Playing cards with Vizing's demon - Brian Rabern and Landon Rabern

Jędrzej Kula - 25th May 2023
Optymalizacja kombinatoryczna

The Solitaire Game

Game number k ($k \geqslant 1$) - how many fixed stacks will be in the game. k-game is a game with a game number k.

The Solitaire Game

Game number k ($k \geqslant 1$) - how many fixed stacks will be in the game. k-game is a game with a game number k.

Card number m ($m \geqslant k$) - how many distinct numbered cards a game will involve.

The Solitaire Game

Game number k (k ⩾ 1) - how many fixed stacks will be in the game. k-game is a game with a game number k.

Card number m (m ⩾ k) - how many distinct numbered cards a game will involve.

E.g. if k = 3, m = 4, then the game will involve 3 1-cards, 3 2-cards, 3 3-cards and 3 4-cards

1	1	1
2	2	2
3	3	3
4	4	4

The Solitaire Game

The demon deals the stacks. He constructs k nonempty stacks such that each stack contains at most one card of each type. The total number of cards put in each stack is up to the demon.

The Solitaire Game

The demon deals the stacks. He constructs k nonempty stacks such that each stack contains at most one card of each type. The total number of cards put in each stack is up to the demon.

Let n_i be the number of cards the demon deals into the stack i.
The sequence (n_1, …, n_k), called the game's stack profile, represents the sizes of the stacks.

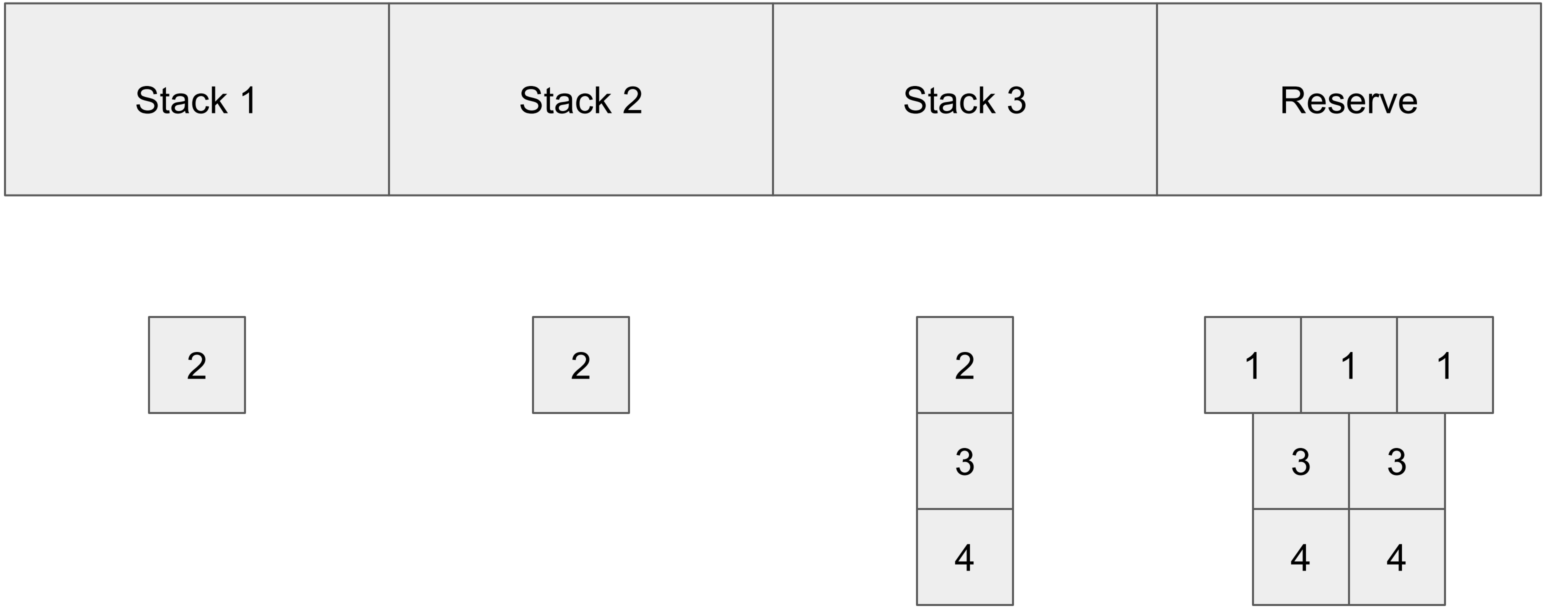
The Solitaire Game

The demon deals the stacks. He constructs k nonempty stacks such that each stack contains at most one card of each type. The total number of cards put in each stack is up to the demon.

