On constructive methods in the theory of colour-critical graphs

Filip Konieczny

Based on
Horst Sachs, Michael Stiebitz (1989).
On constructive methods in the theory of colour-critical graphs

May 18, 2023

Colorability

Graph $G = (V, E)$ is *k-colourable* if there is $c : V \to \{1, 2, \ldots, k\}$ such that for every $e \in E |c(e)| > 1$ (i.e. there is no monochromatic edge).

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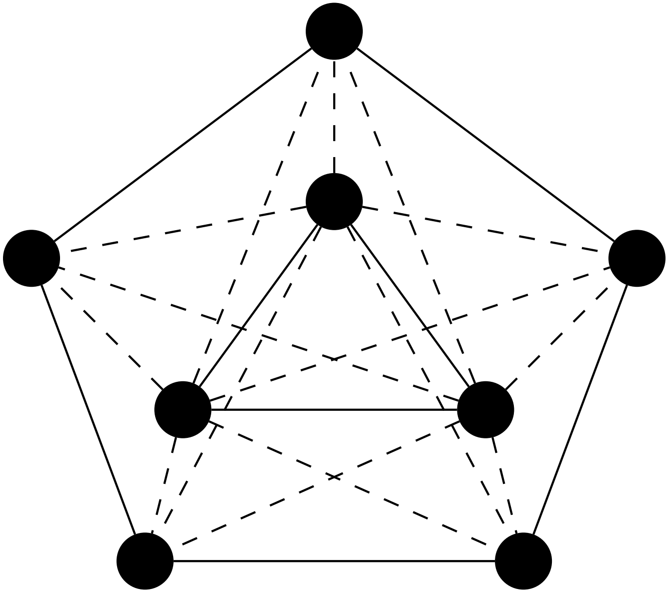
Examples

Odd cycles, K_n but also:



Basic properties

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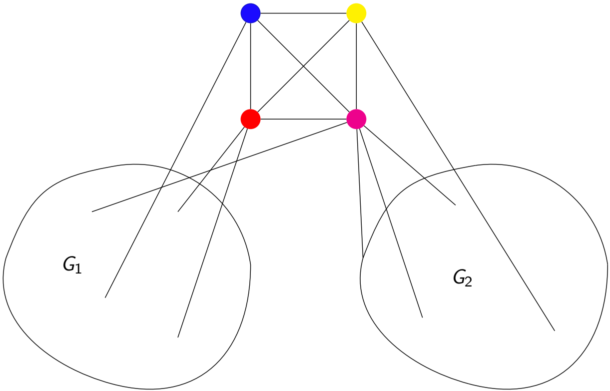

Constructions

Dirac's construction

Given G_1, G_2 let G be graph $G_1 \cup G_2$ with additional edges between every pair of vertices from G_1 and G_2. Then

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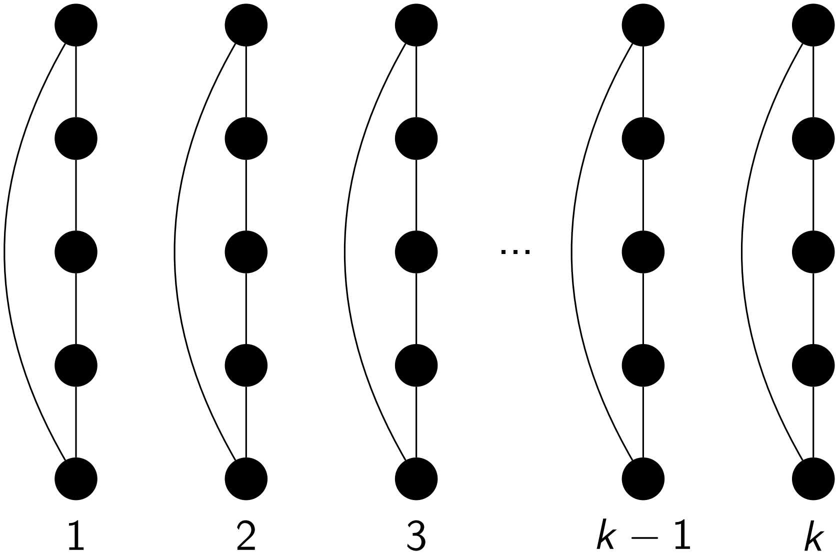
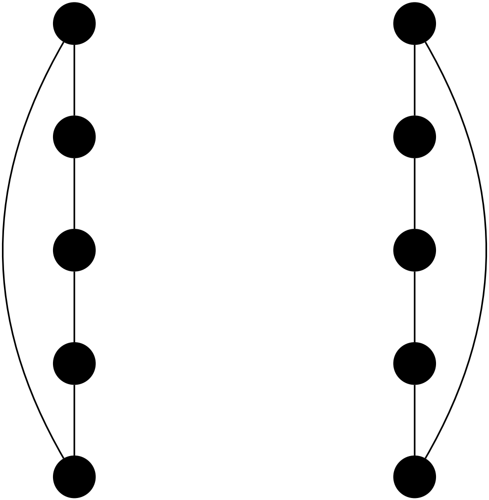
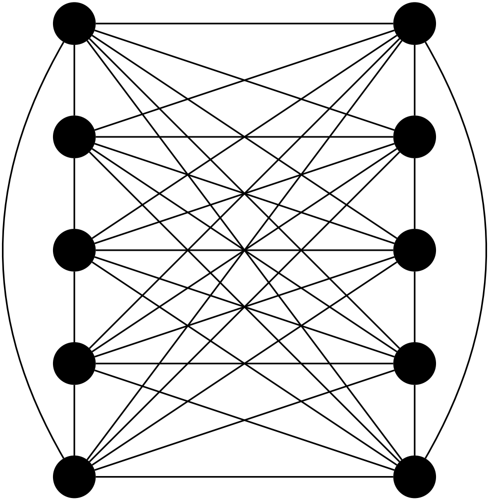
Corollary

