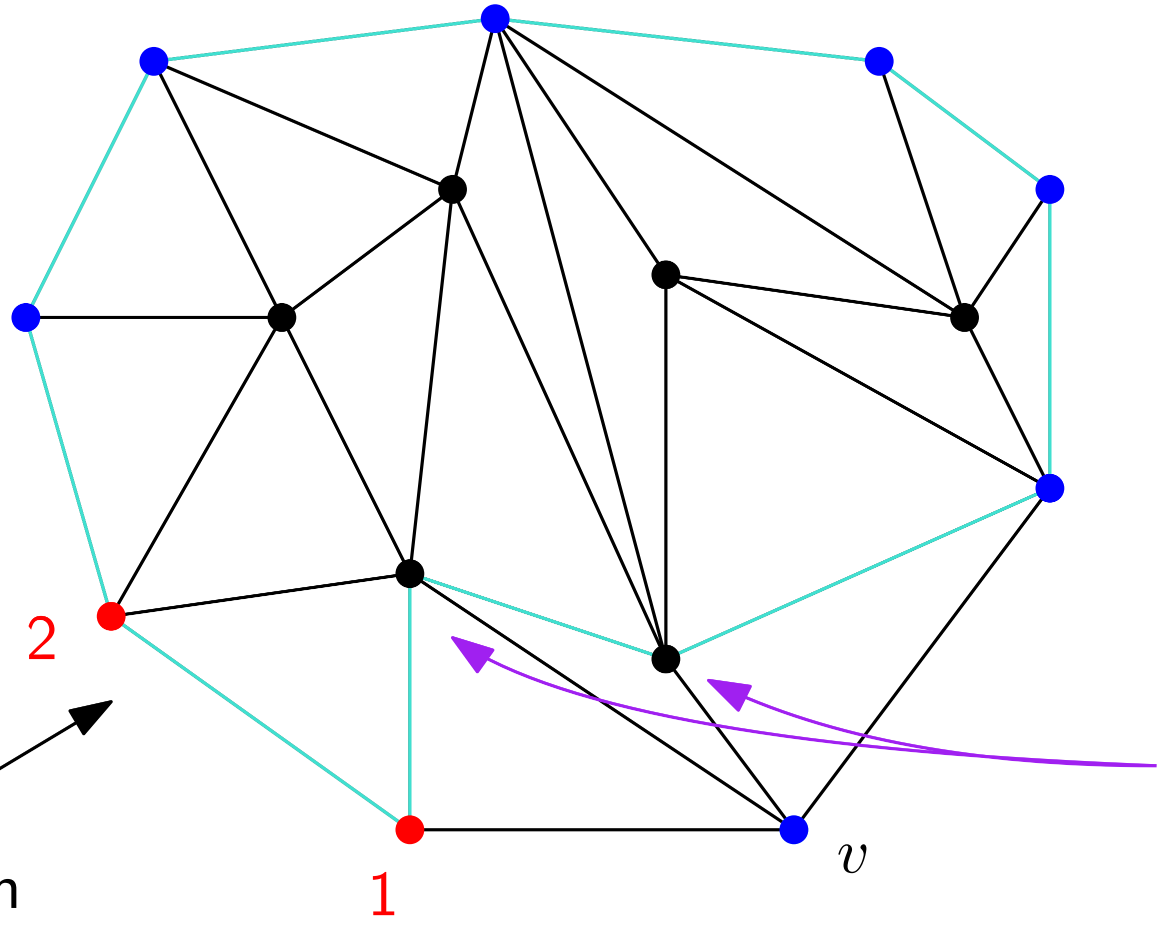

Near triangulation

Induction (by $|V(G)|$): G is 5-choosable even, if:

- We precolor two consecutive vertices from the outer face.
- We limit the lists of other vertices from the outer face to only 3 colors.

If outer face has a chord - induction.

Let v be incident to our vertex precolored by 1

Let $x, y \in L(v) - \{1\}$

Forbid x, y from those lists.

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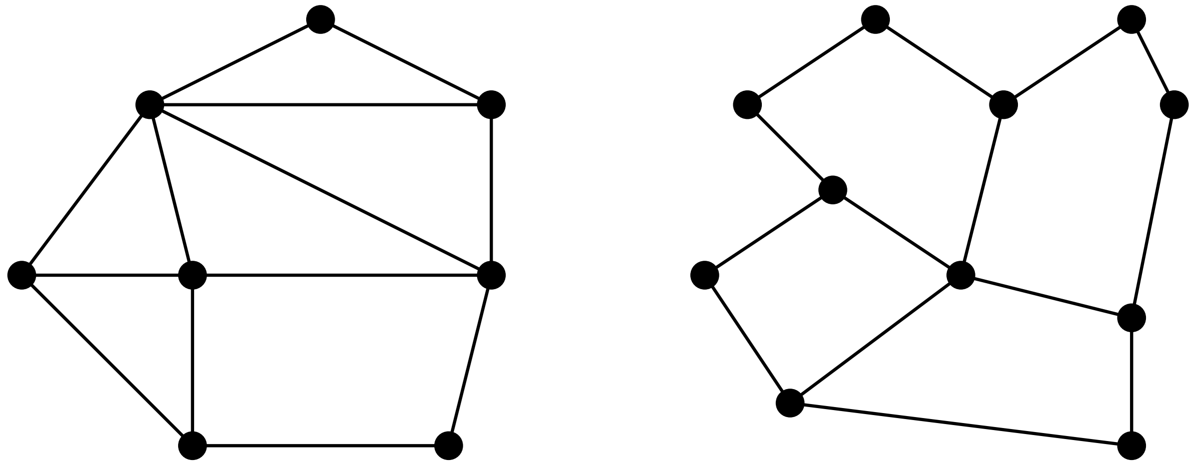
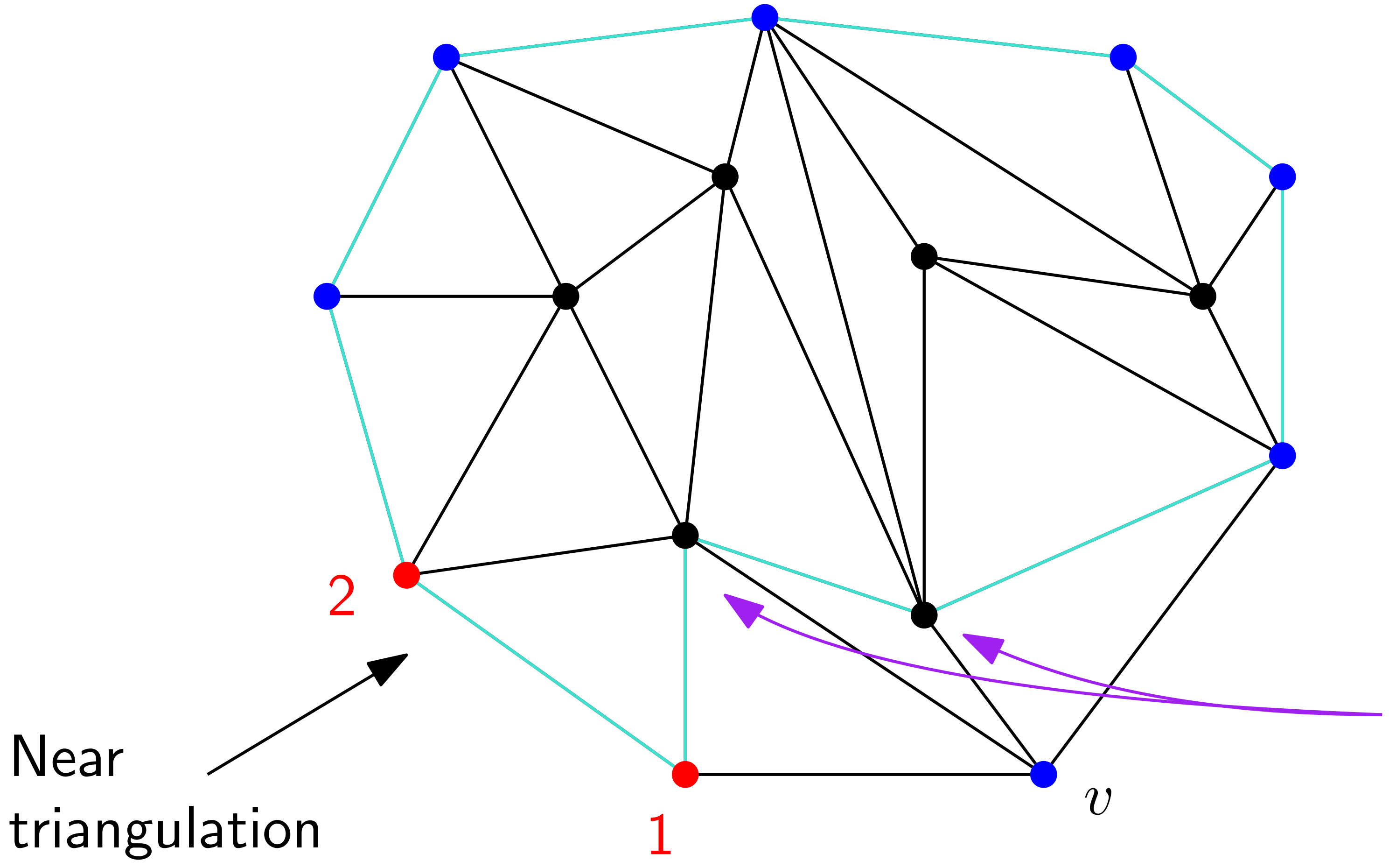


girth 3 girth 4



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Color by induction, then select for v either x or y so as not to collide with its other neighbour.

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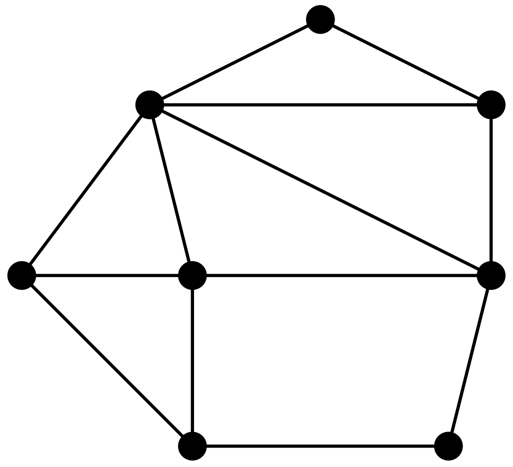

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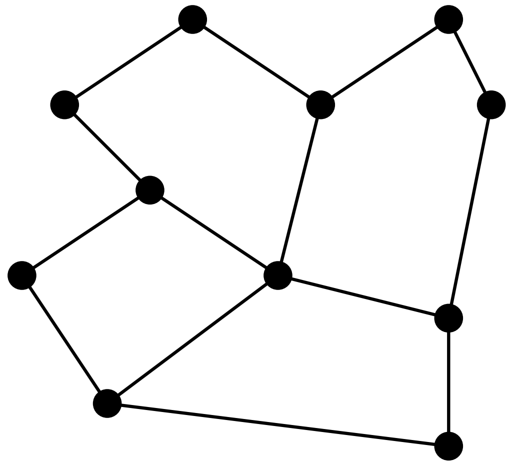

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  hard (random lemmas until we suddenly get what we wanted)

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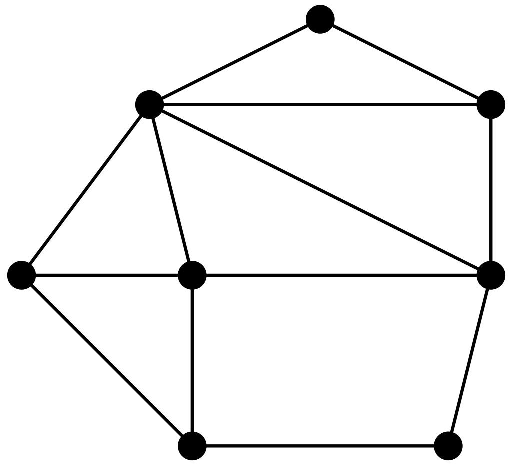

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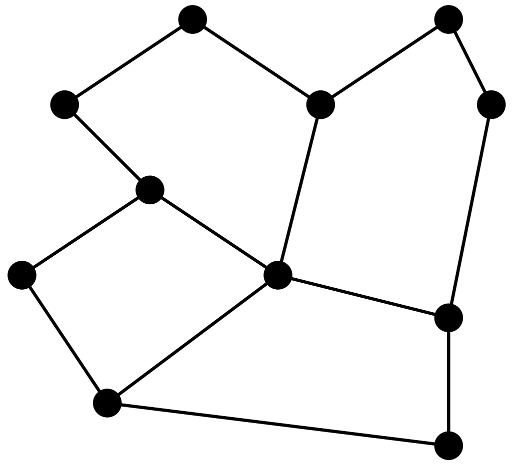

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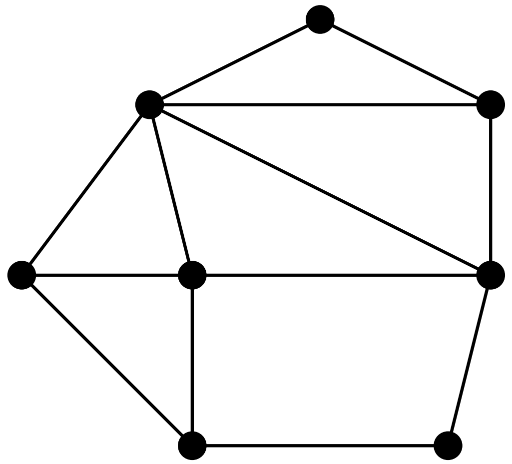

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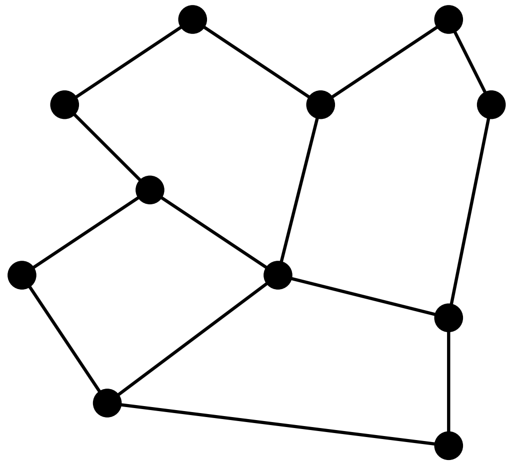

girth 4

$$\max\{k : \exists_G G\text{-planar}, \ g(G) = 4, \mathsf{ch}(G) = k\} \in \{3, 4, 5\}$$

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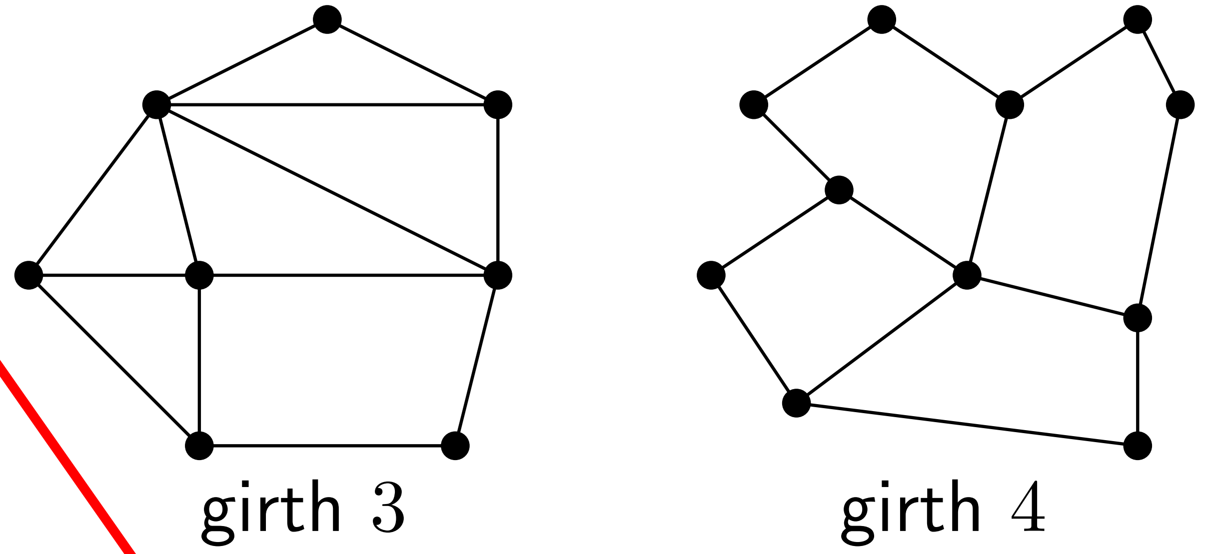

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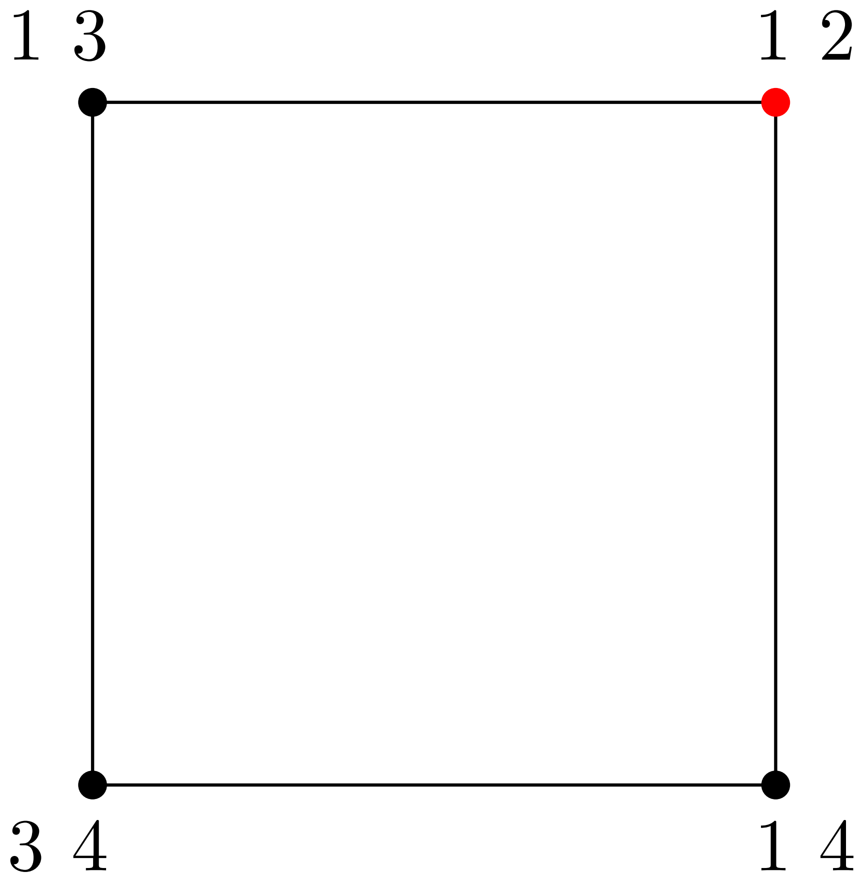

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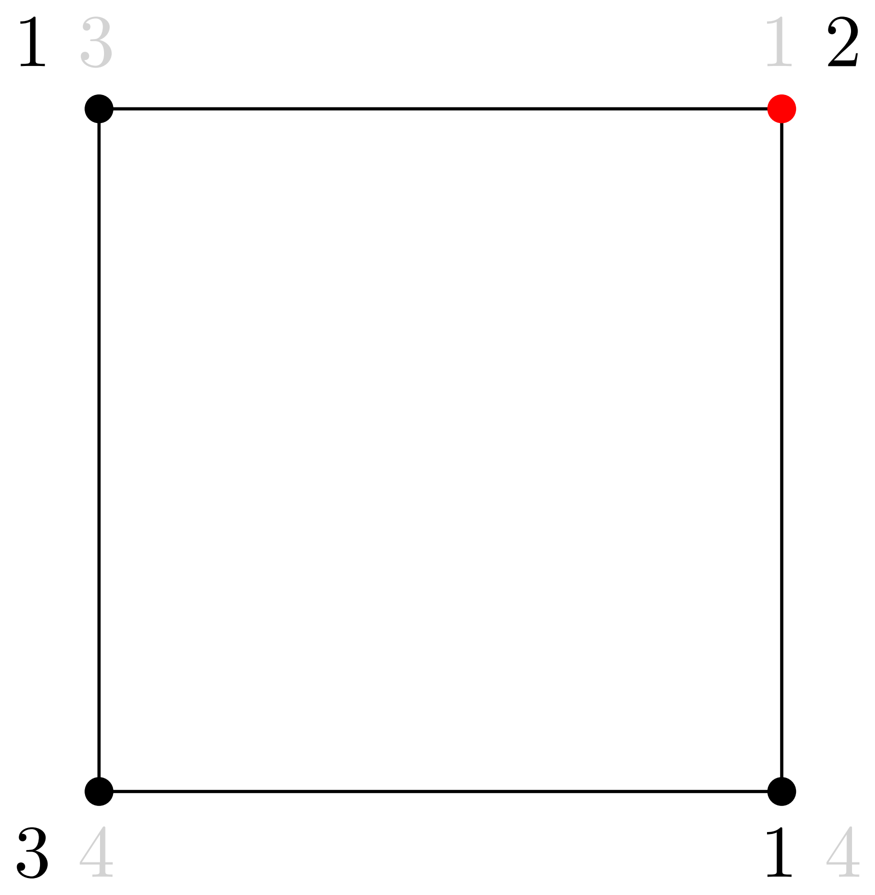