Let n_i be the number of cards the demon deals into the stack i.
The sequence $(n_1, \ldots, n_k)$, called the game's stack profile, represents the sizes of the stacks.

The leftover cards in the deck form the reserve.

Game example



The Solitaire Game

Player’s move – pick some stack containing an a-card but no b-card and then swap the a-card for a b-card from the reserve.

The Solitaire Game

Player’s move – pick some stack containing an a-card but no b-card and then swap the a-card for a b-card from the reserve.

Demon’s move – rearrange some cards.

The Solitaire Game

Player’s move – pick some stack containing an a-card but no b-card and then swap the a-card for a b-card from the reserve.

Demon’s move – rearrange some cards.

Win – The player wins the game if at the start of a turn he can make a hand of k differently numbered cards by picking one card from each stack.

The Solitaire Game

Player’s move – pick some stack containing an a-card but no b-card and then swap the a-card for a b-card from the reserve.

Demon’s move – rearrange some cards.

Win – The player wins the game if at the start of a turn he can make a hand of k differently numbered cards by picking one card from each stack.

Existence of a winning strategy will depend on how the demon plays.

Extreme demons

Lazy demon – does nothing.

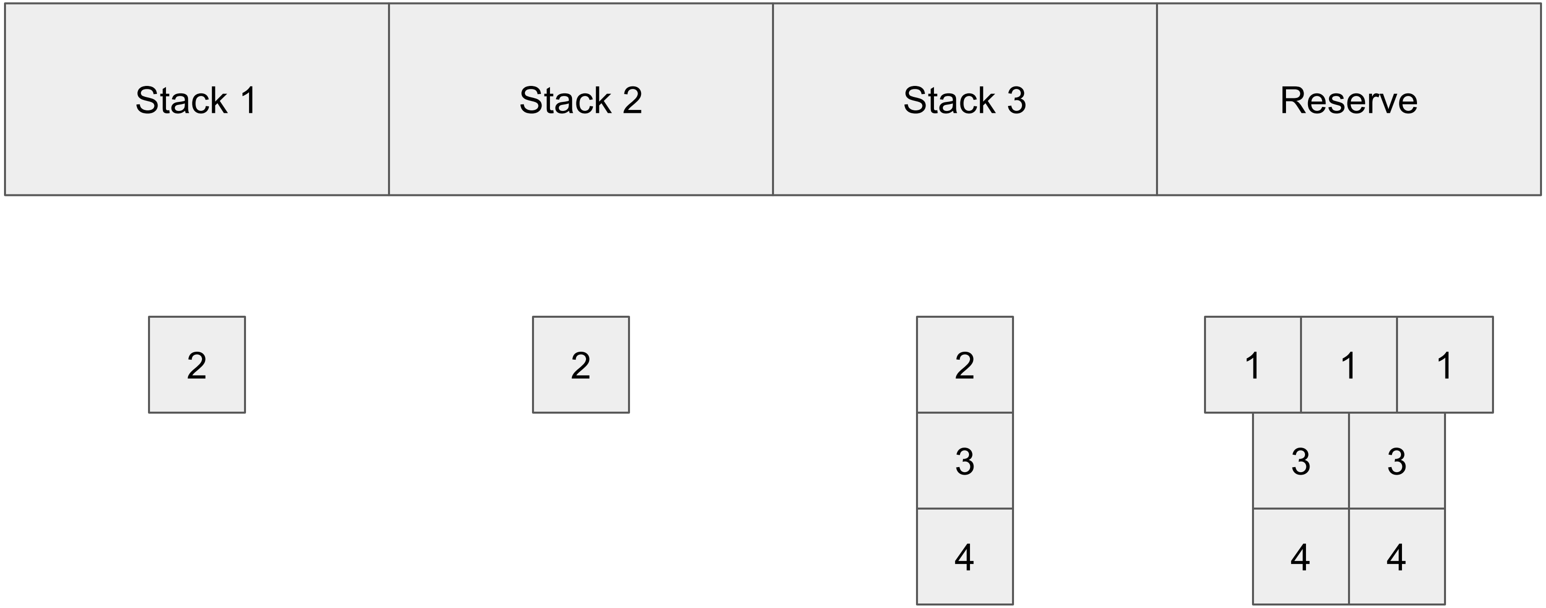
Player can win by ensuring that stack i contains i-card.