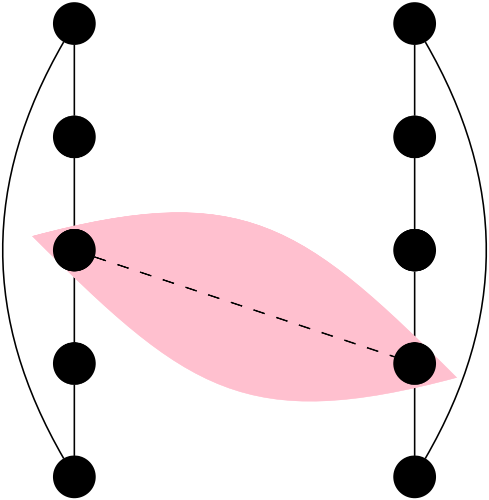
There are k-critical graphs with $f(k)|V|^2$ edges.

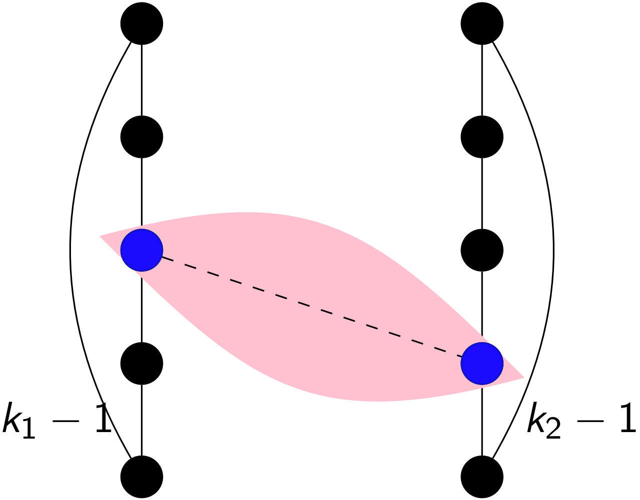


3k-critical



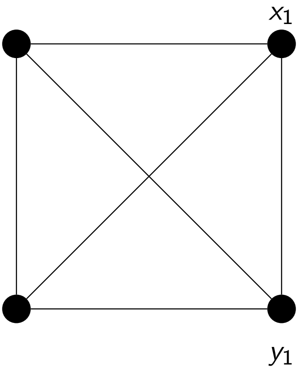




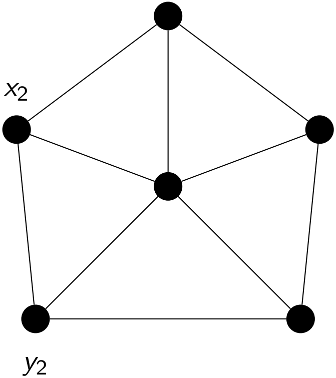

$k_1 - 1$
$k_2 - 1$

Dirac-Hajós' construction

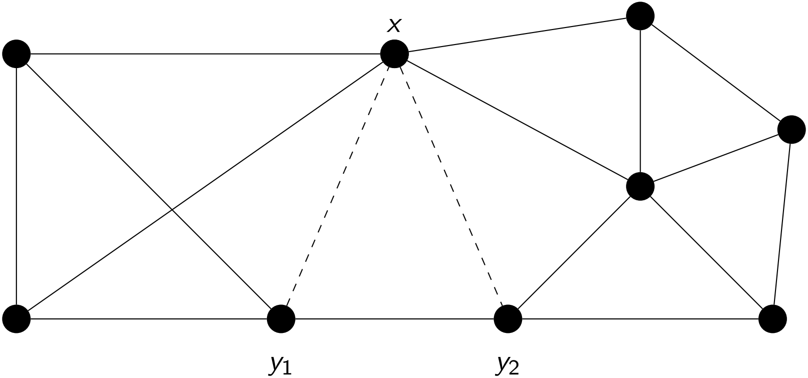
Let G_1, G_2 be k-critical graphs and $\{x_i, y_i\}$ be edges in these graphs. Let G be given by sum of G_1 and G_2 where x_1, x_2 are identified, edges $\{x_i, y_i\}$ are removed and edge $\{y_1, y_2\}$ is added. Then G is k-critical.