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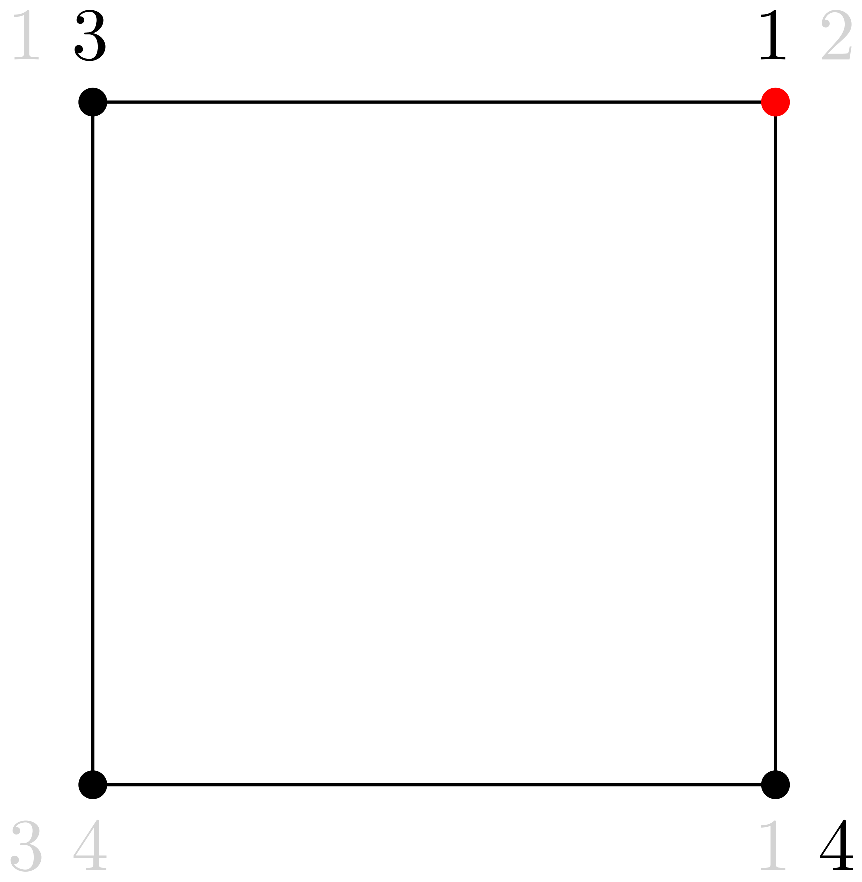

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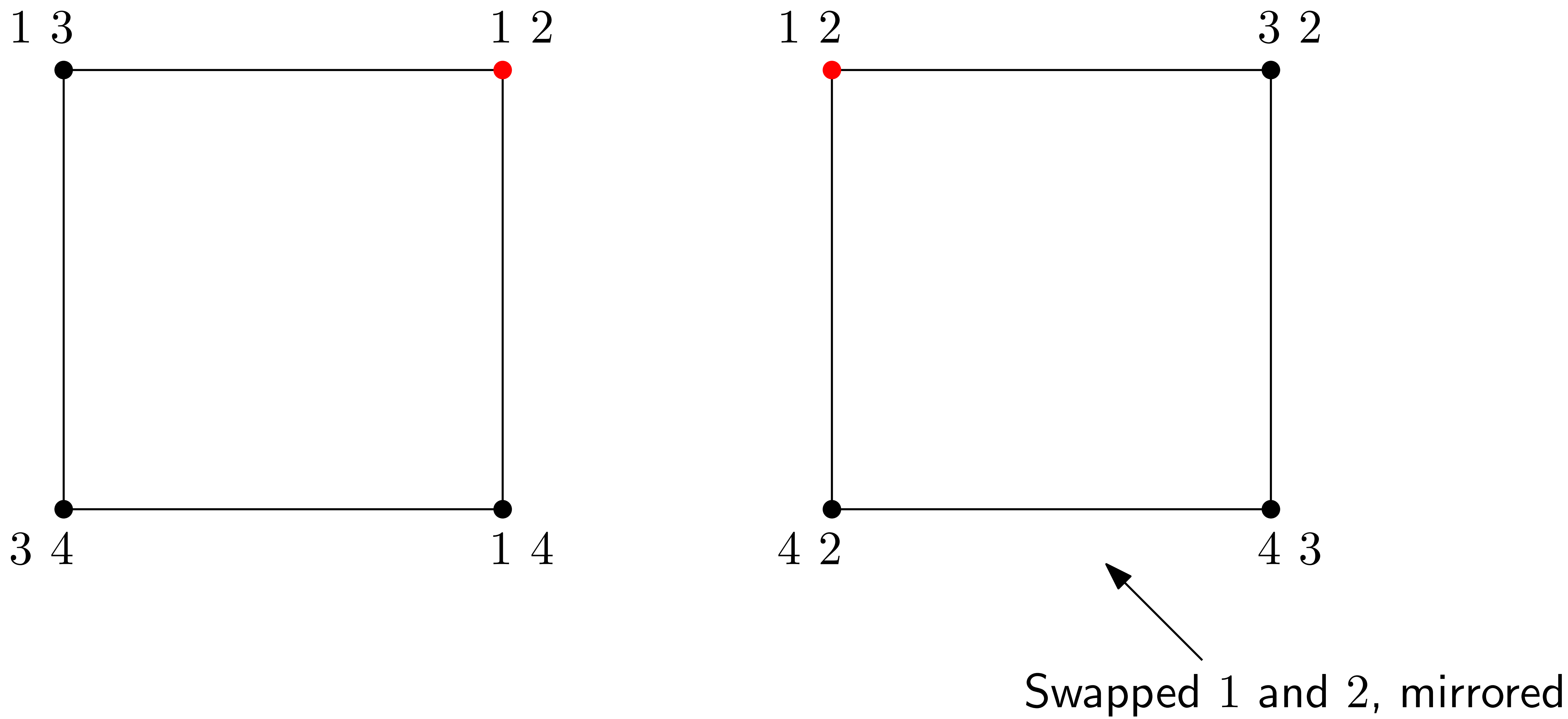

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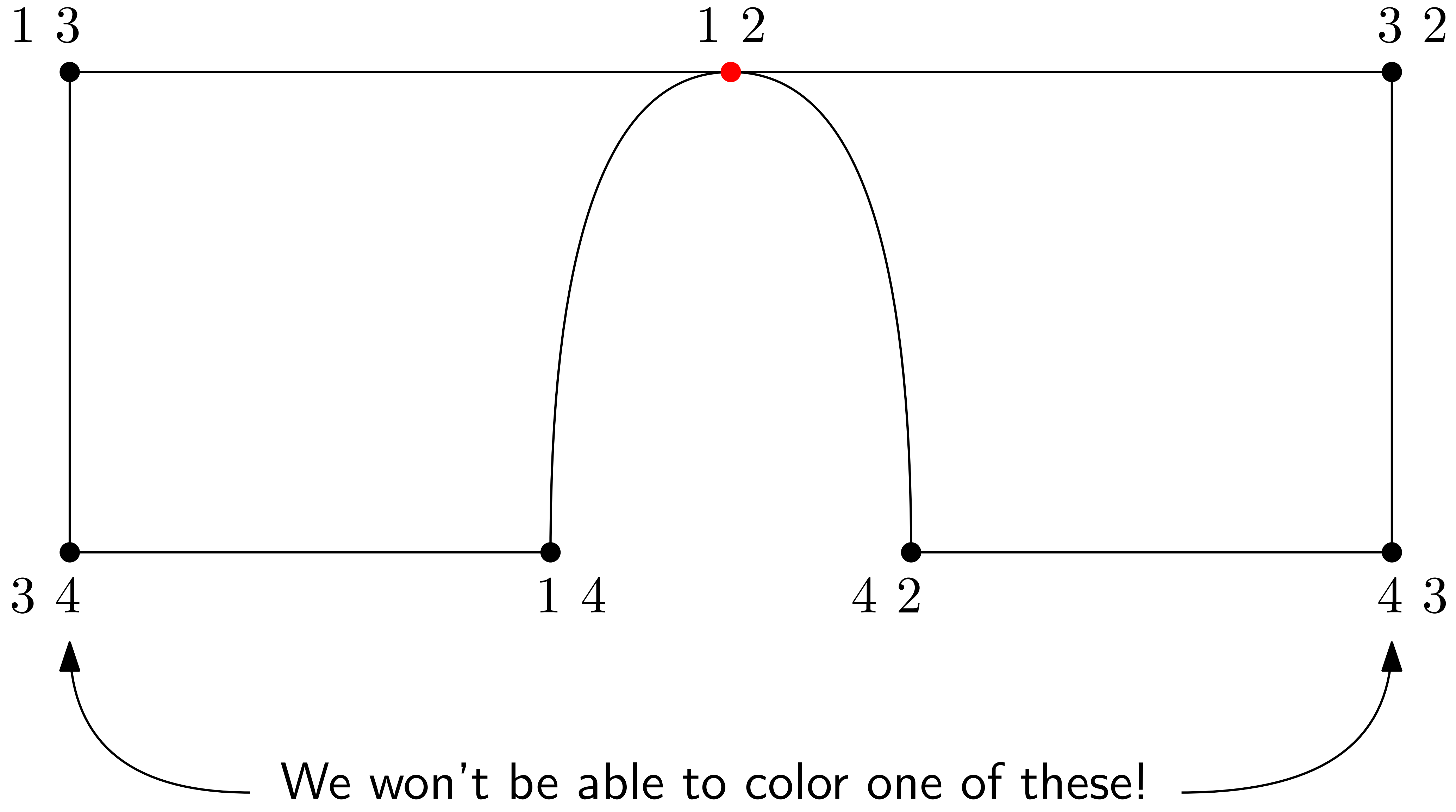

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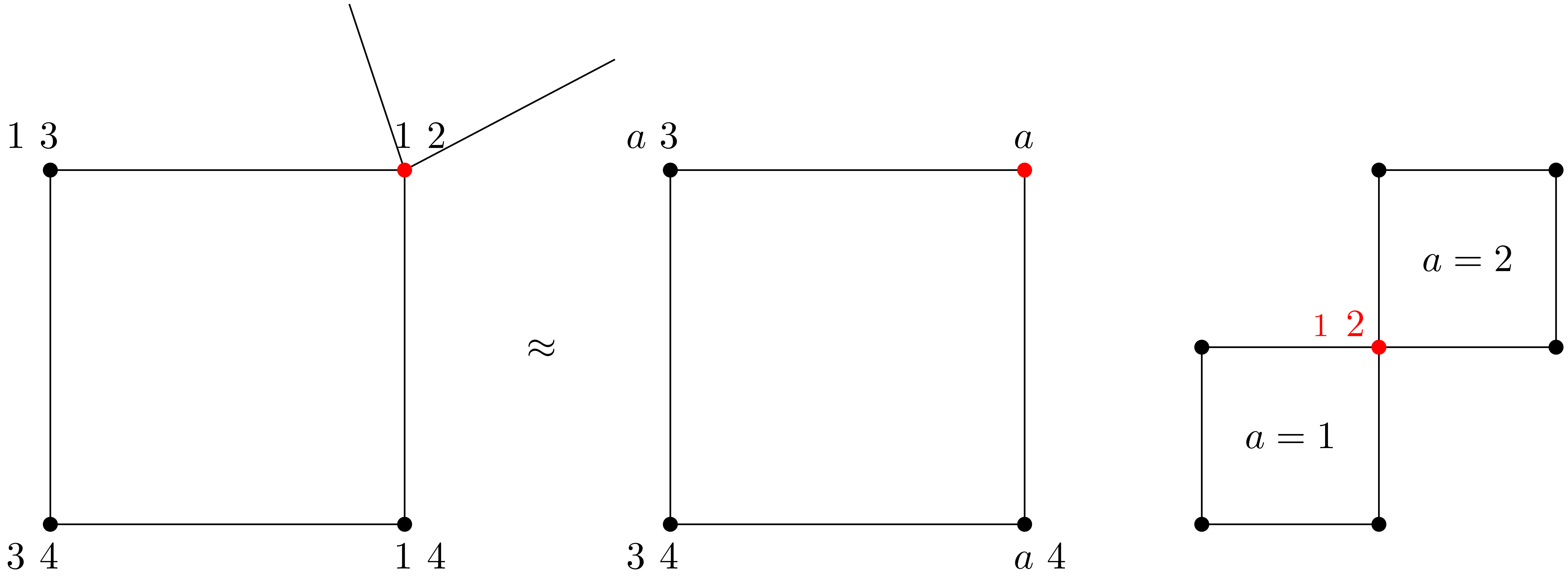

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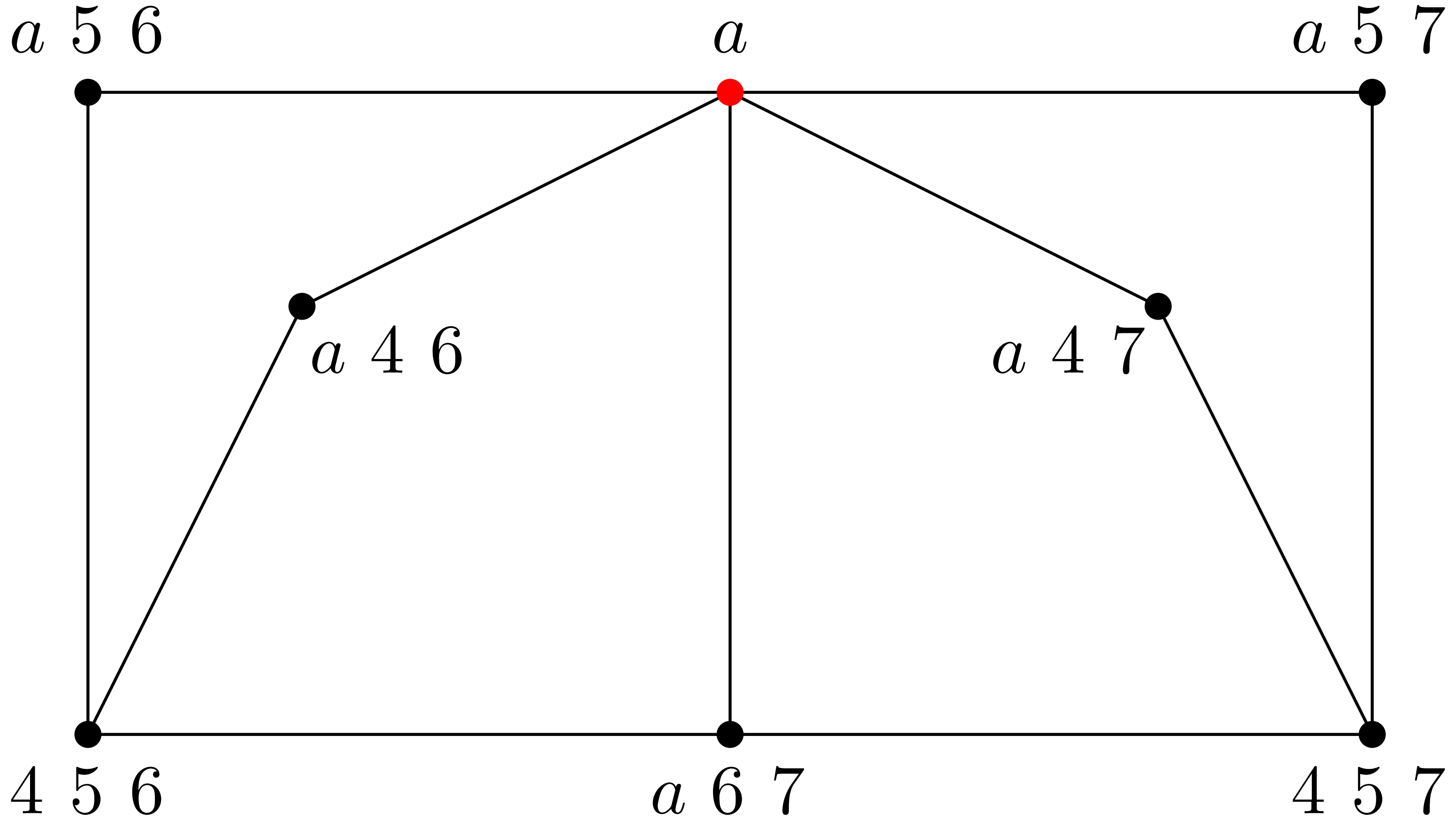

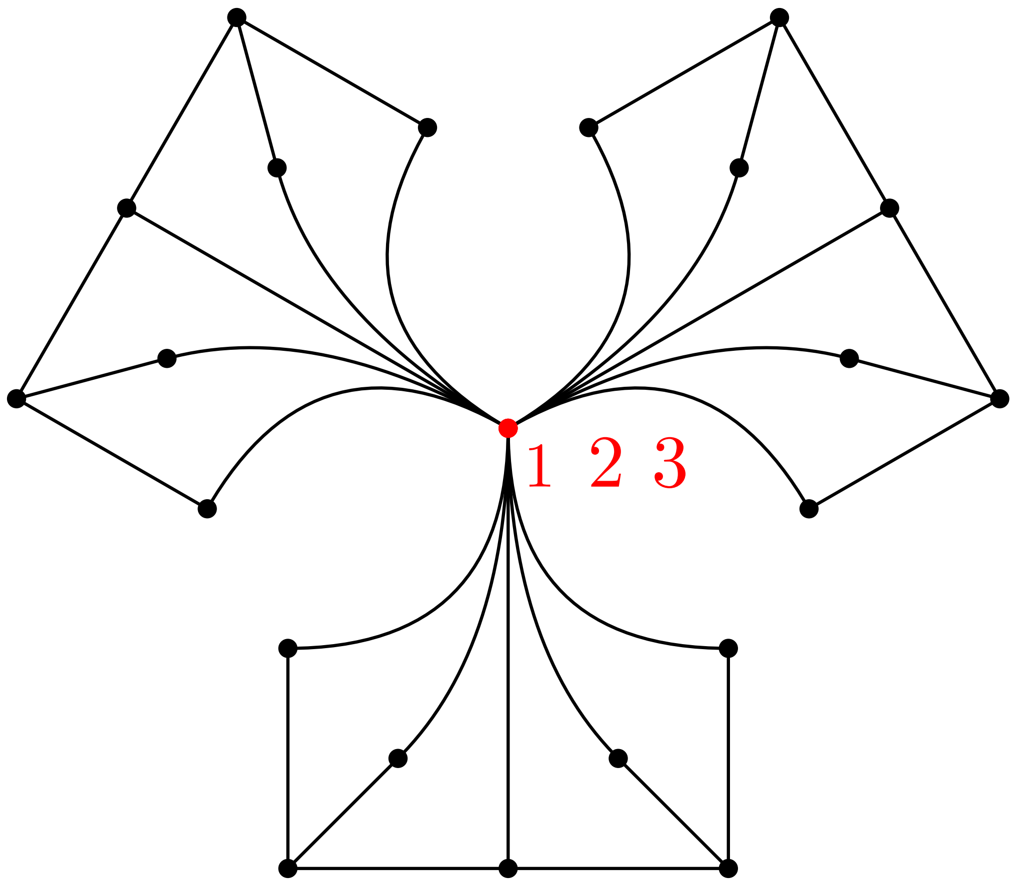

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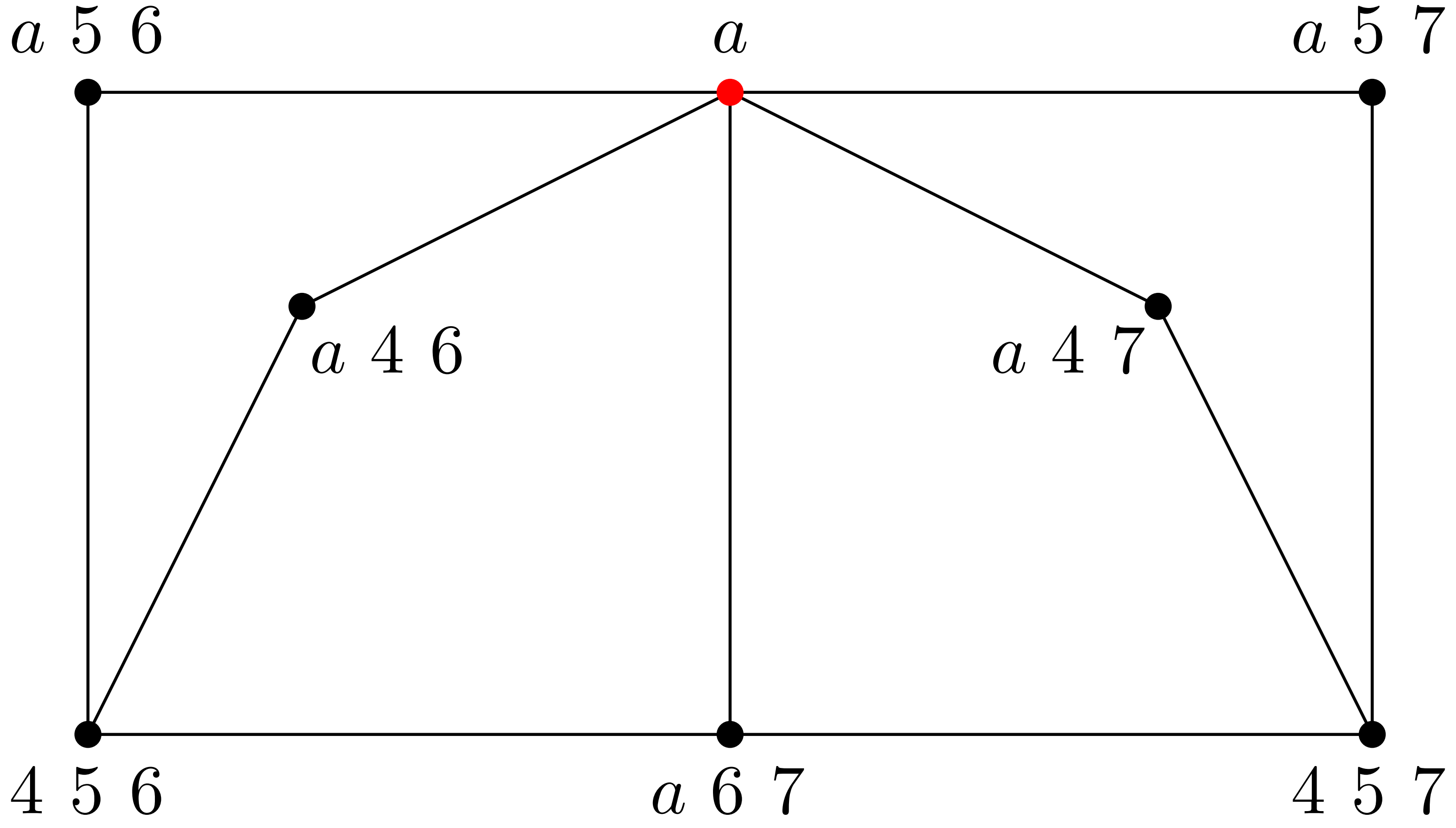