Extreme demons

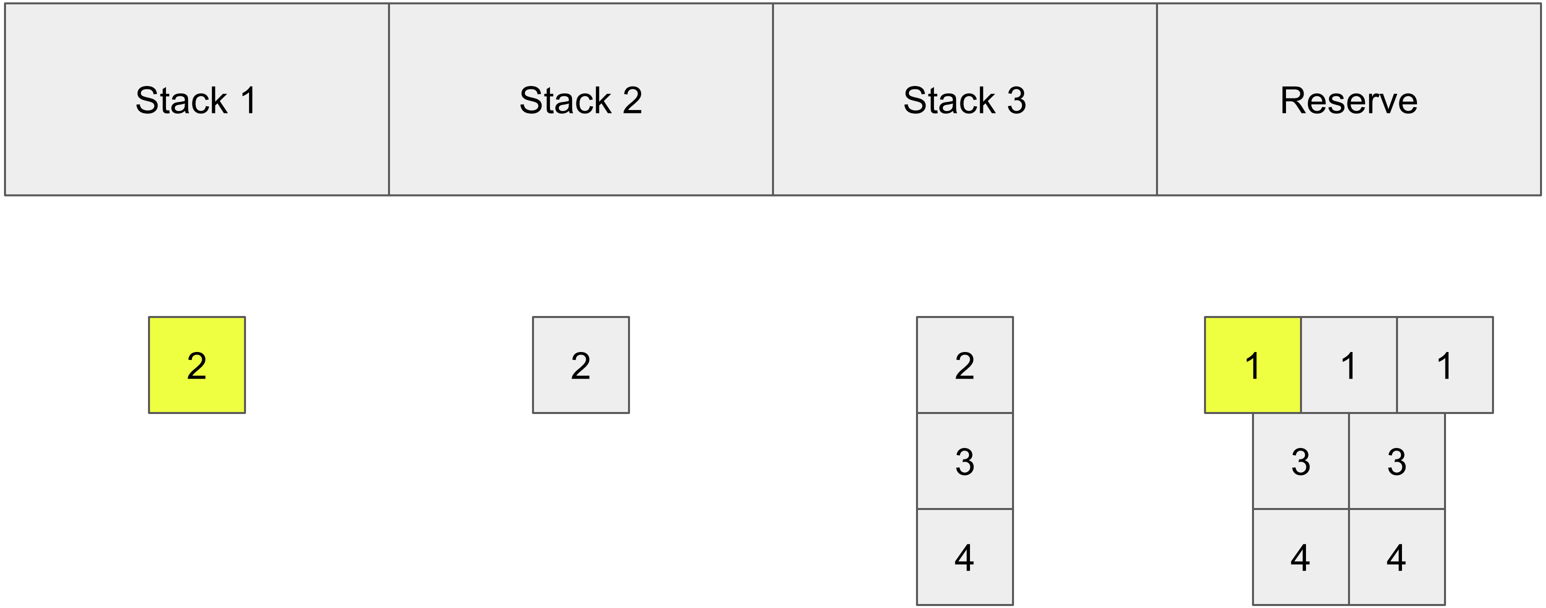
Lazy demon – does nothing.
Player can win by ensuring that stack i contains i-card.

Contrary demon – after each turn, undoes what was just done.
It is impossible to win against such demon.