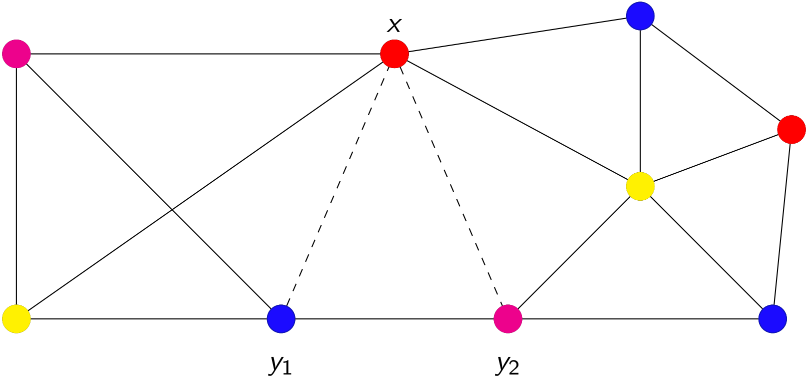


x_1
y_1

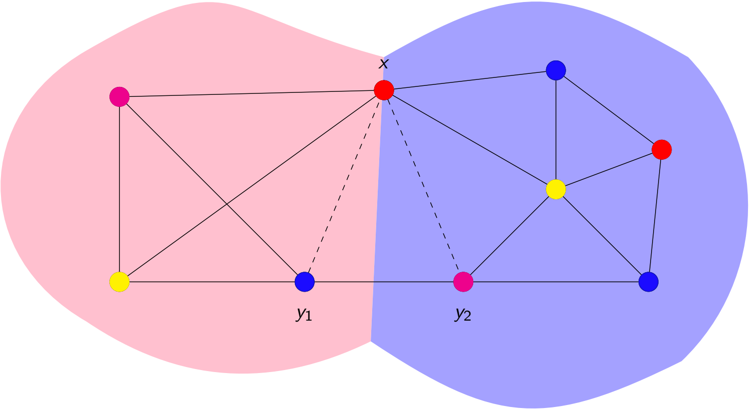


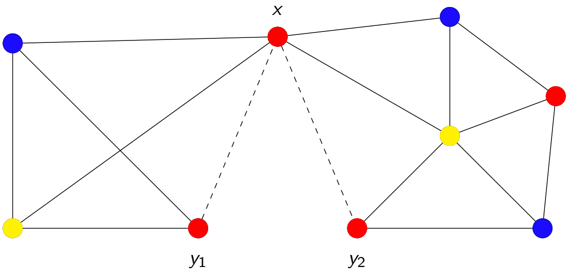
x_2
y_2


x
y1
y2



x
y_1
y_2


x
y1
y2


x
y_1
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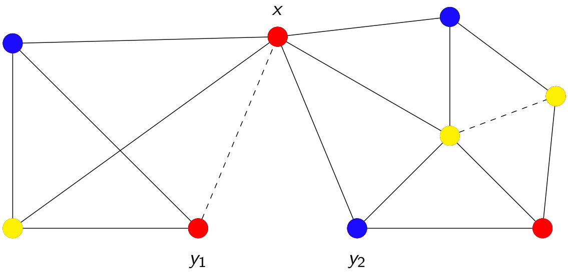
x
y1
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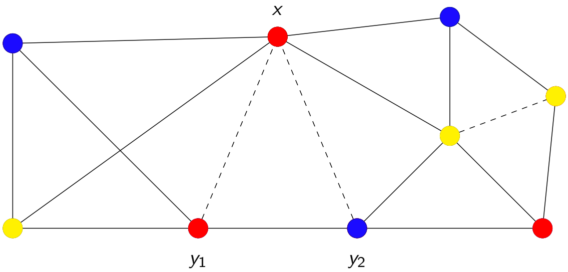
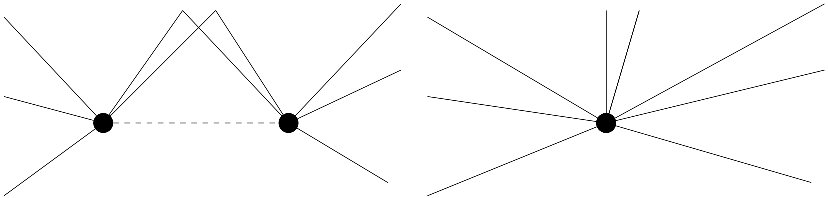

x
y1
y2

Hajós theorem

Every k-critical graph is obtained this way from two smaller k-critical graphs or by identifing two non-adjacent vertices in k-critical graph.



Instead for single vertex x_1 we can have whole clique.