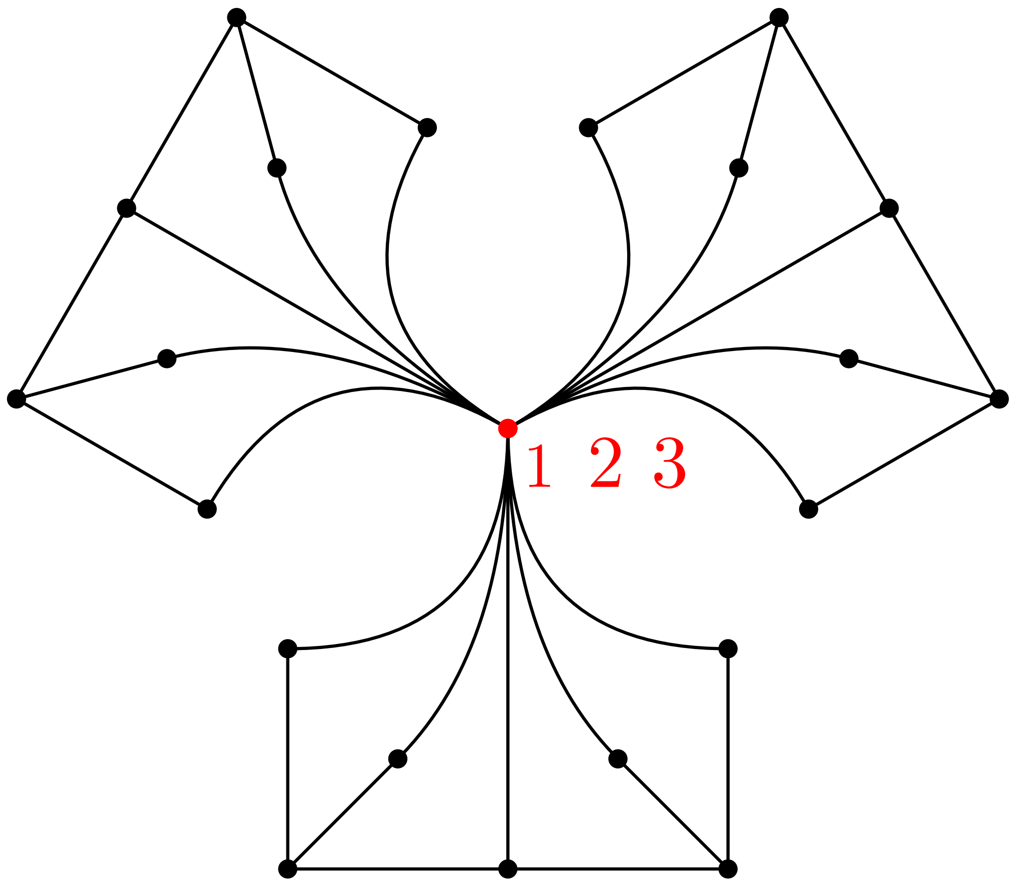

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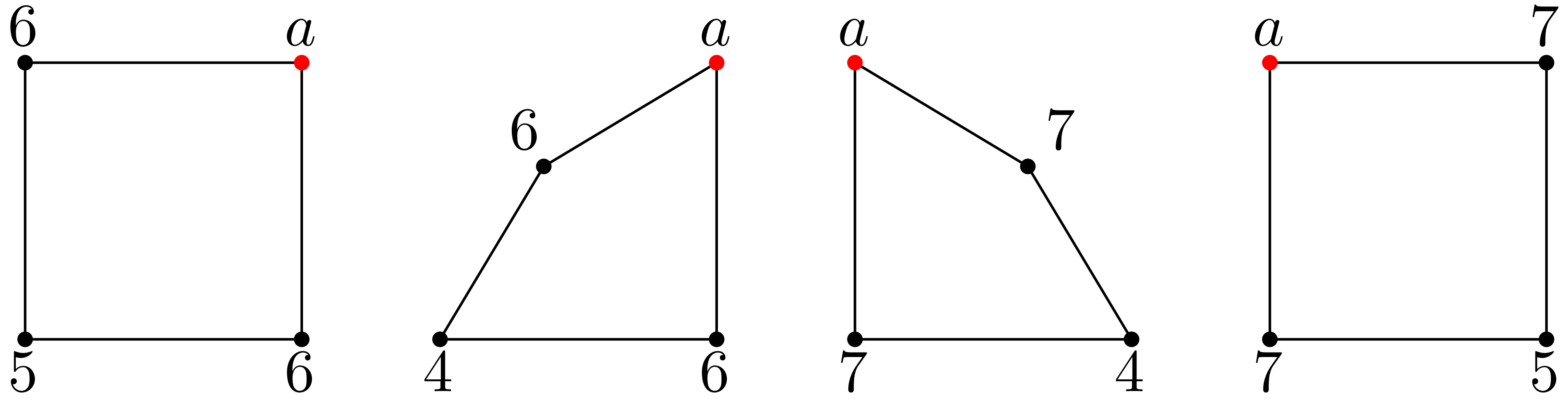

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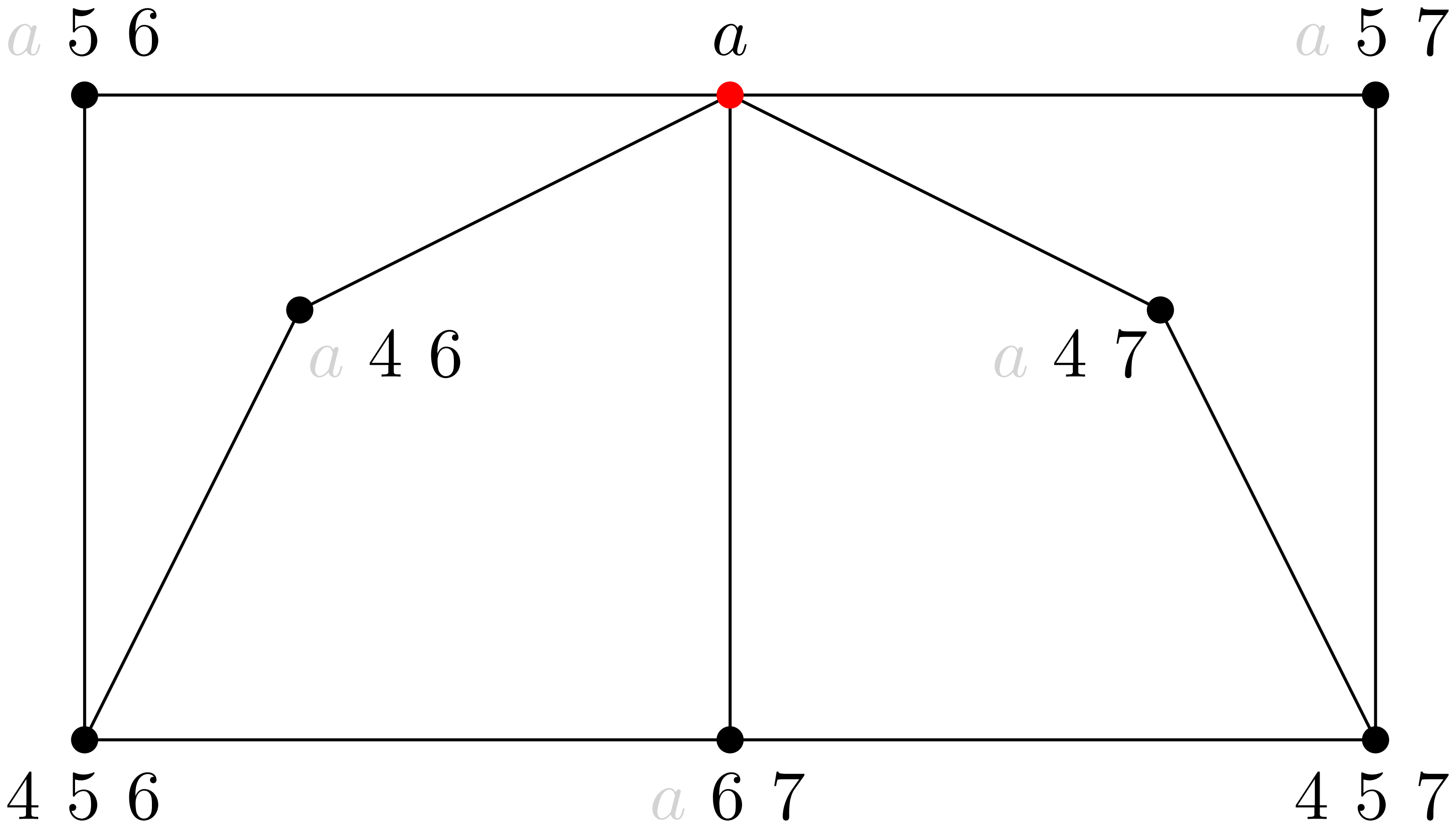

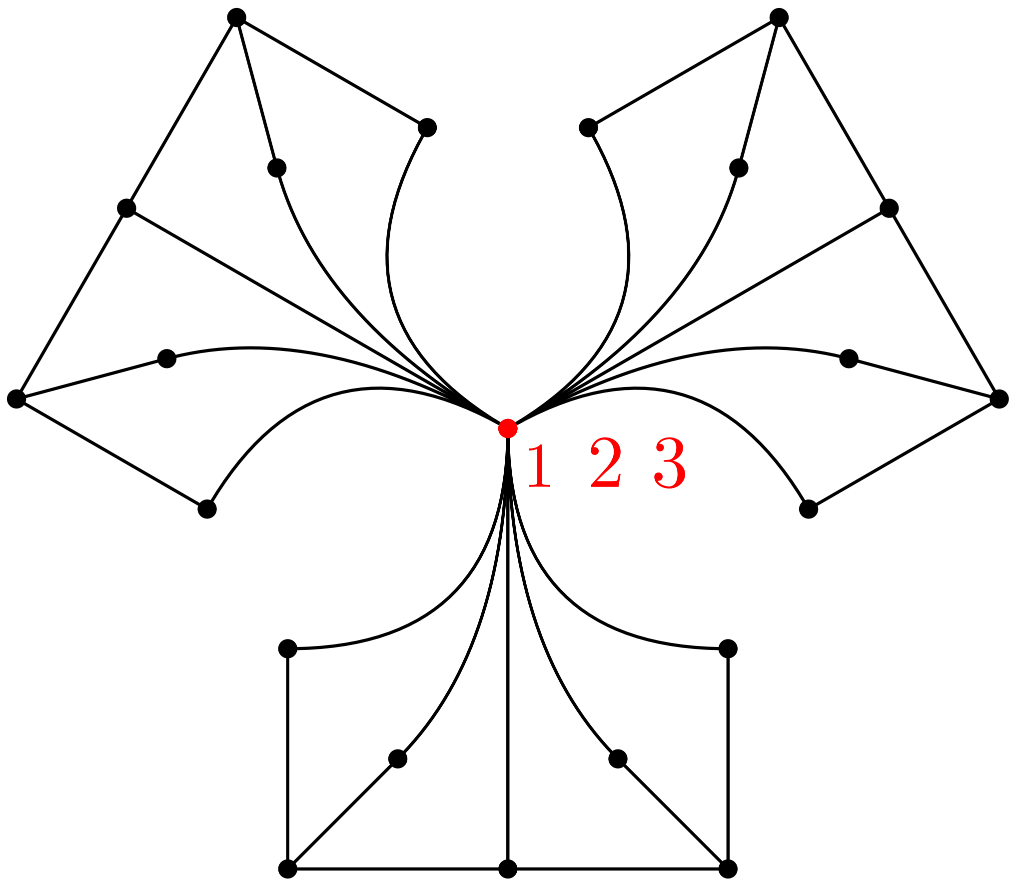

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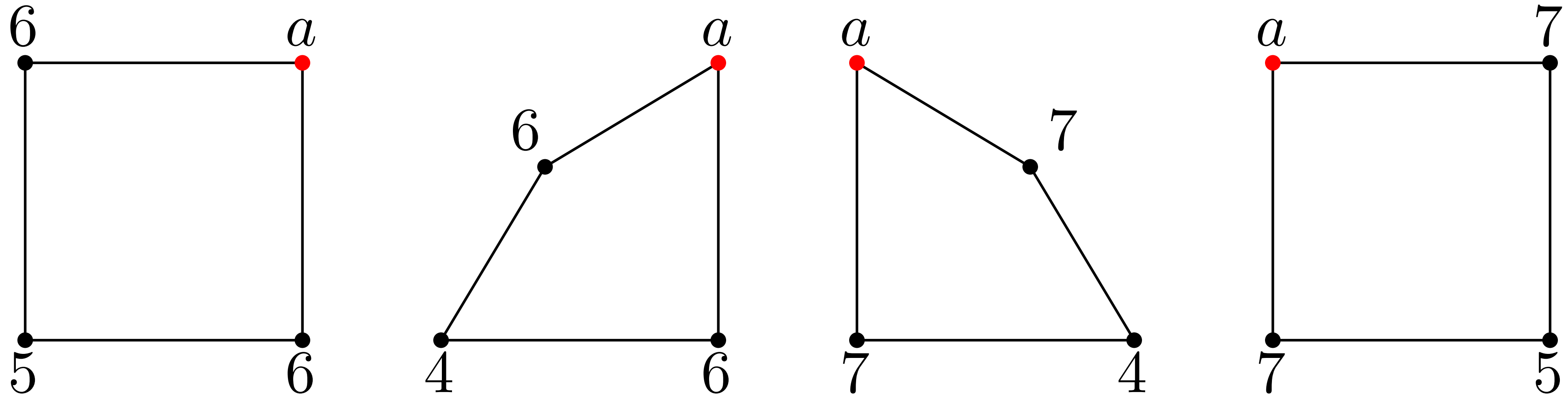

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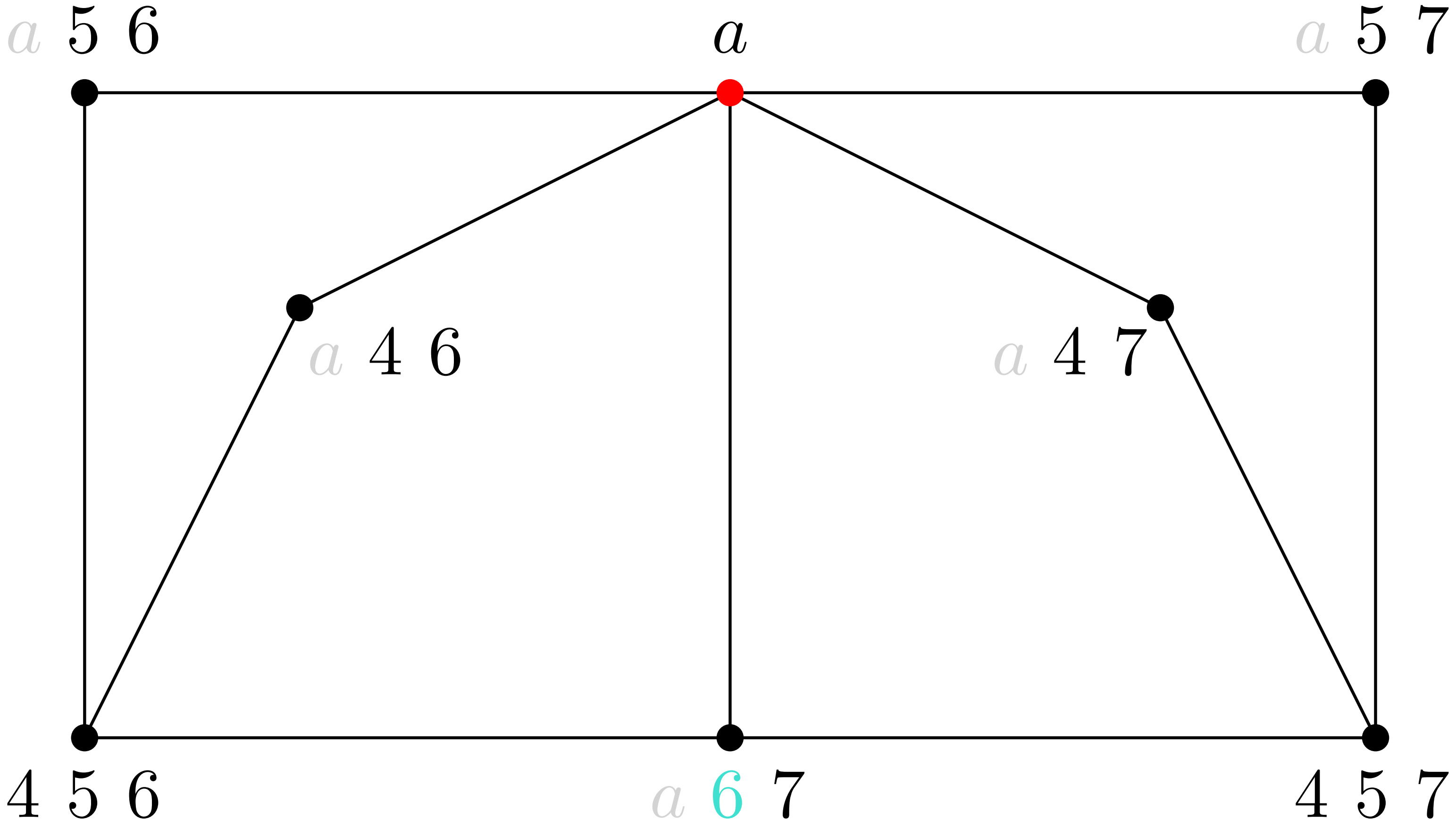

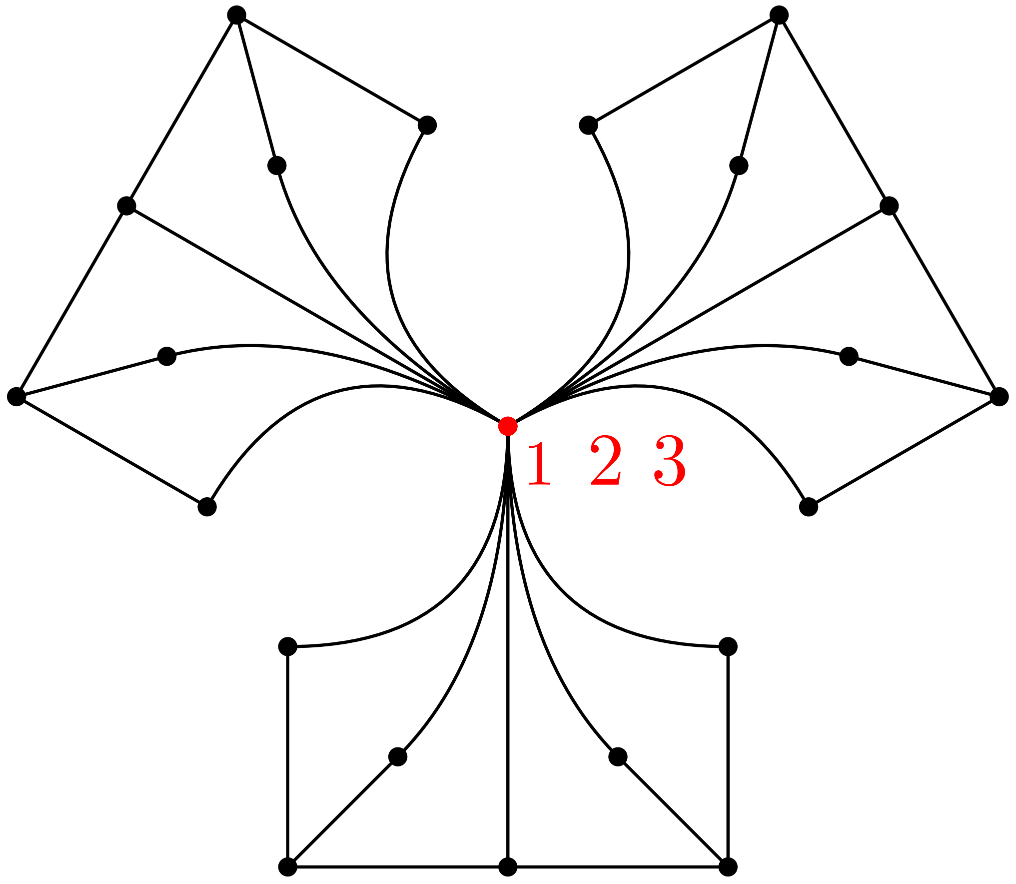

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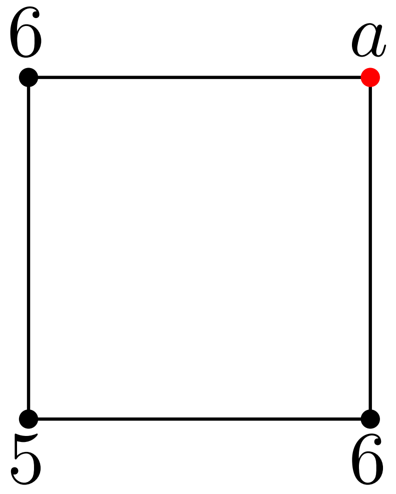