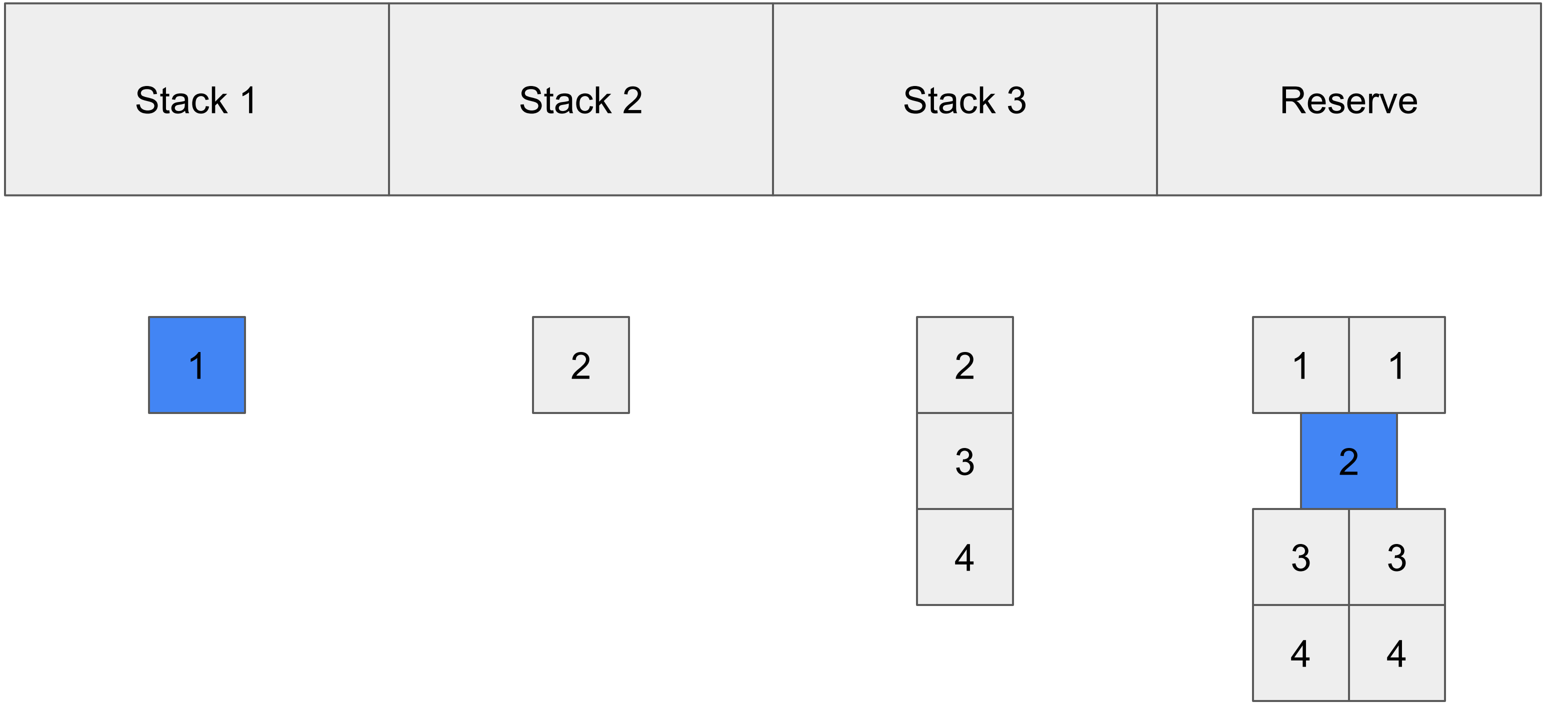
Game example against lazy demon



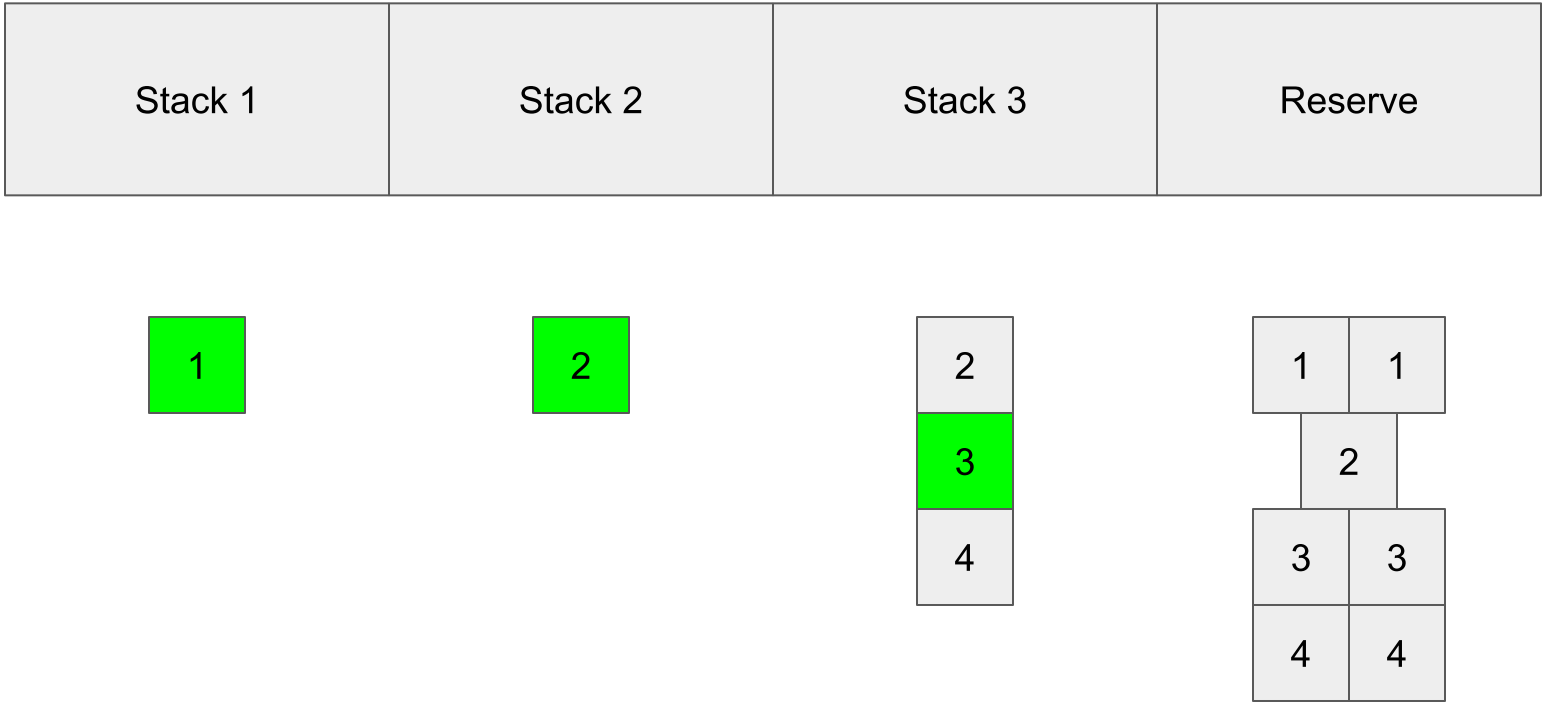
Game example against lazy demon