Generalization

Let q be positive integer, and G_1, G_2 be containing vertices $\{x_i^1, x_i^2, \ldots, x_i^q, y_i\}$ for $i \in \{1, 2\}$ and

1. $\{x_i^1, y_i\}$ is an edge in G_i,

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If G_1, G_2 is k-critical then G is also k-critical.

Further generalization

Let $k \geq 4, 1 \leq p \leq q \leq k-1-p$. Let $G_1, G_2, \ldots, G_p$ be k-critical graphs which satisfy, for every $1 \leq i \leq p+1$:

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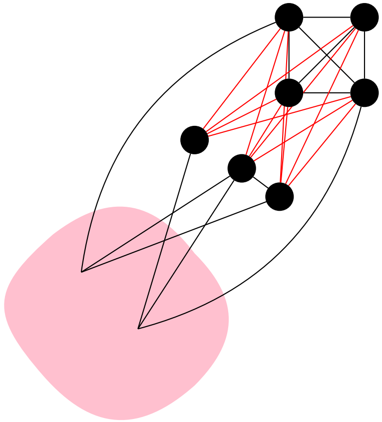
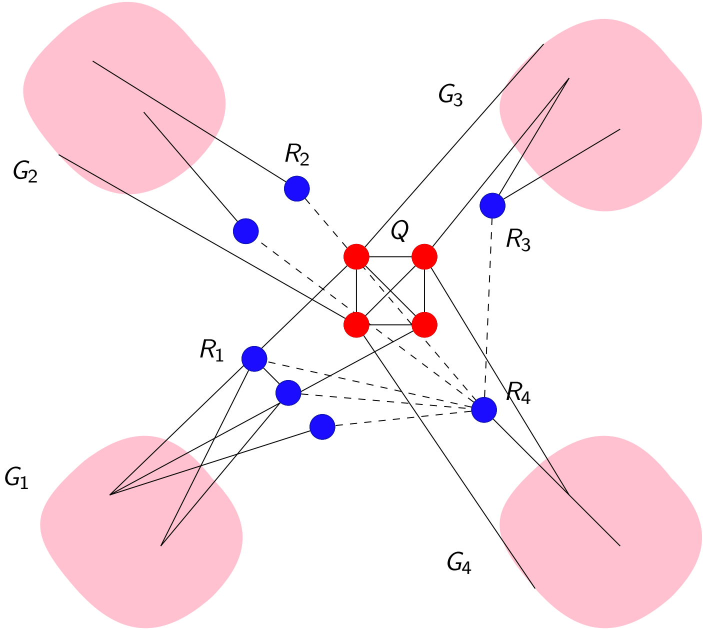
Resulting graph G is k-critical.