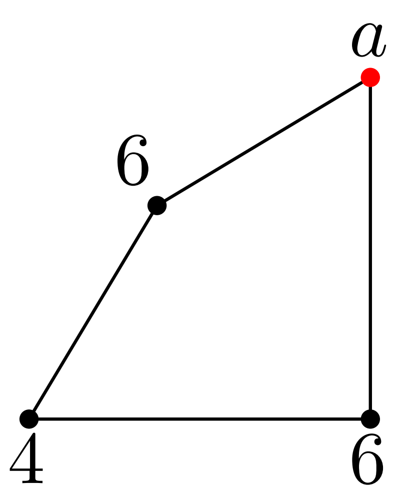

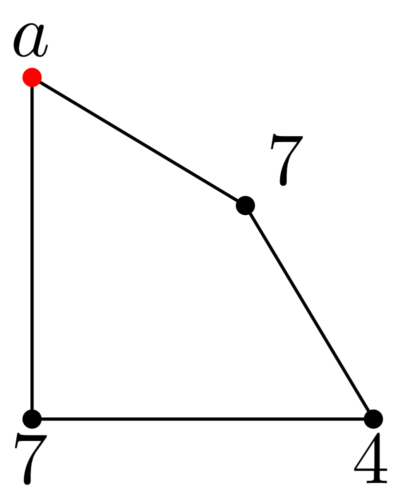

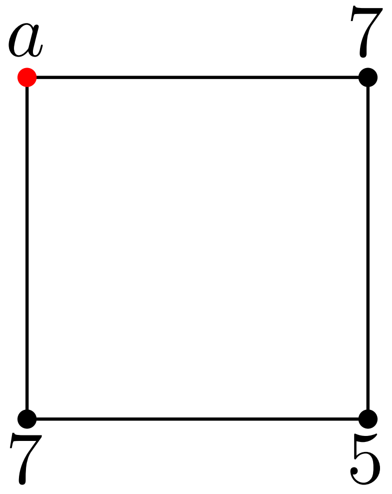

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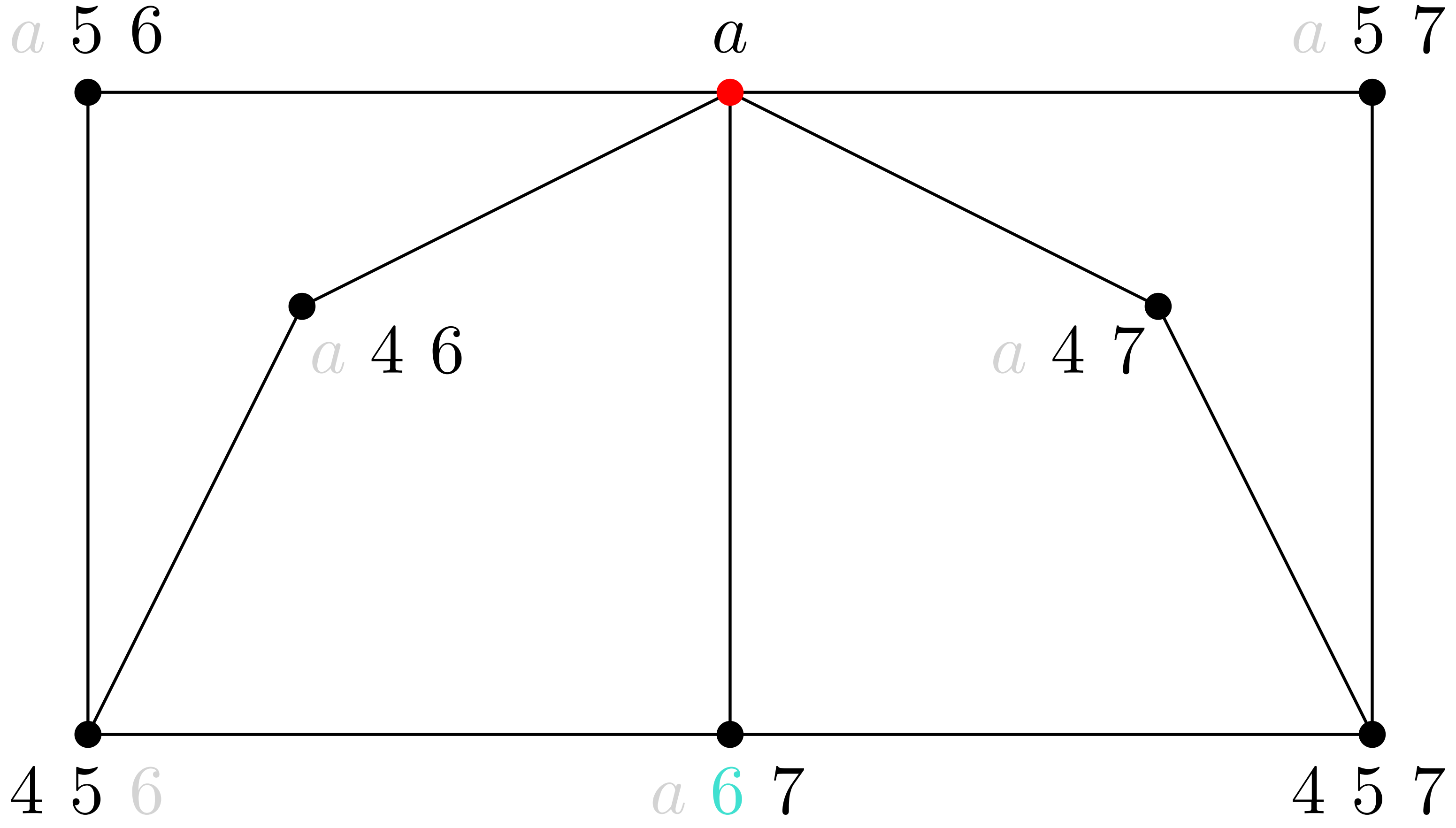

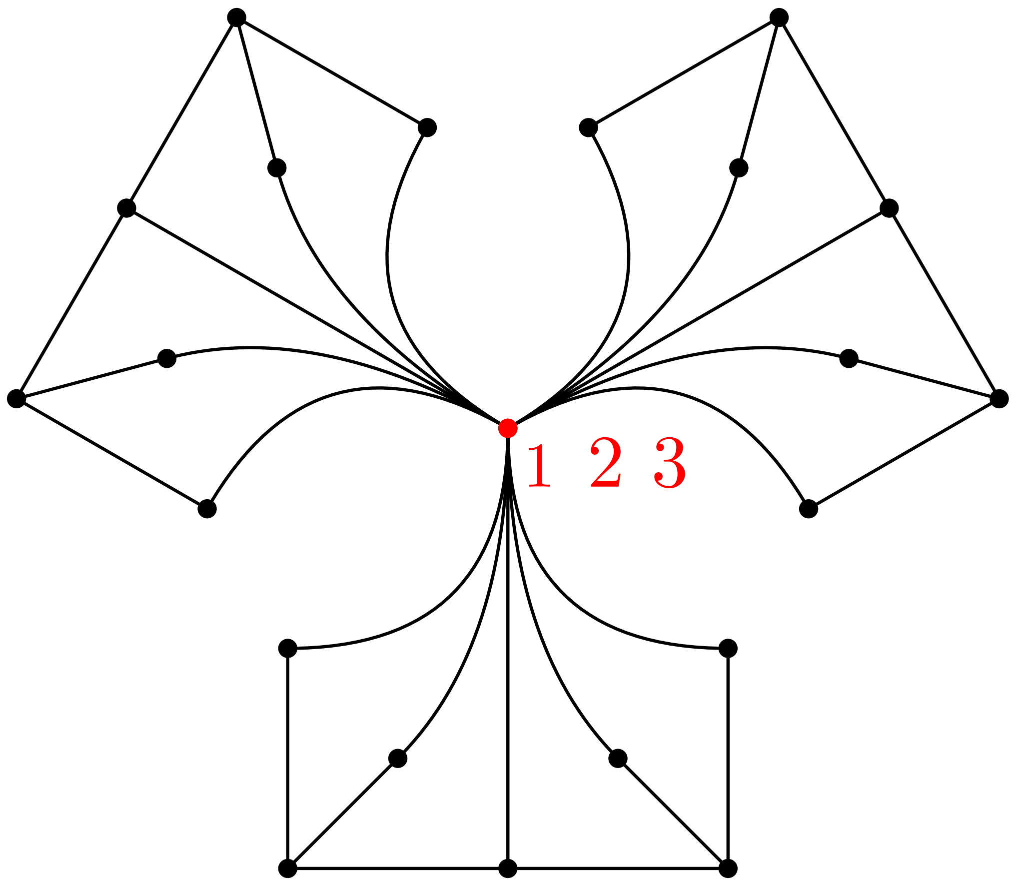

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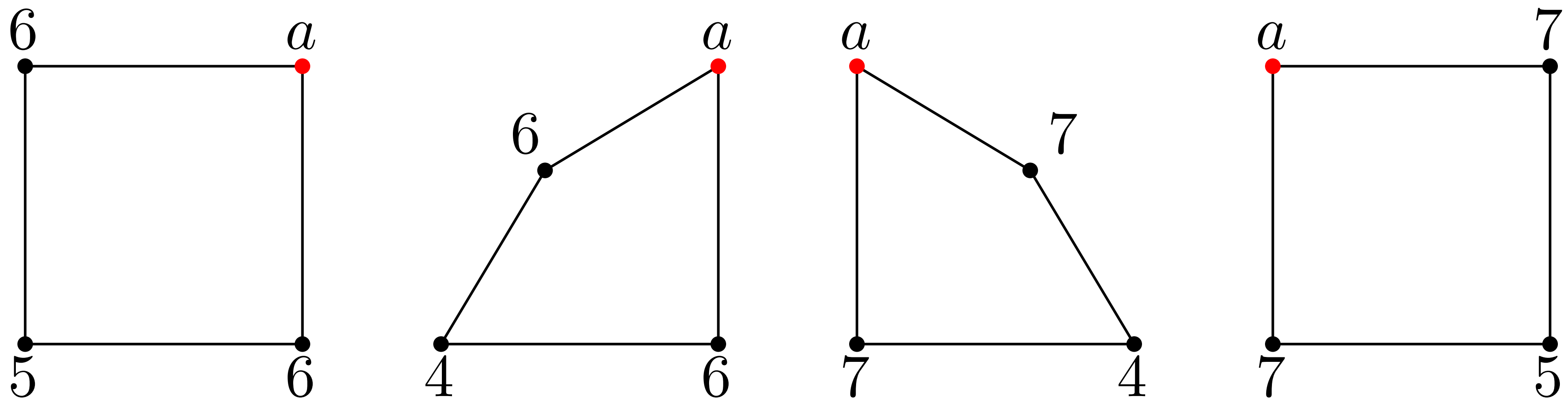

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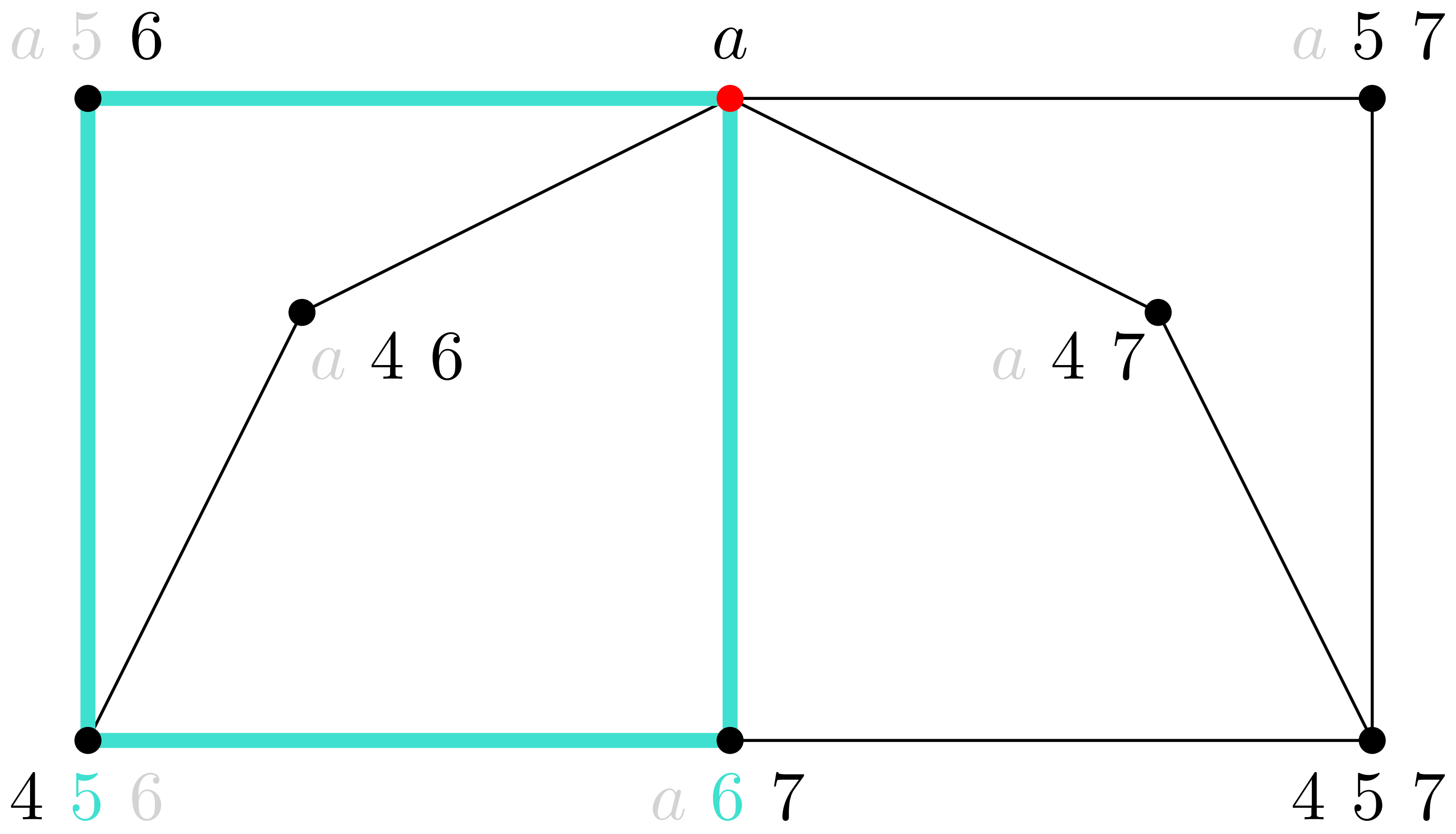

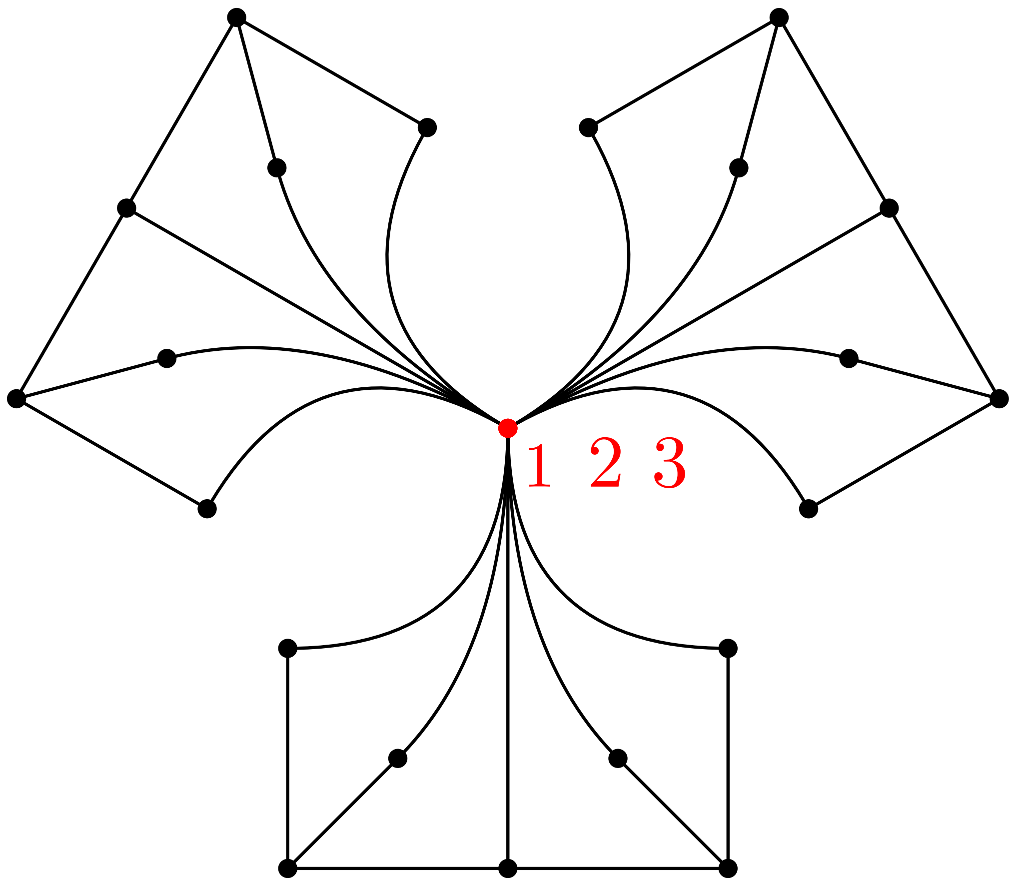

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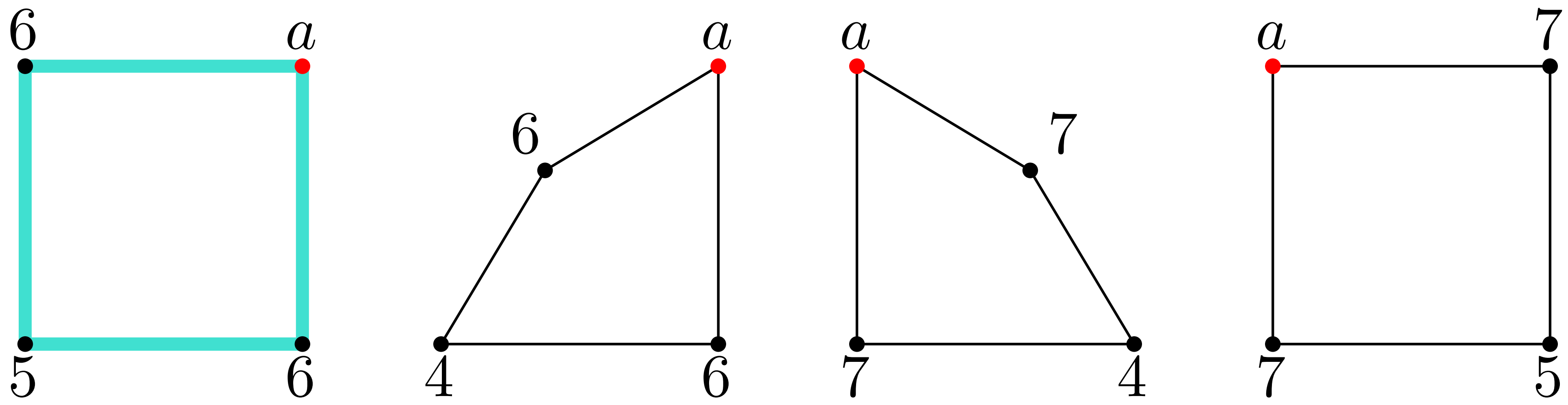

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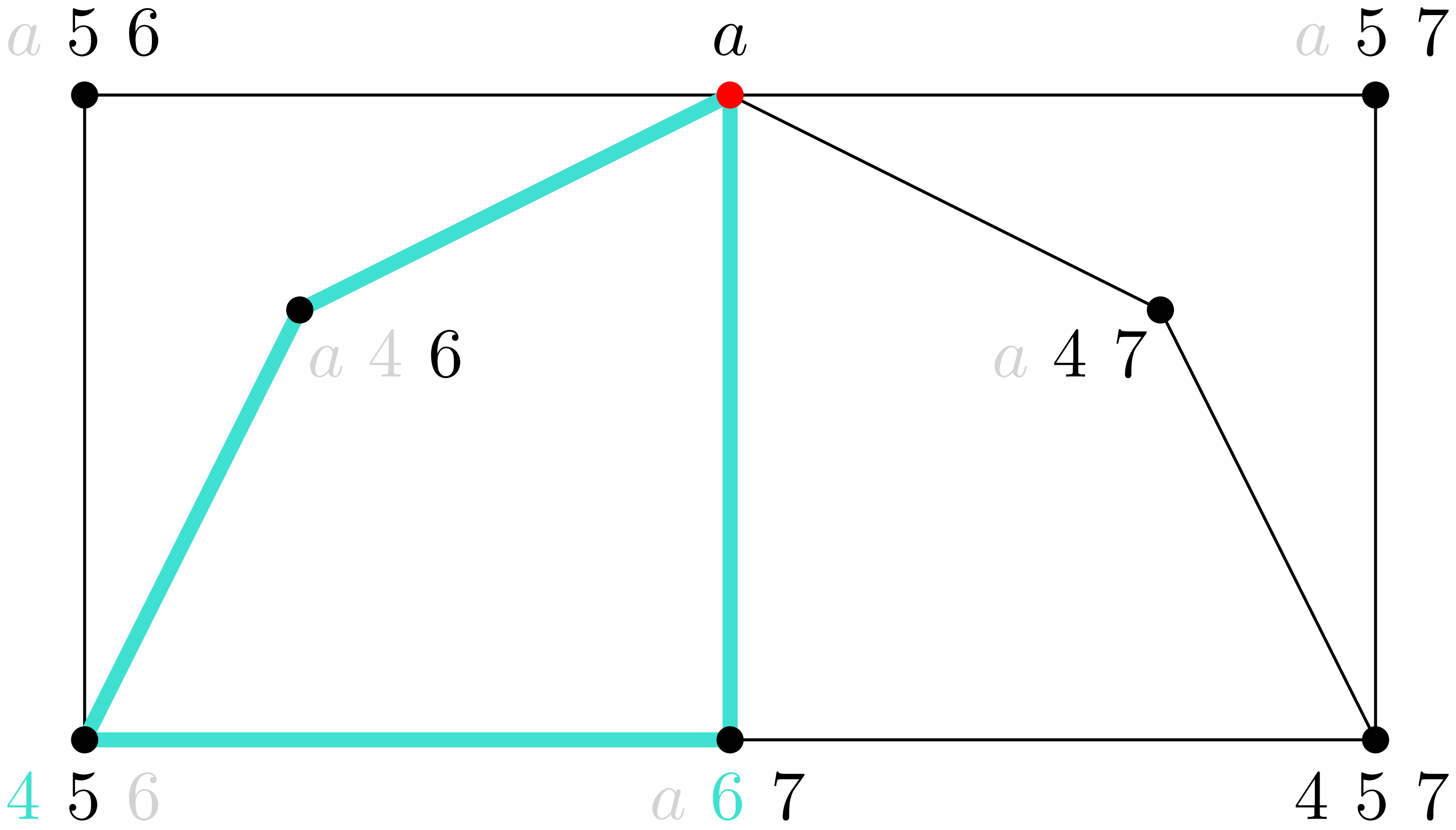