Game example against lazy demon

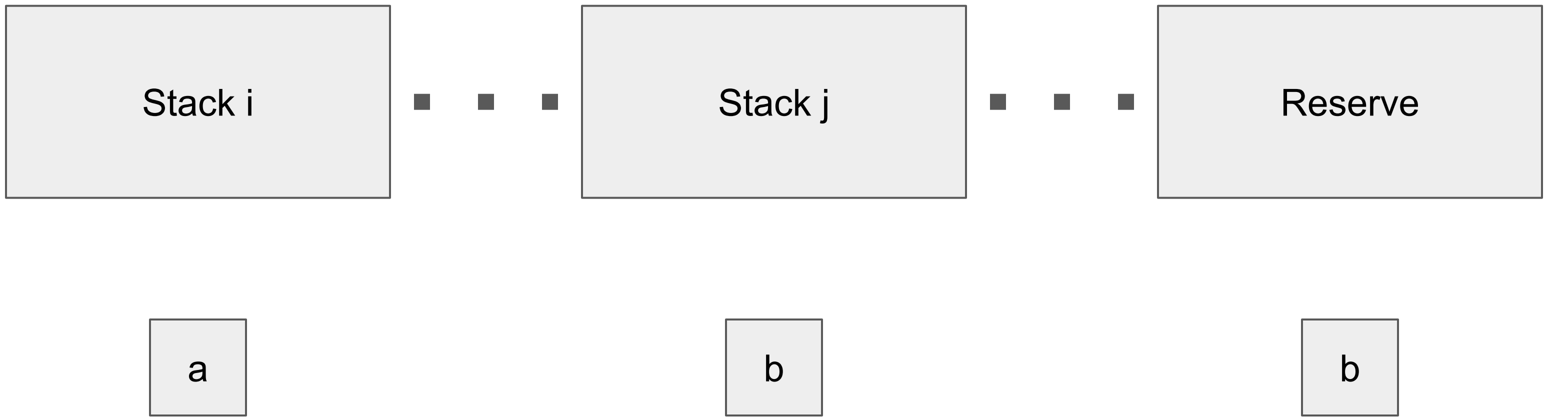


Game example against lazy demon