G_2
R_2
G_3
Q
R_3
R_1
R_4
G_1
G_4

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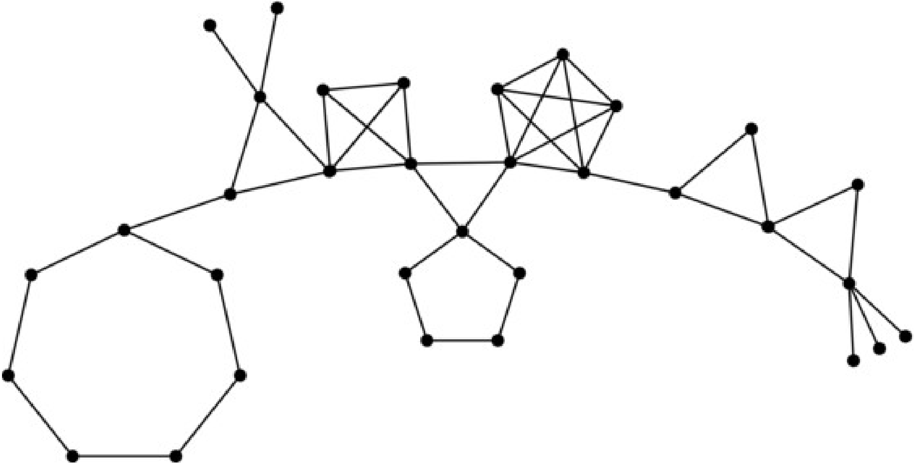
Reverse theorem

Every k-Gallai-forest without K_k as component is subgraph induced by low vertices of some k-critical graph.



Source: Cranston, Daniel & Rabern, Landon. (2014). Brooks' Theorem and Beyond. Journal of Graph Theory. 80. 10.1002/jgt.21847.

Mycielski construction

Let $X_i = \{x^i_1, x^i_2 \dots x^i_n\}$ $i \in \{1, 2 \dots r\}$ be r copies of vertices of G. Let $M_r(G)$ be graph with $V(M_r(G)) = \{z\} \cup \bigcup X_i$ be graph with edges:

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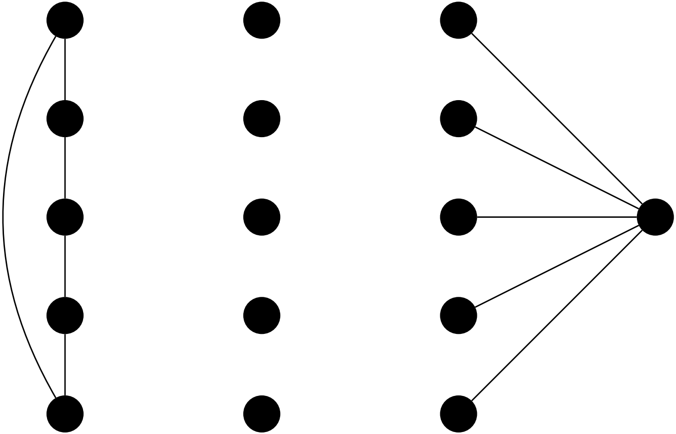
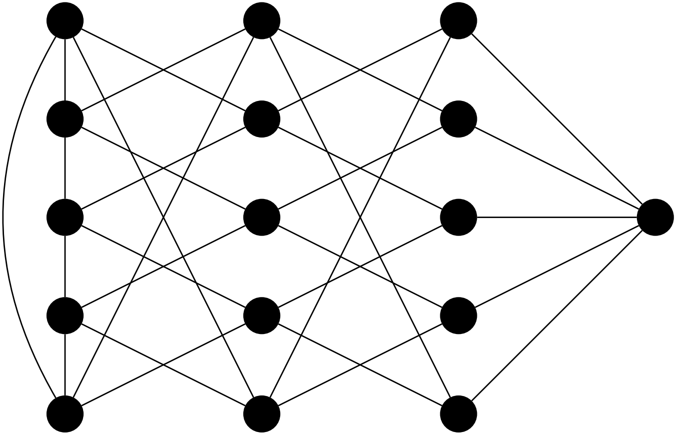
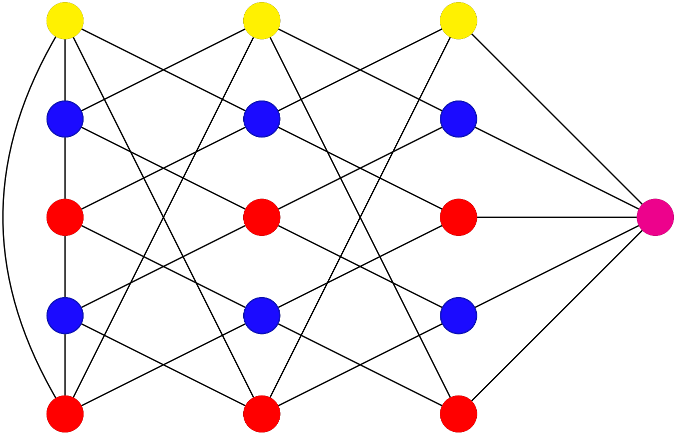
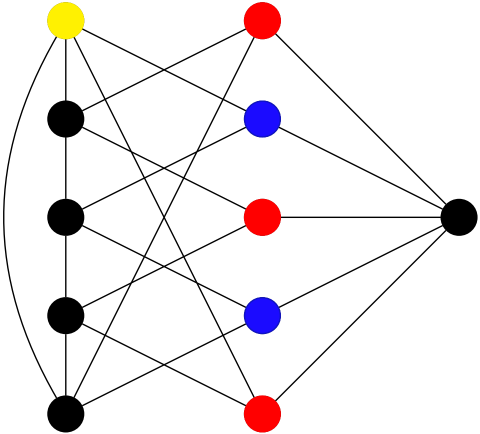
3. $\{z, x_i^r\}$ for every $i \in \{1, 2, \dots, n\}$.