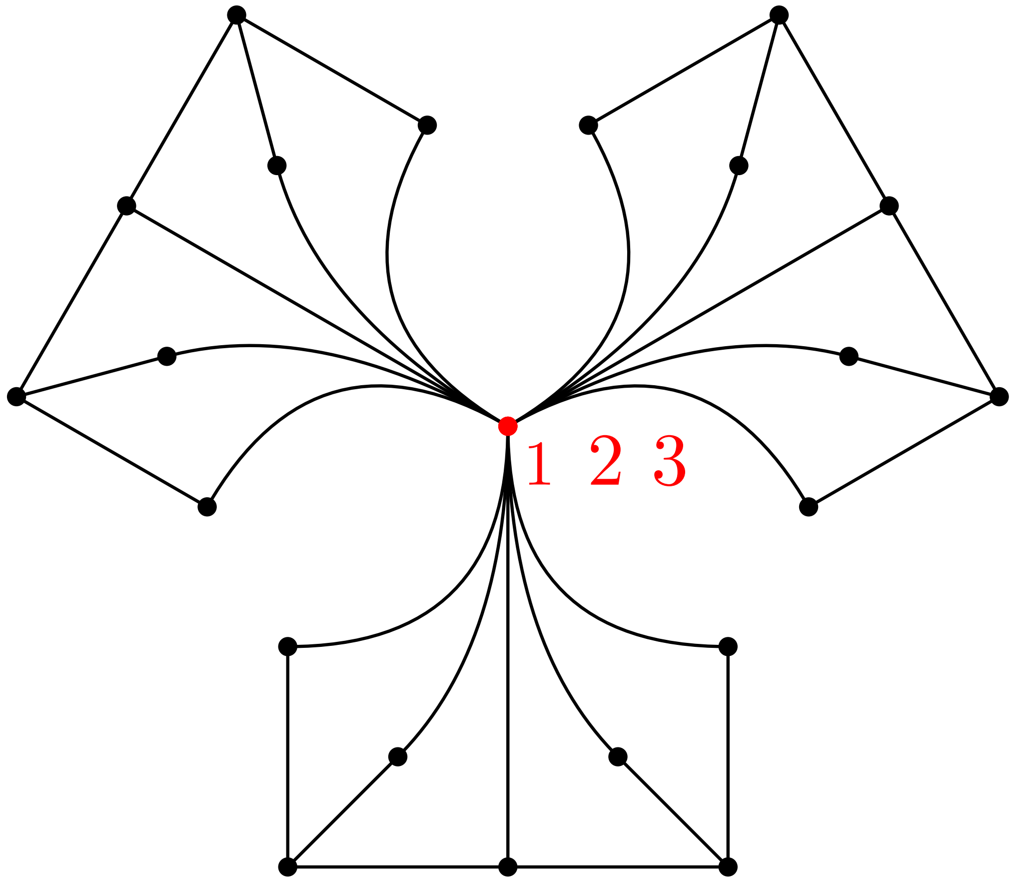

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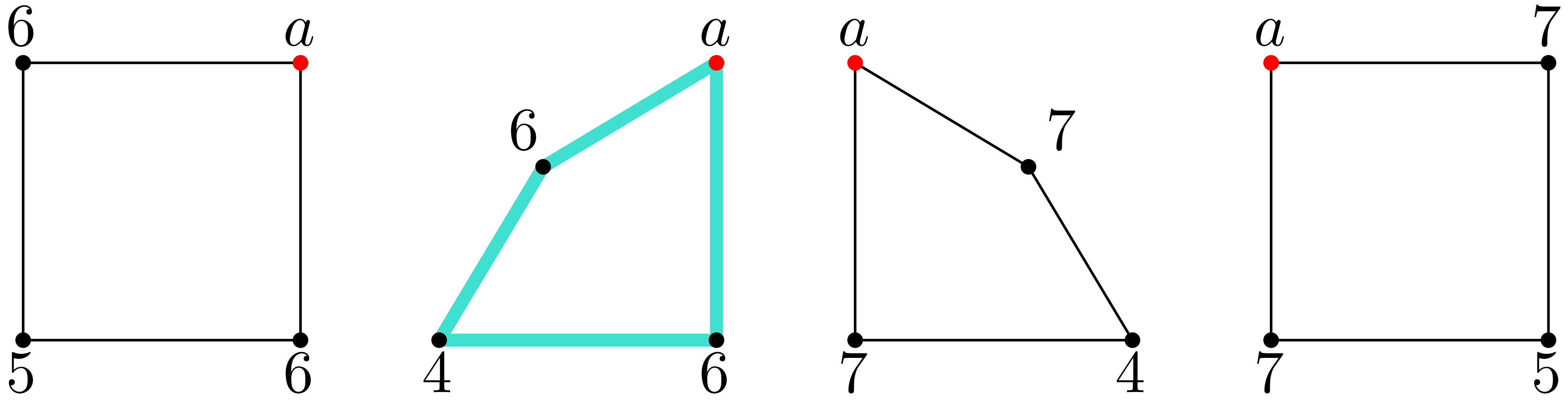

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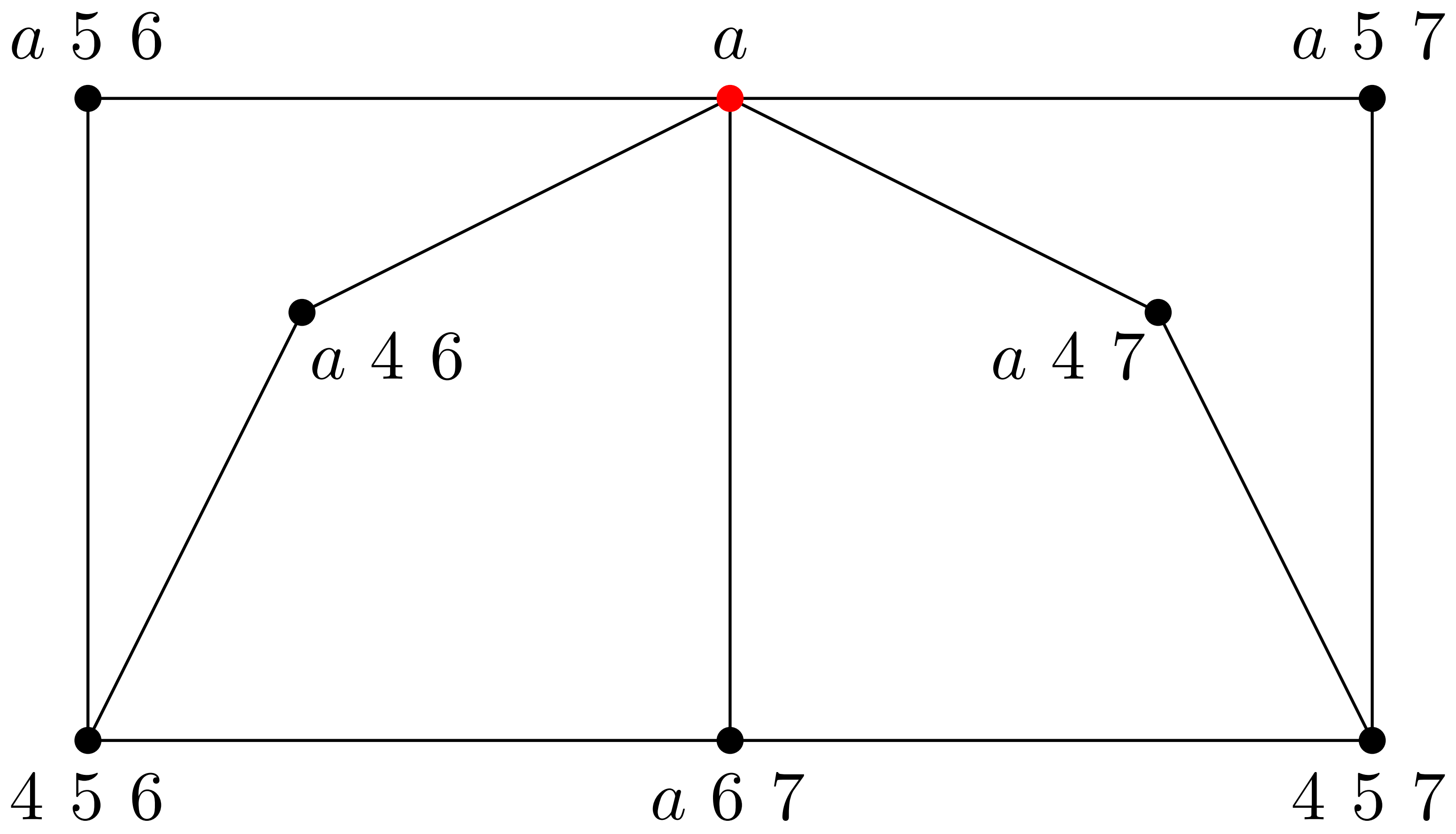

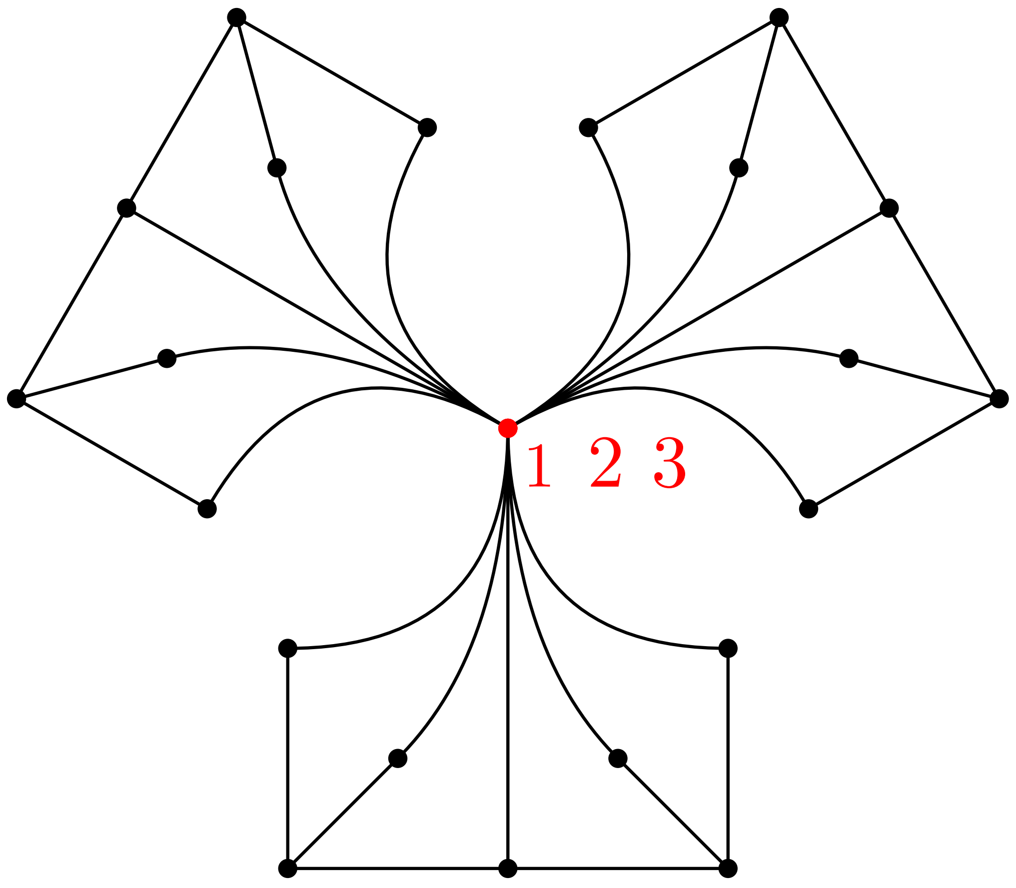

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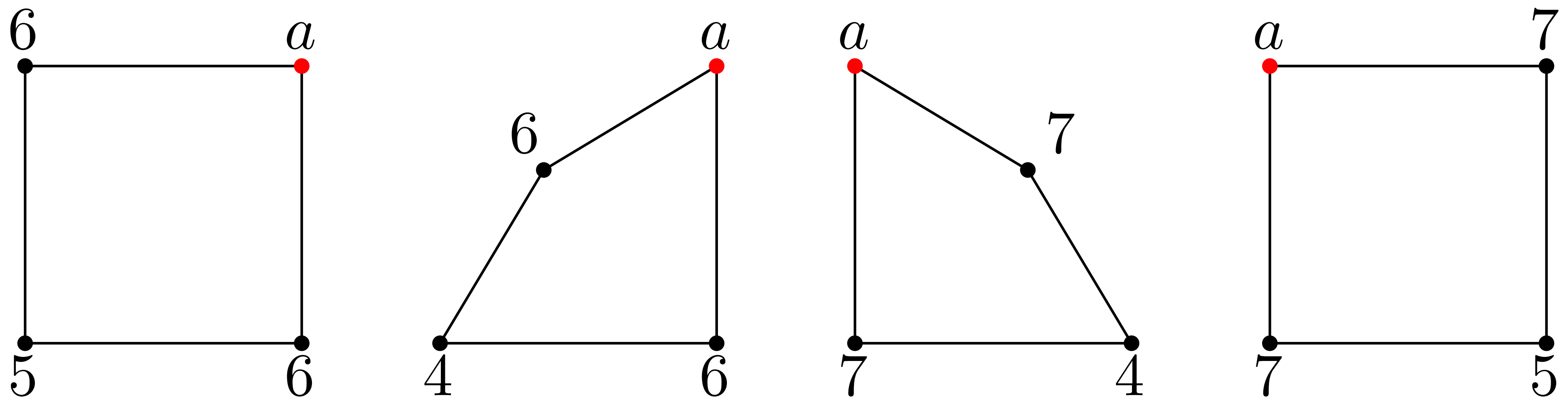

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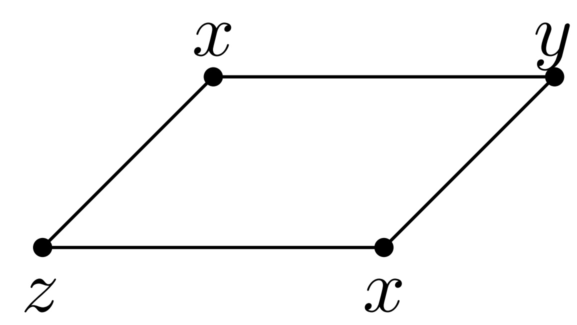

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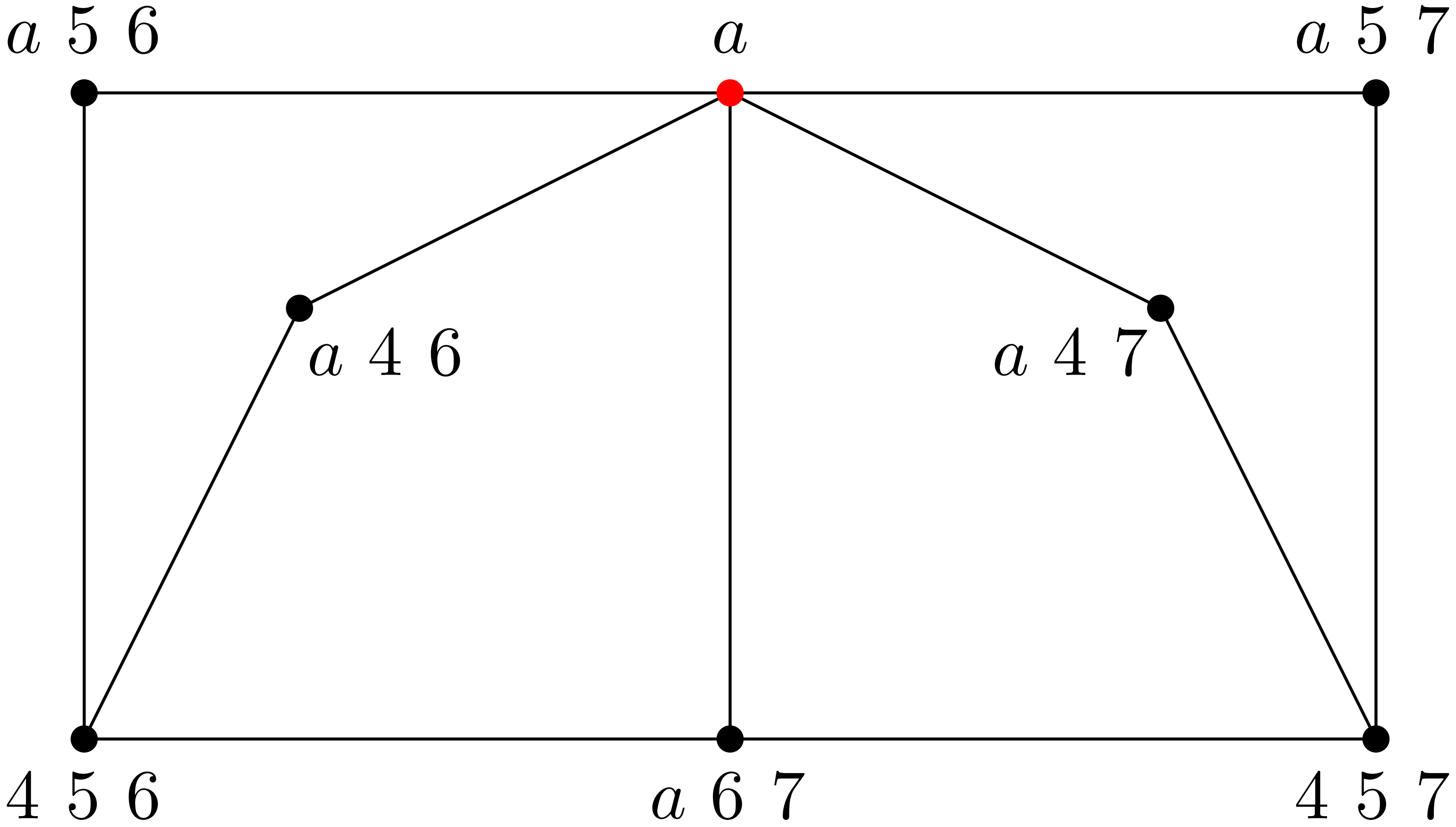

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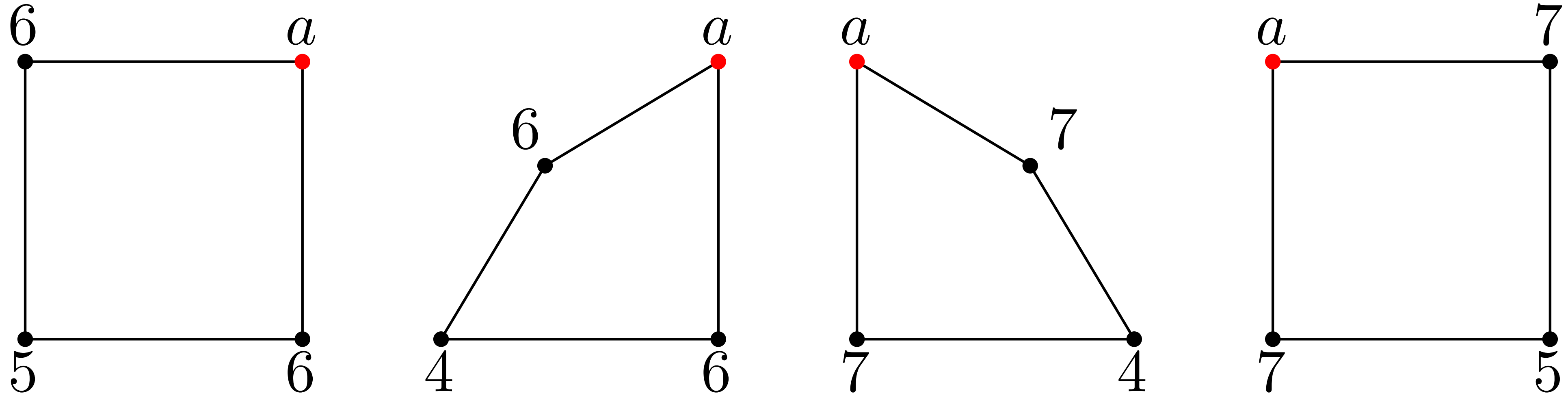