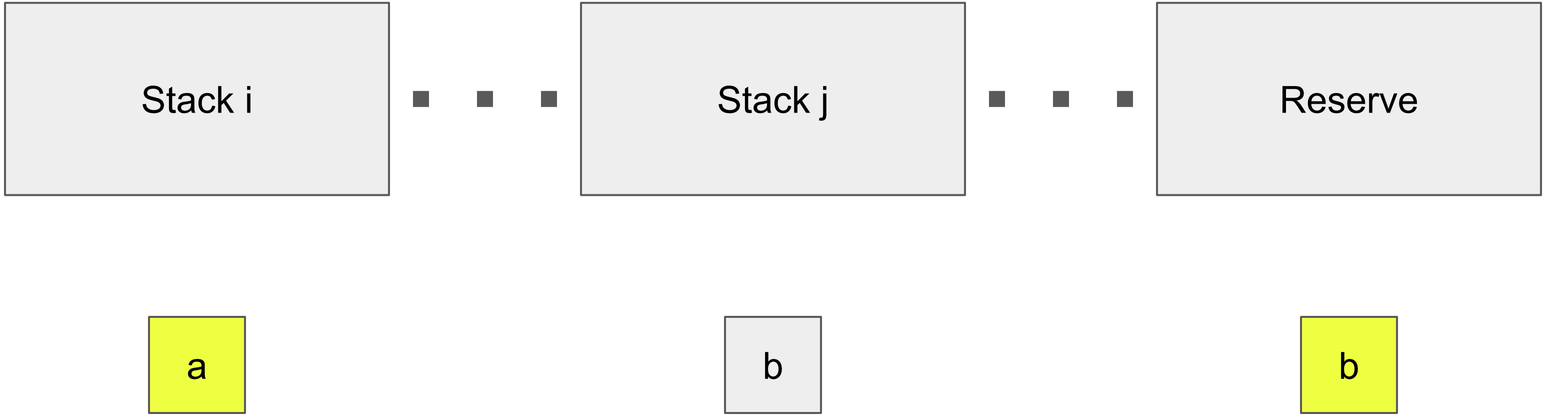


Kőnig’s demon

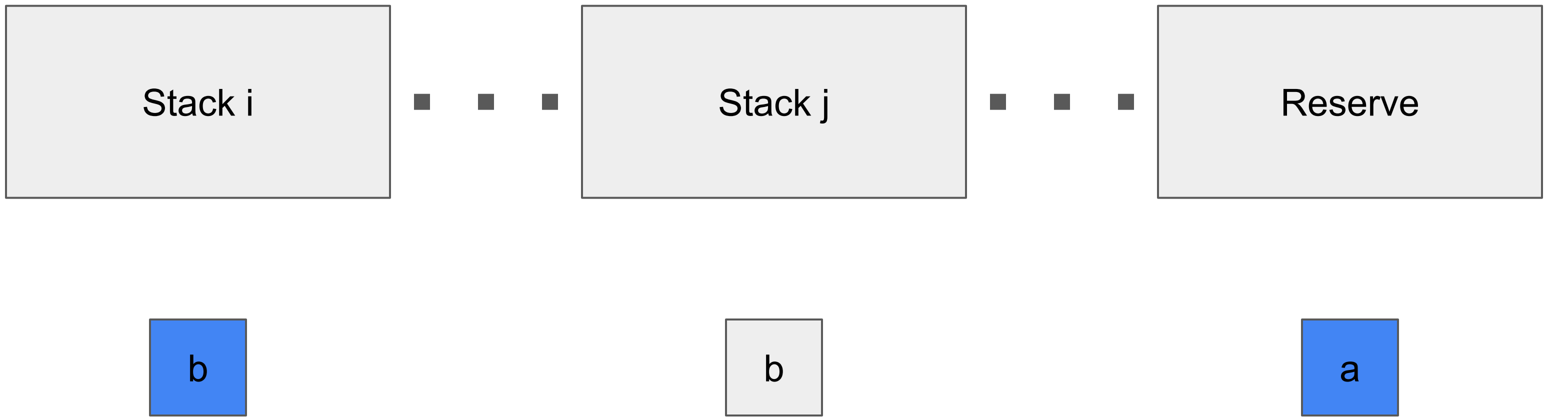
Kőnig’s demon moves – nothing or move below