Theorem

If $k \geq 2$ and $\chi(G) = k$ then $\chi(M_2(G)) = k + 1$.

Theorem

If $k \geq 2$ and G is k-critical then $M_2(G)$ is $(k+1)$-critical.

Theorem

$M_r(K_k)$ is $(k+1)$-critical for every $r \geq 1$. As corollary, there are k-critical graphs which can be made bipartite by removing only $\binom{k}{2}$ edges. This result is proved optimal (Tuza, Rodl 1985).

In general it is not true that $\chi(M_r(G)) = \chi(G) + 1$. However if $M(k+1) = \{M_r(G) \mid G \in M(k), r \geq 1\}$ for $k > 3$ and $M(2)$ are odd cycles, then $M(k)$ are k-critical.

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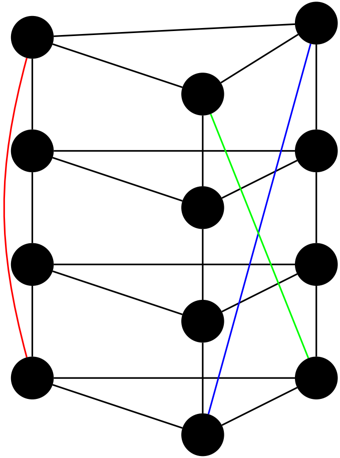

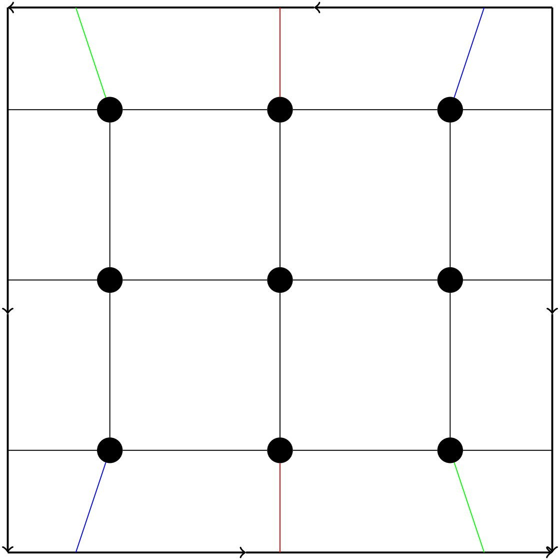

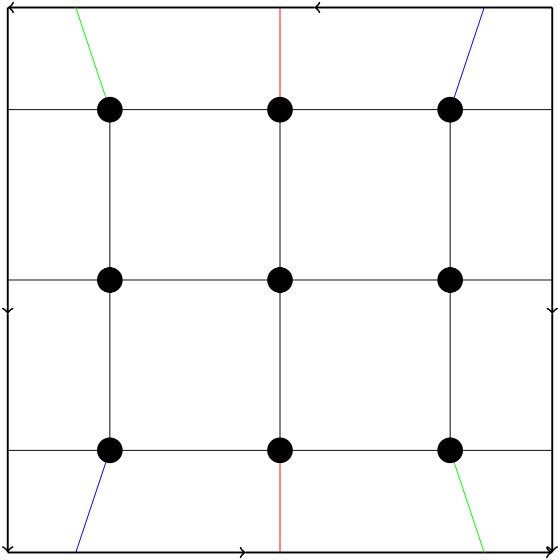

Thank you for your attention!