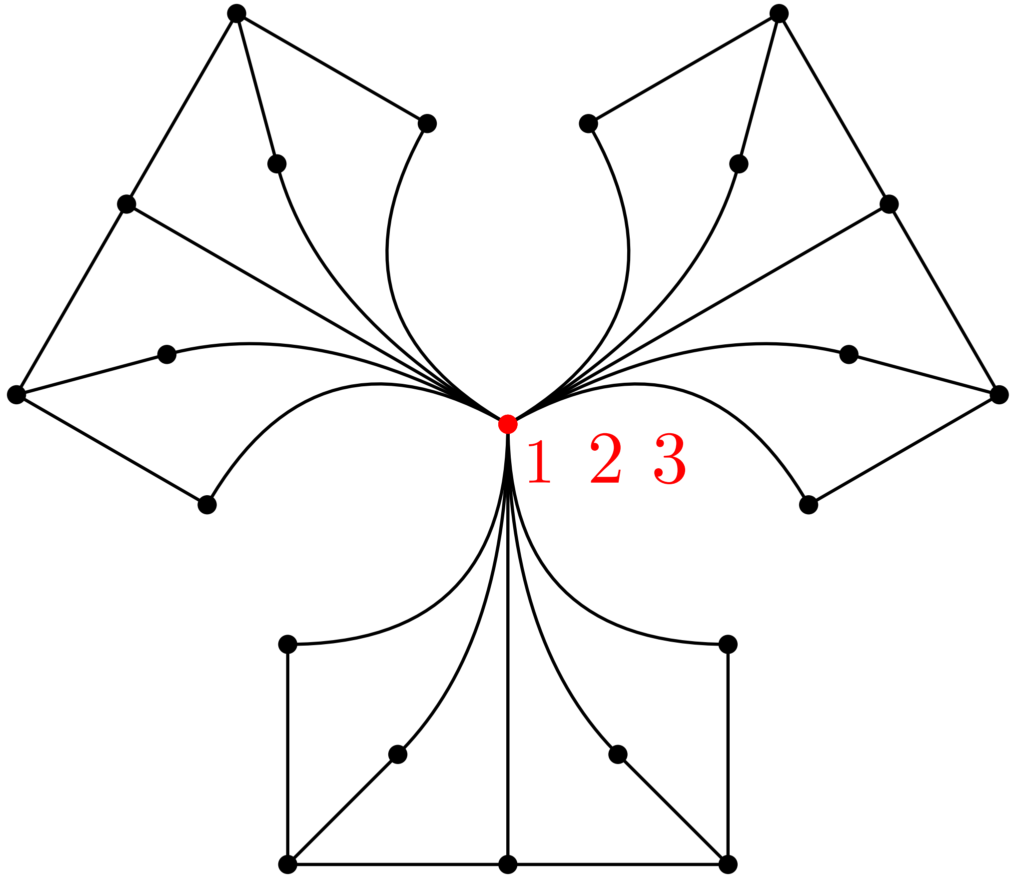

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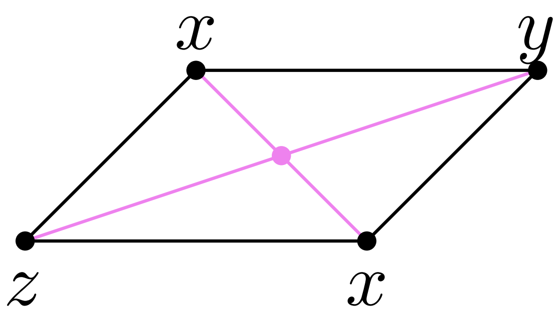

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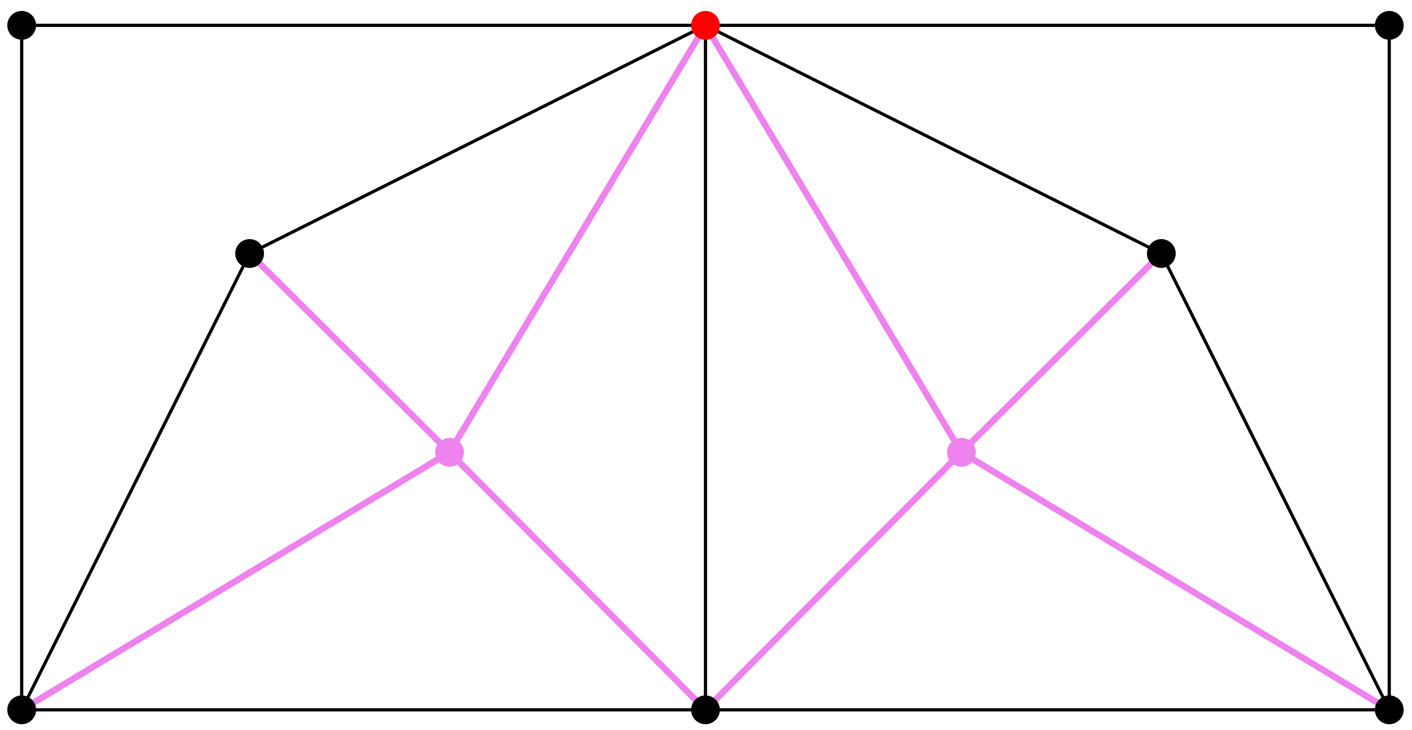

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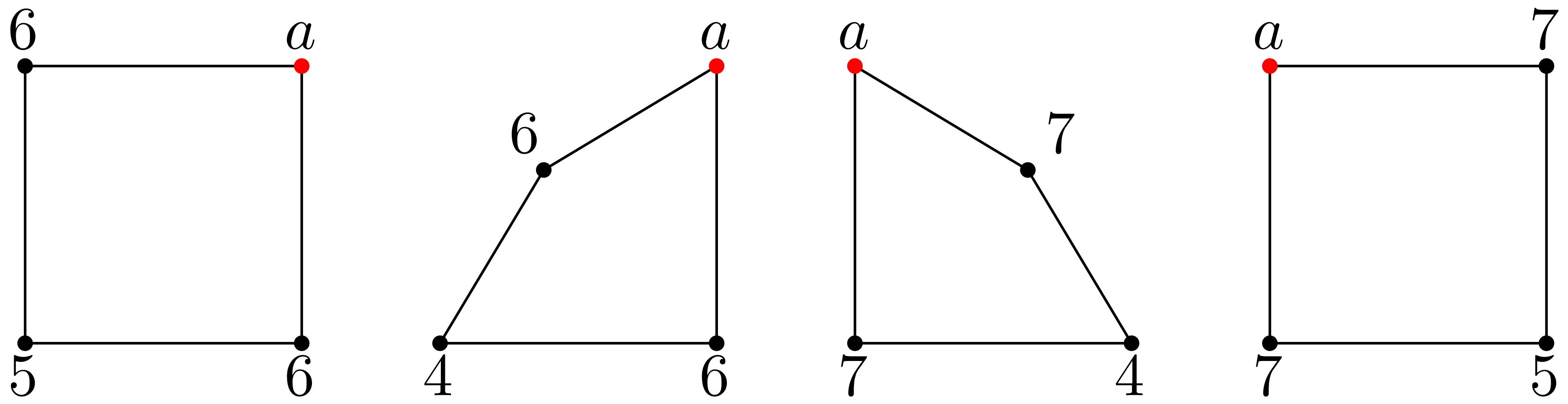

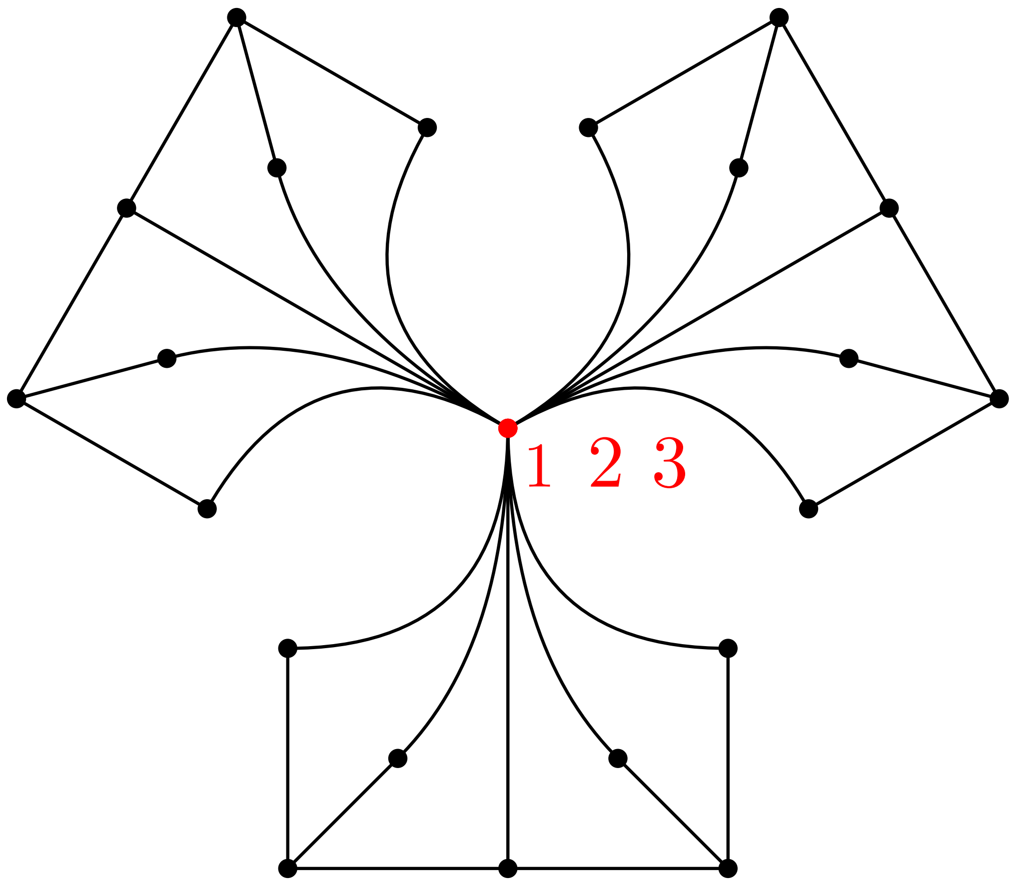

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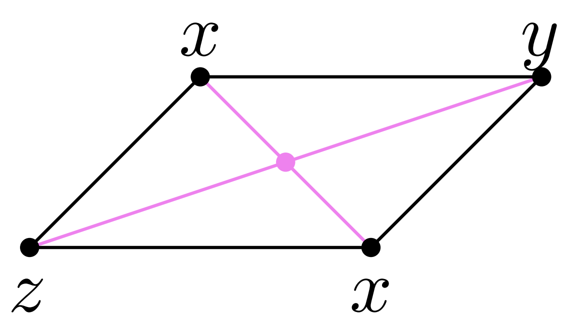

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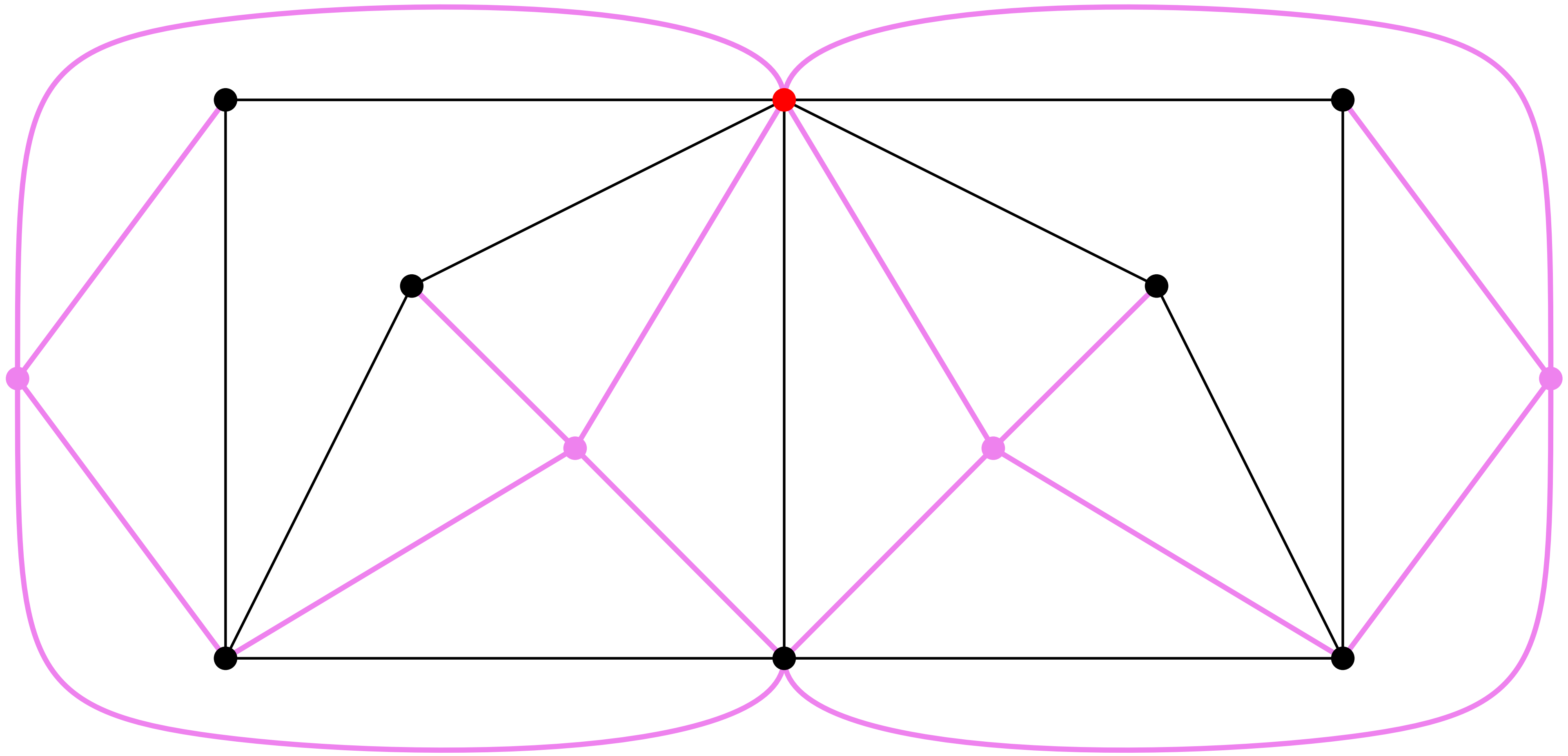

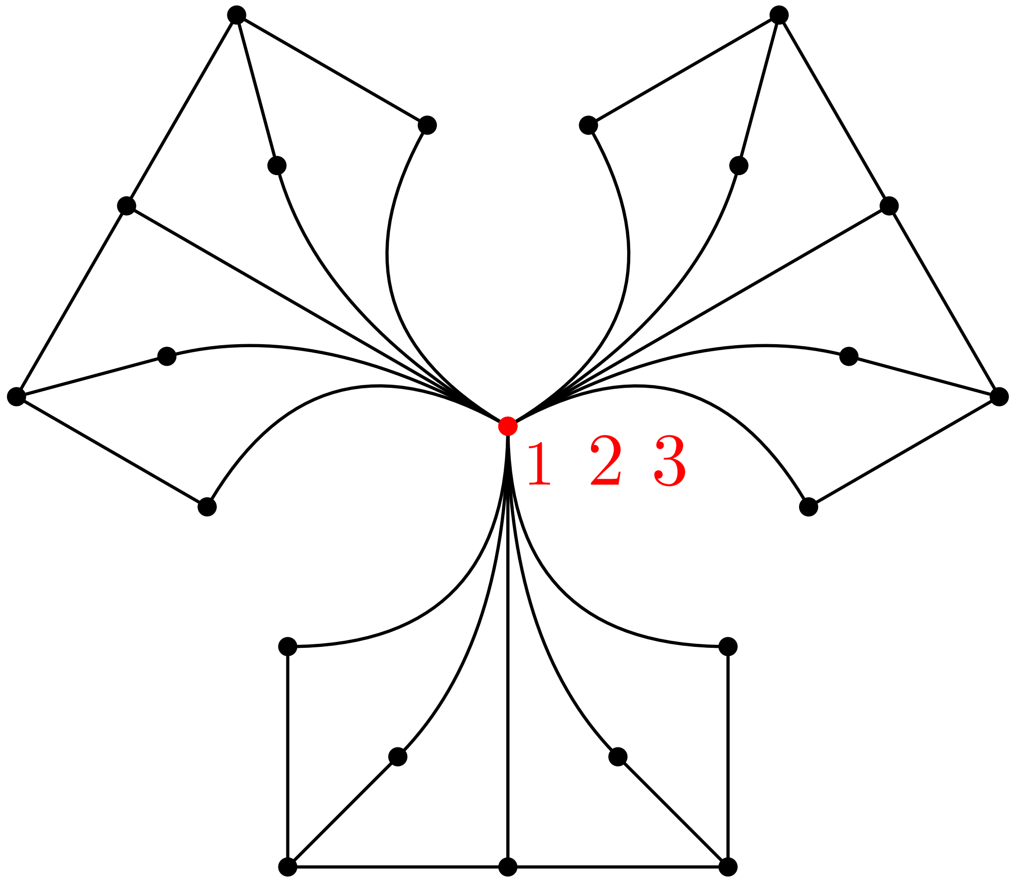

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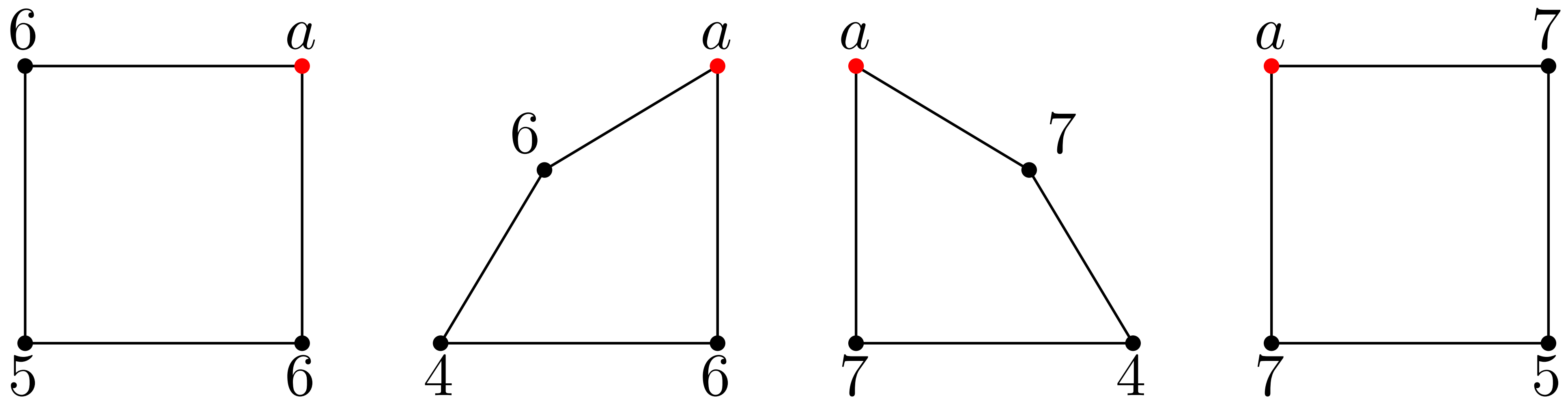

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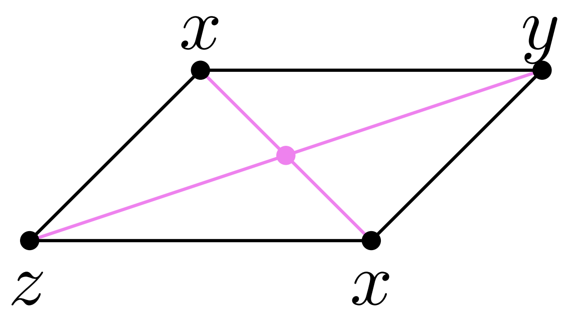

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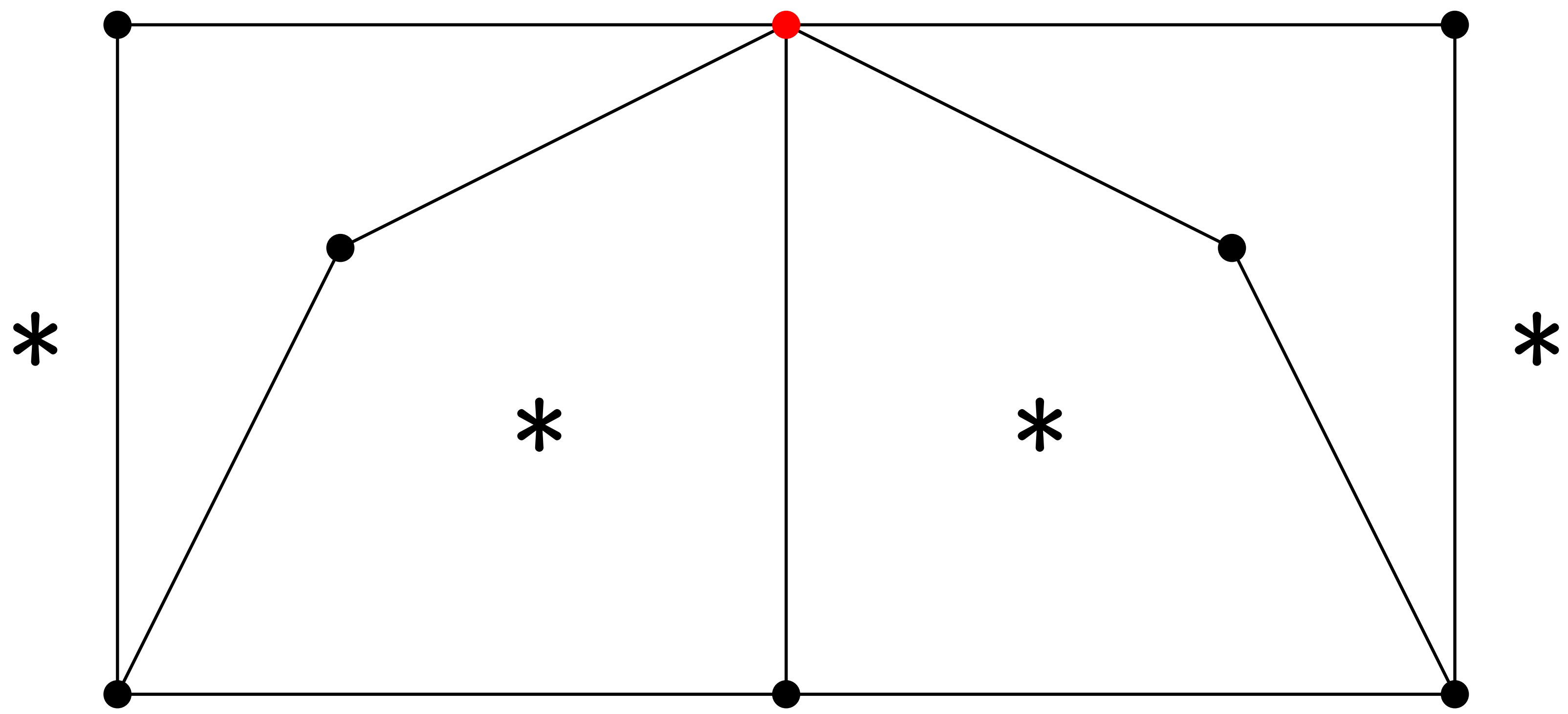

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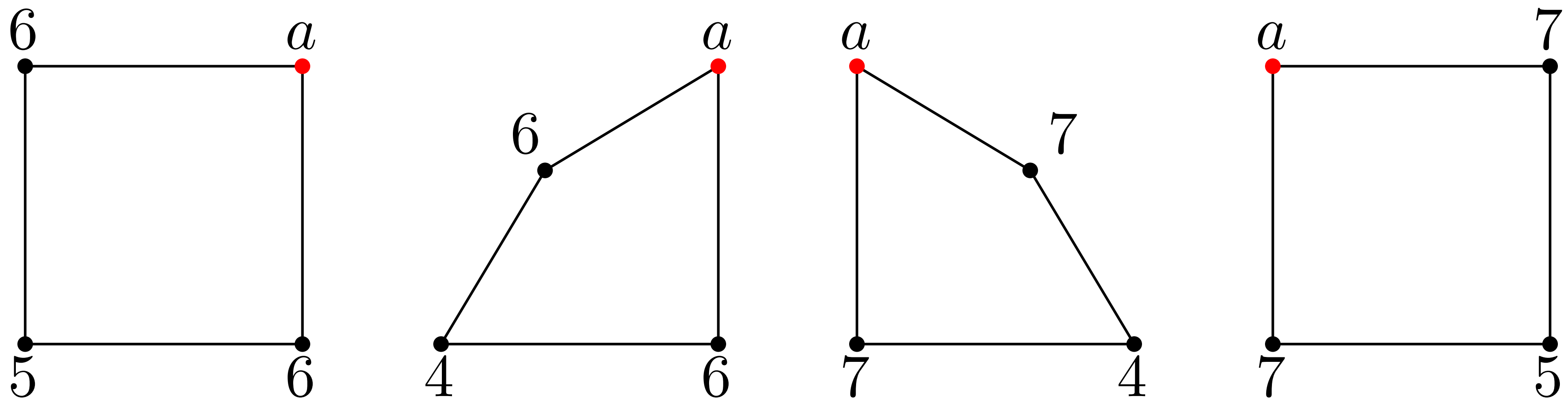
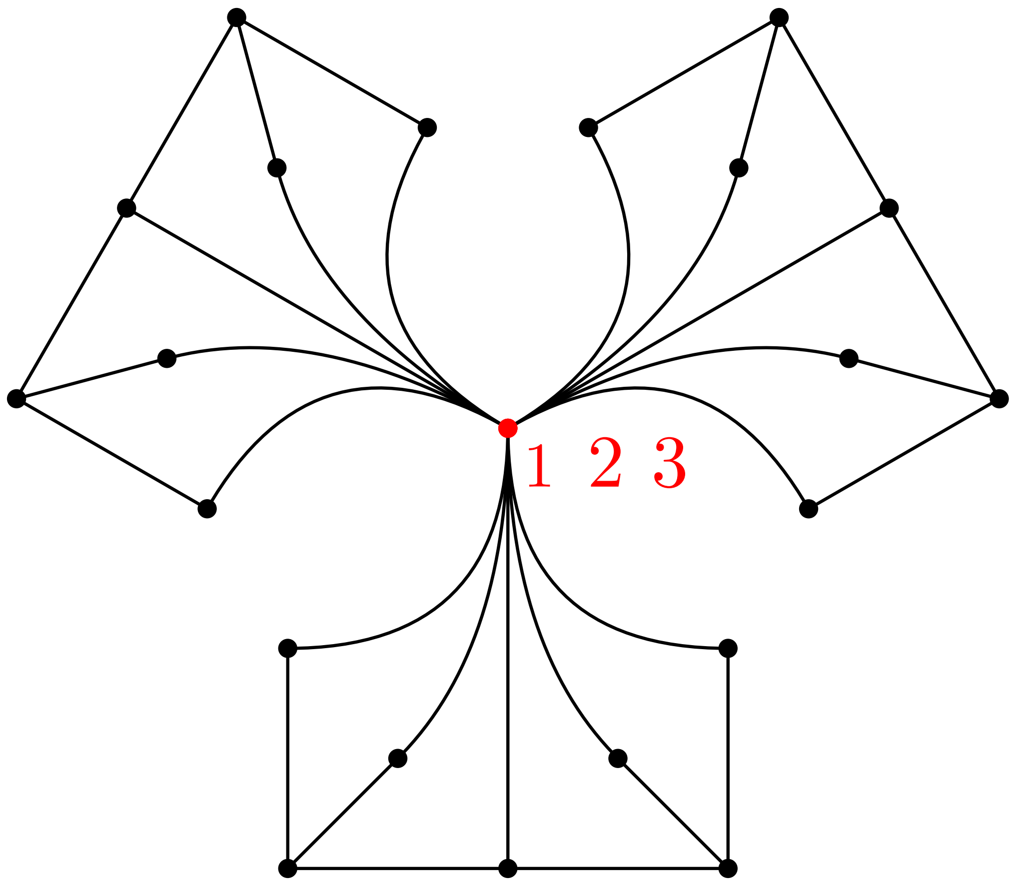
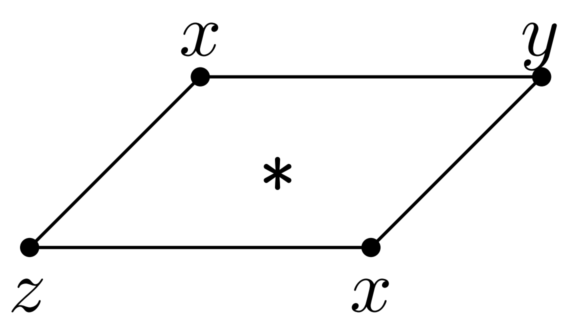




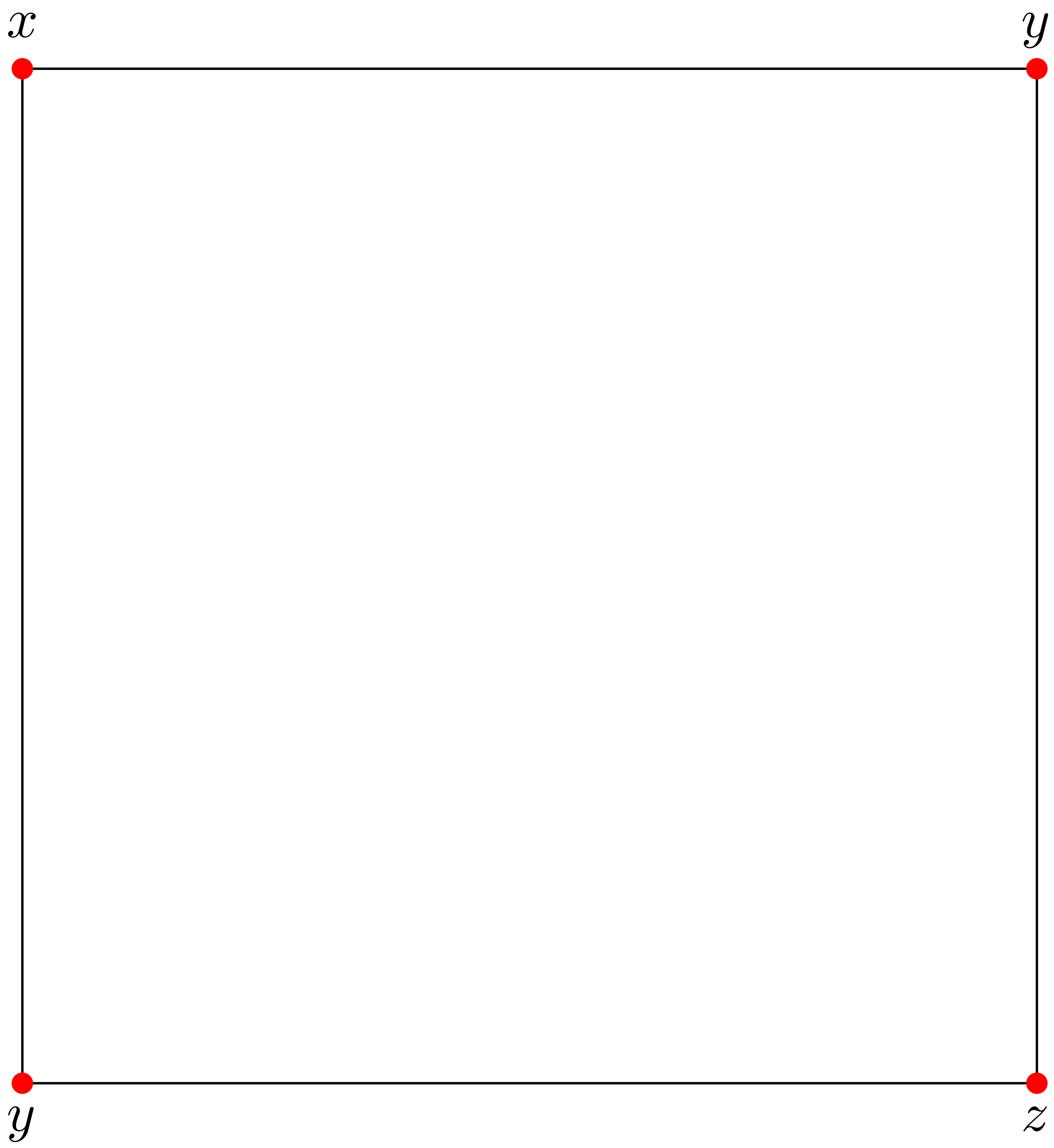
In other words, we can assign numbers x, y, z to some 4-cycles, and in ANY coloring, one of them will be colored:



The counterexample

So, we will have a 4-cycle colored like this:

$x, y, z \in \{1, 2, 3, 4, 5, 6, 7\}$



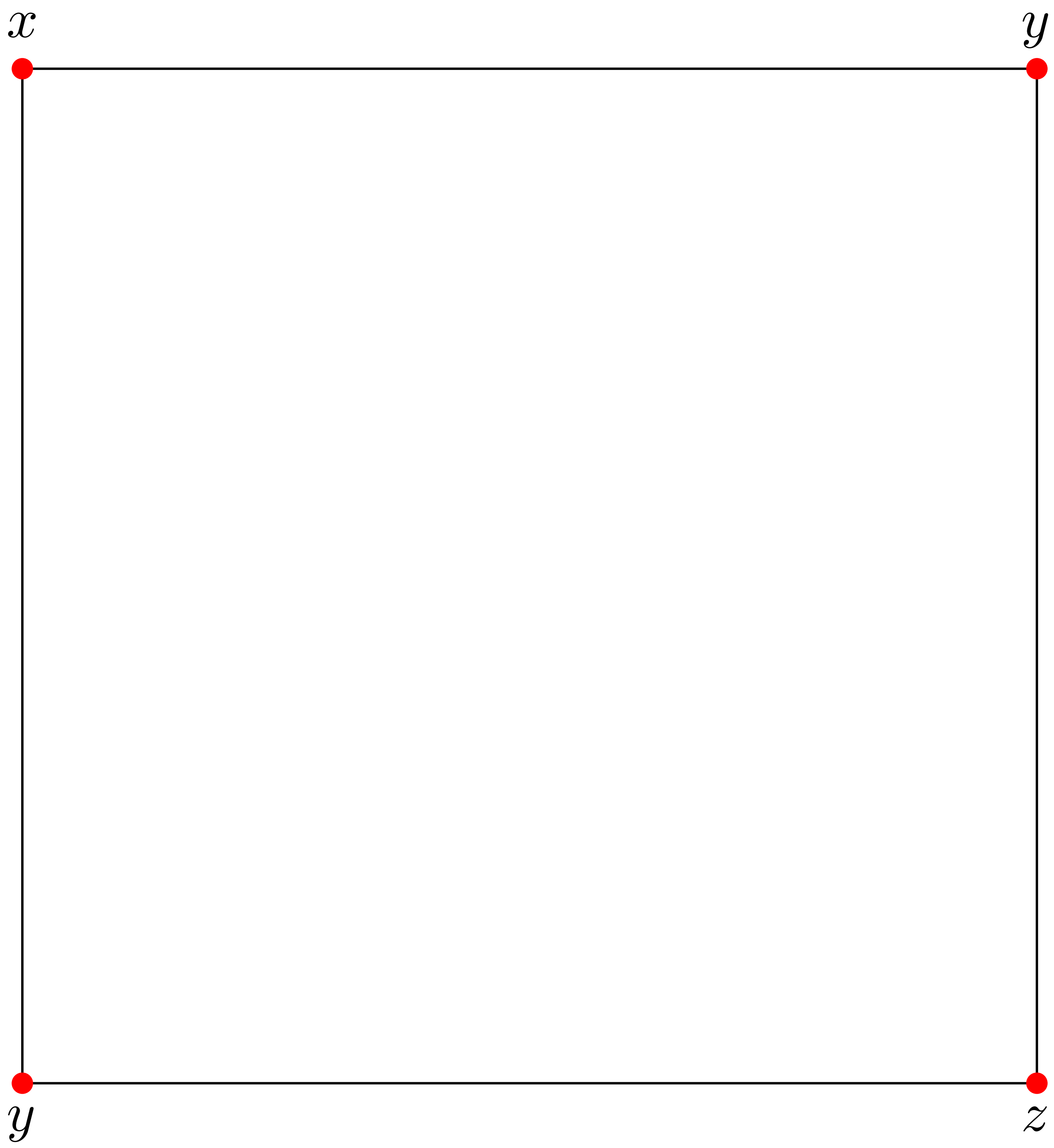
The counterexample

So, we will have a 4-cycle colored like this:

~~$x, y, z \in \{1, 2, 3, 4, 5, 6, 7\}$~~

$x, y, z \in \{5, 6, 7, 8, 9, 10, 11\}$

Disclaimer: in the paper they reused colors when they could, I decided to replace them with fresh ones for clarity.



The counterexample

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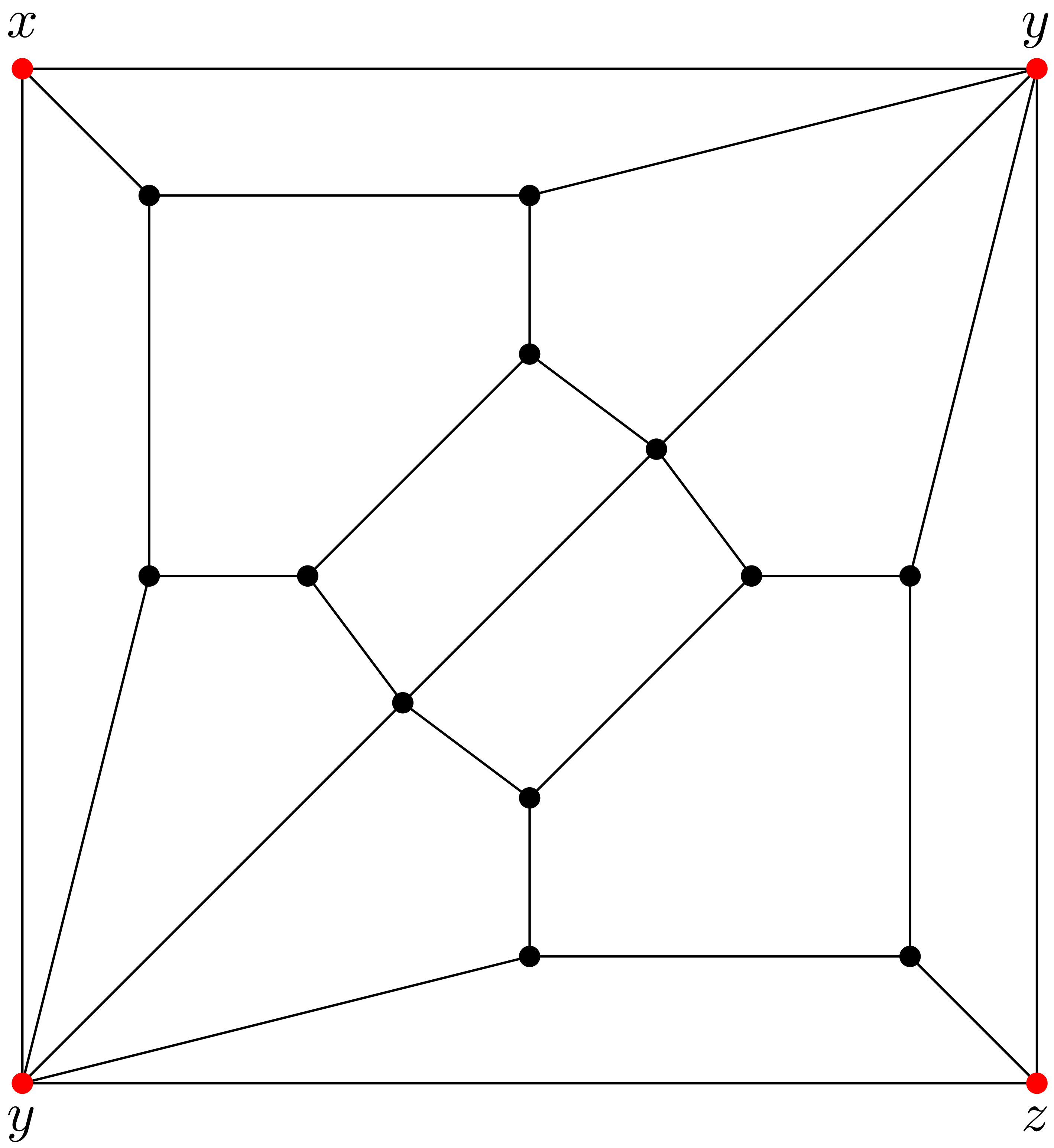
Disclaimer: in the paper they reused colors when they could, I decided to replace them with fresh ones for clarity.

This will be our $*$



The counterexample

So, we will have a 4-cycle colored like this:

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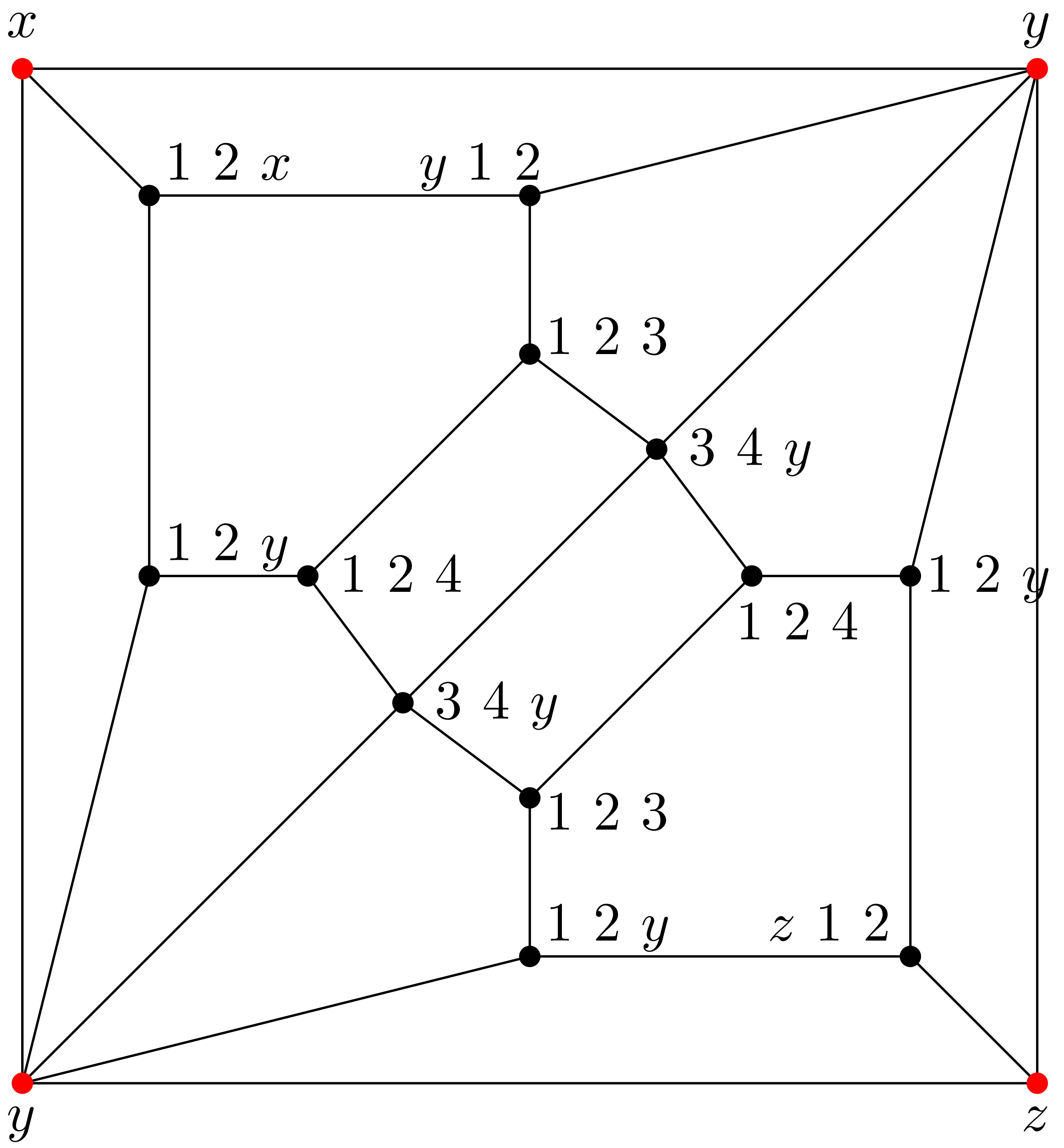

The counterexample

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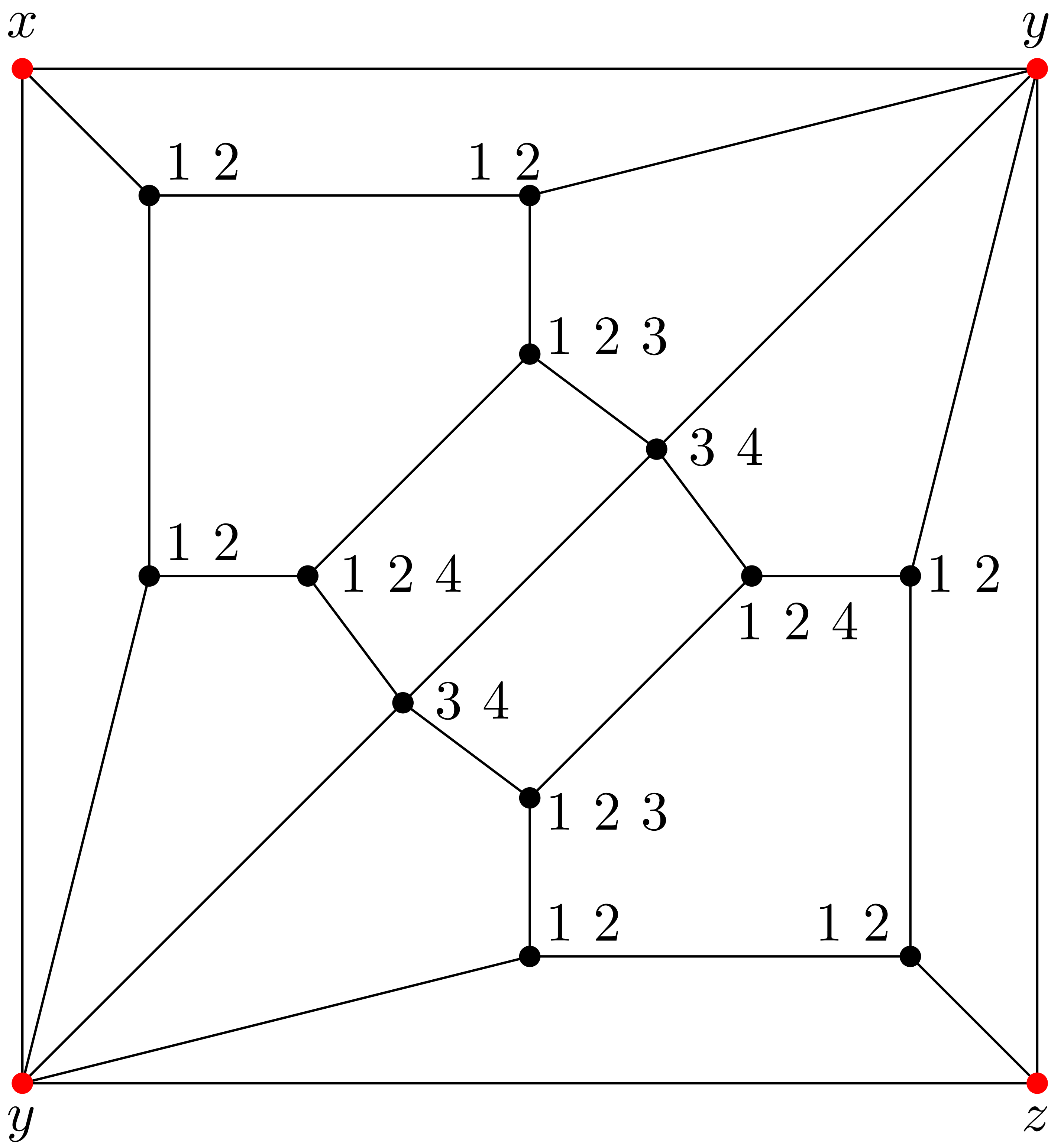

The counterexample

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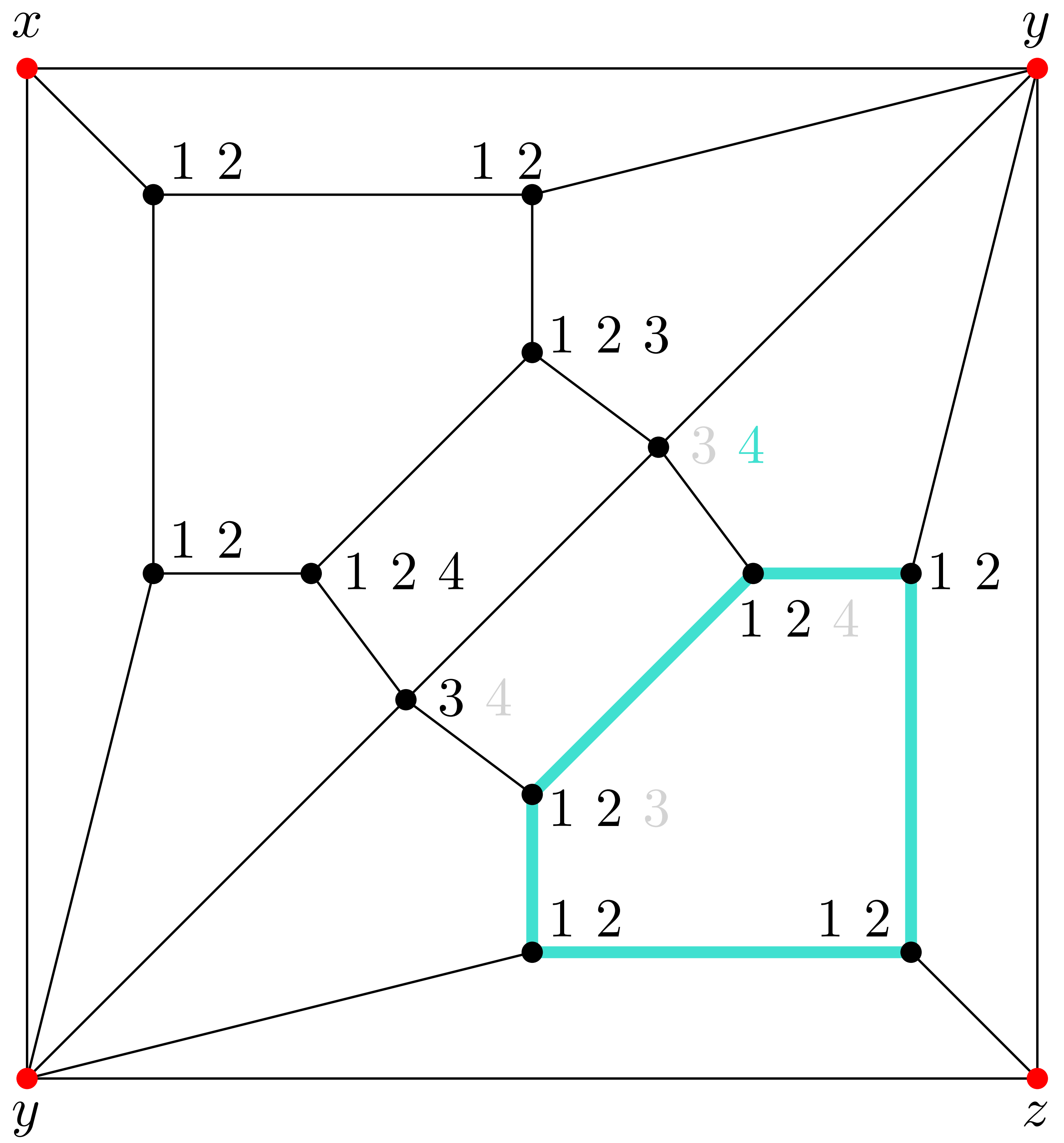


The counterexample

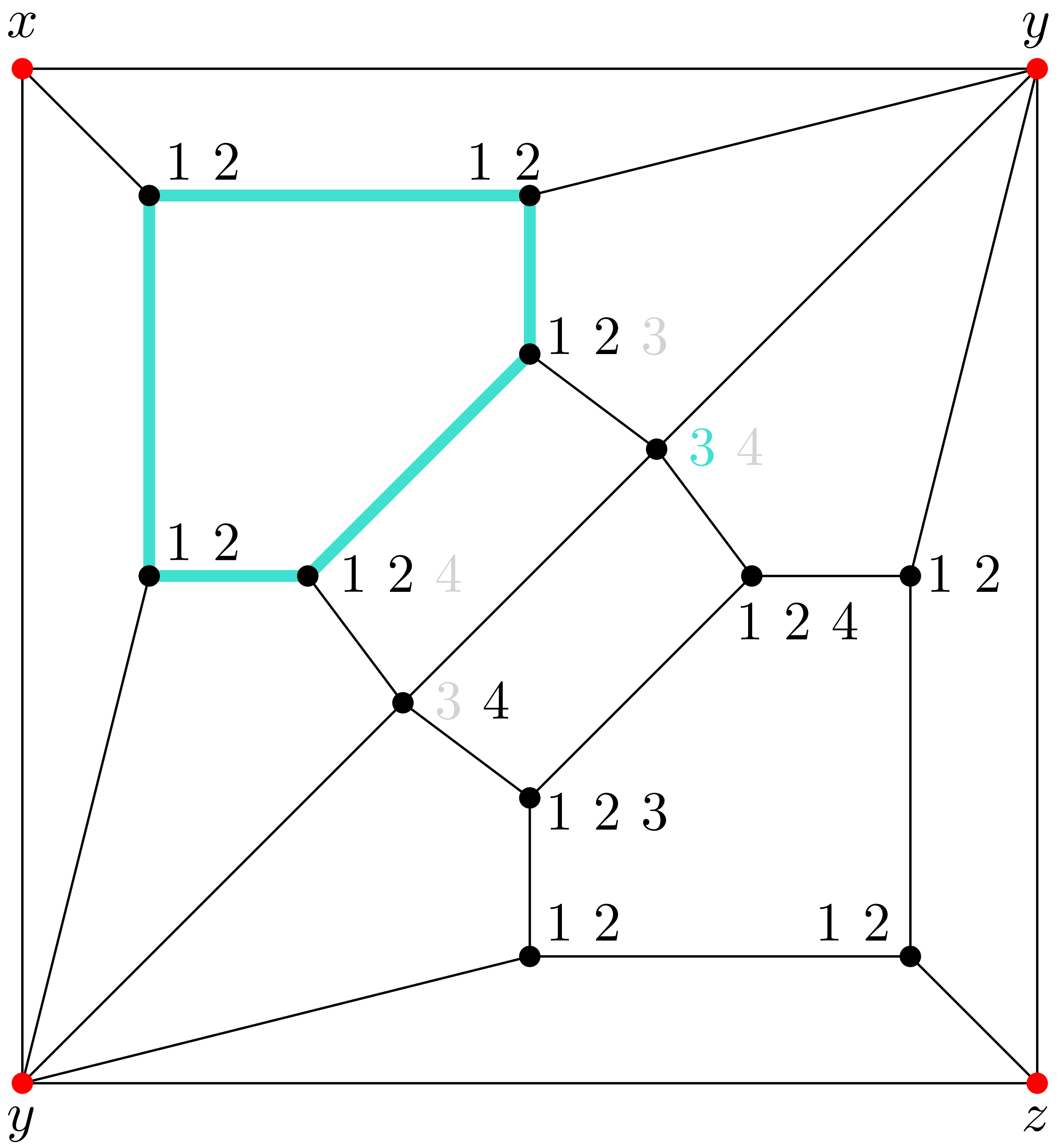
So, we will have a 4-cycle colored like this:

~~$x, y, z \in \{1, 2, 3, 4, 5, 6, 7\}$~~

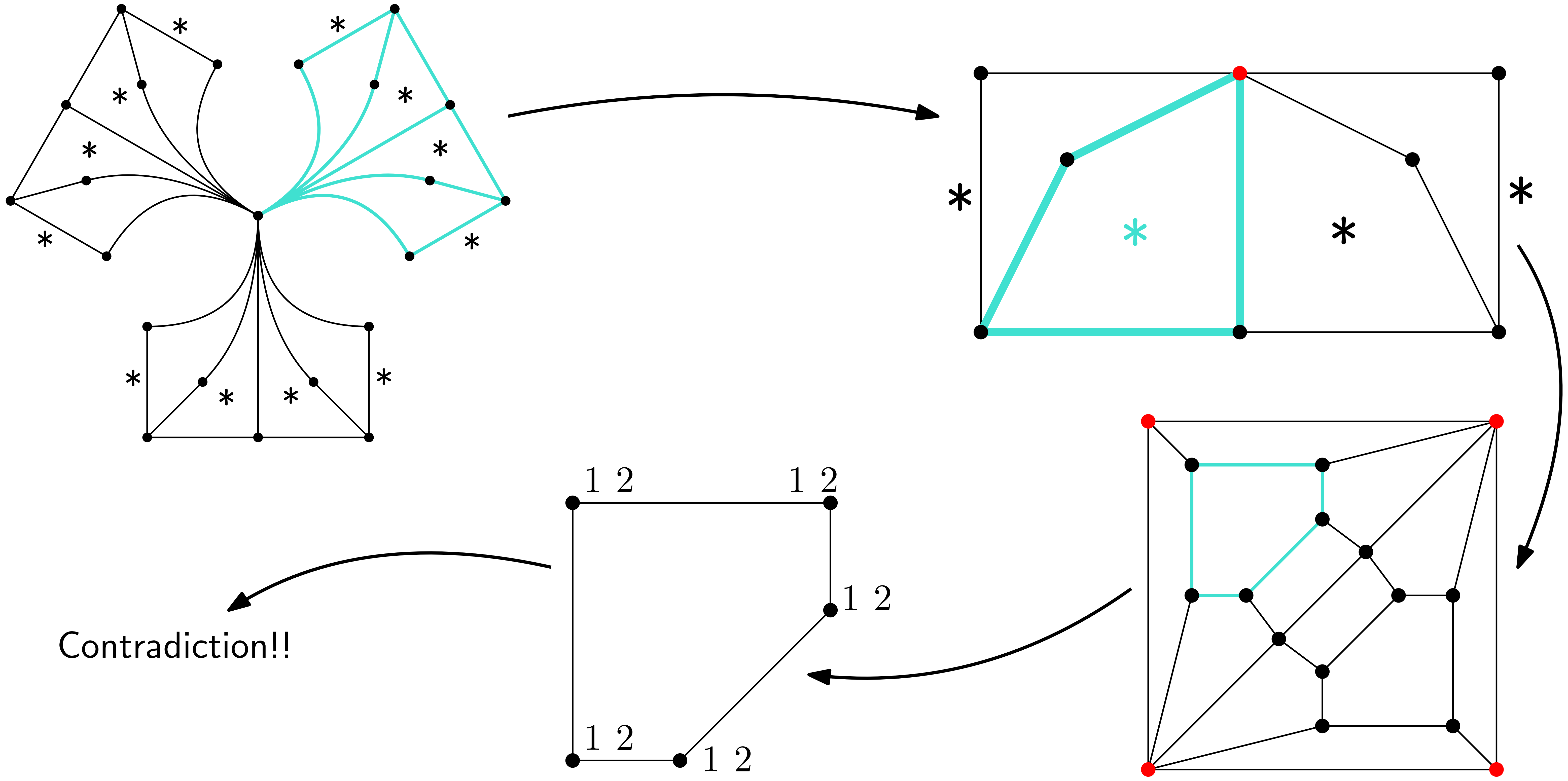
$x, y, z \in \{5, 6, 7, 8, 9, 10, 11\}$

Disclaimer: in the paper they reused colors when they could, I decided to replace them with fresh ones for clarity.

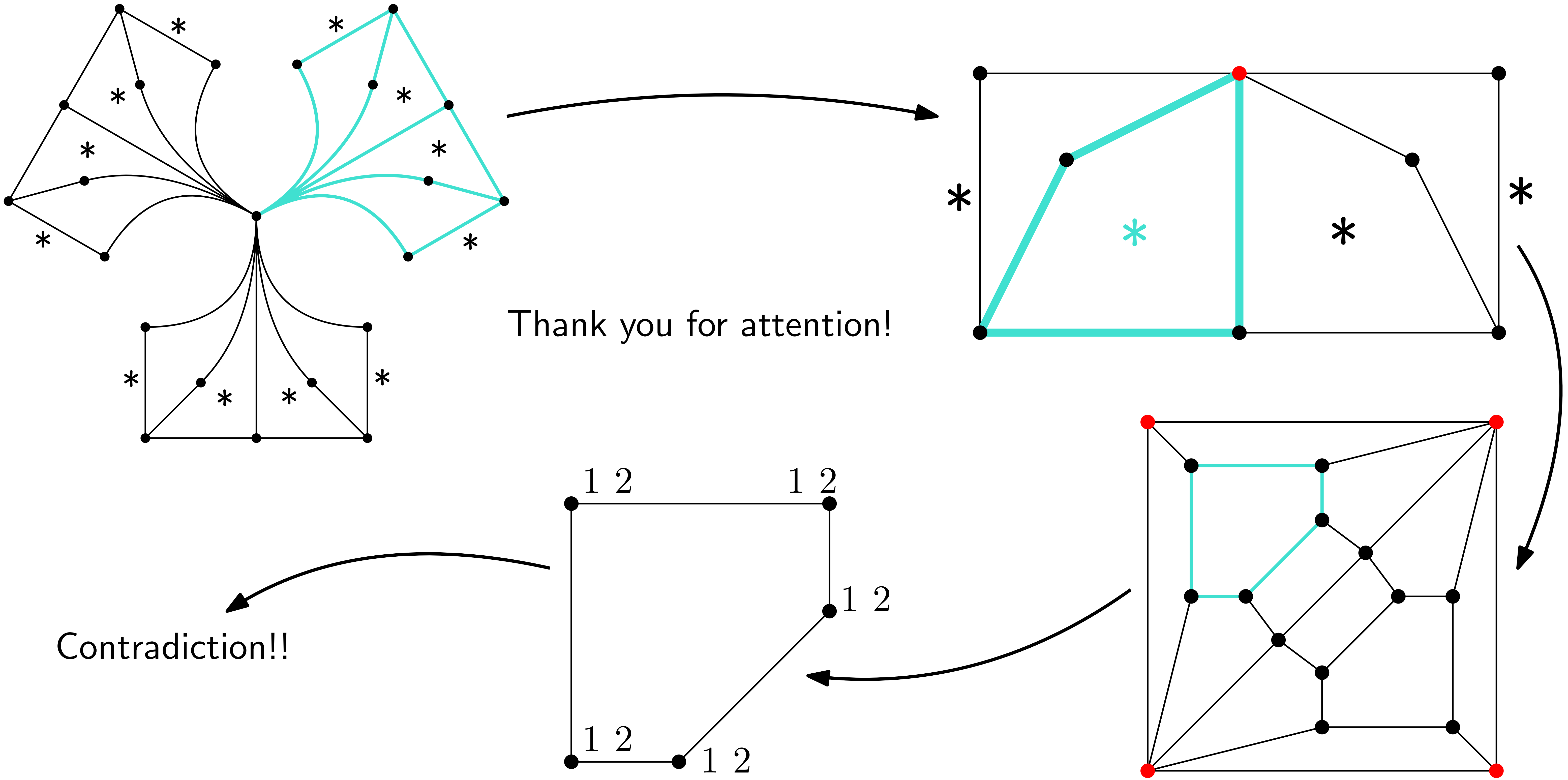
This will be our $*$



Summary



Summary



Thank you for attention!