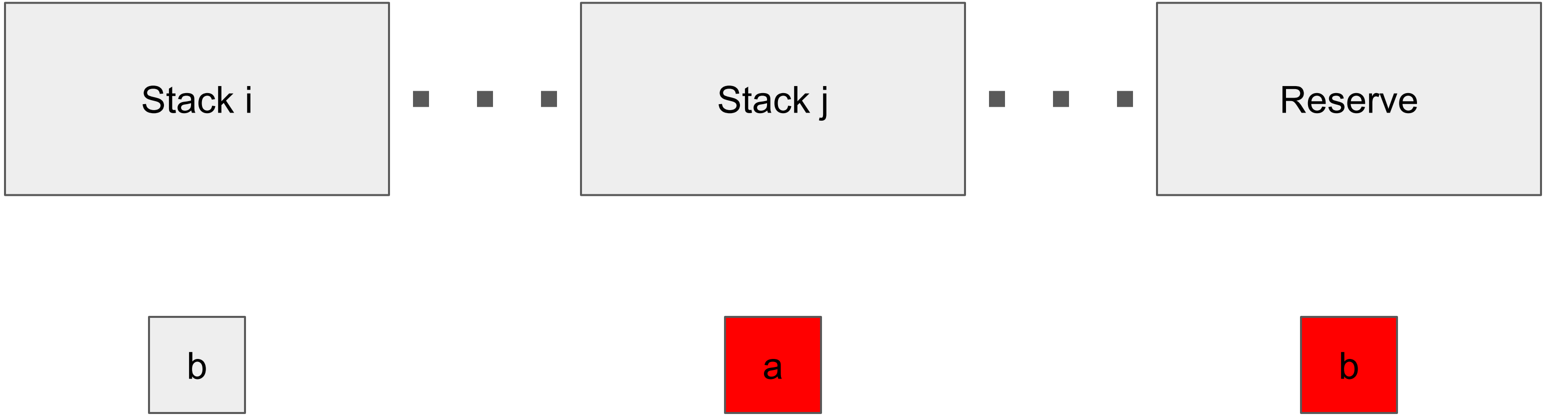
Kőnig’s demon moves – nothing or move below



Kőnig’s demon moves – nothing or move below



Kőnig’s demon moves – nothing or move below



Winning strategy against Kőnig's demon

k-game with stack profile $(n_1, \ldots, n_k)$.

Winning strategy against Kőnig's demon

k-game with stack profile $(n_1, \ldots, n_k)$.

Player consider the largest hand of differently numbered cards he can make by picking one card from each stack.

Winning strategy against Kőnig’s demon

k-game with stack profile (n_1, …, n_k).

Player consider the largest hand of differently numbered cards he can make by picking one card from each stack.

If he can make a hand of size k, he wins.

Winning strategy against Kőnig’s demon

If player is not currently winning, then there exist stack i from which he is not picking a card and b-card ($b \leqslant m$) that is not in his hand.

Winning strategy against Kőnig’s demon

If player is not currently winning, then there exist stack i from which he is not picking a card and b-card ($b \leqslant m$) that is not in his hand.

Since $n_i \geqslant 1$, stack i contains an a-card for same a.

Winning strategy against Kőnig's demon

If player is not currently winning, then there exist stack i from which he is not picking a card and b-card ($b \leqslant m$) that is not in his hand.

Since $n_i \geqslant 1$, stack i contains an a-card for same a.

Player can swap an a-card with b-card and make a larger hand.

Winning strategy against Kőnig's demon

If player is not currently winning, then there exist stack i from which he is not picking a card and b-card ($b \leqslant m$) that is not in his hand.

Since $n_i \geqslant 1$, stack i contains an a-card for same a.

Player can swap an a-card with b-card and make a larger hand.

Since the hand uses a b-card from only the stack i, the demon swapping out a b-card in another stack cannot decrease the size of the player's hand.

Winning strategy against Kőnig's demon

If player is not currently winning, then there exist stack i from which he is not picking a card and b-card ($b \leqslant m$) that is not in his hand.

Since $n_i \geqslant 1$, stack i contains an a-card for same a.

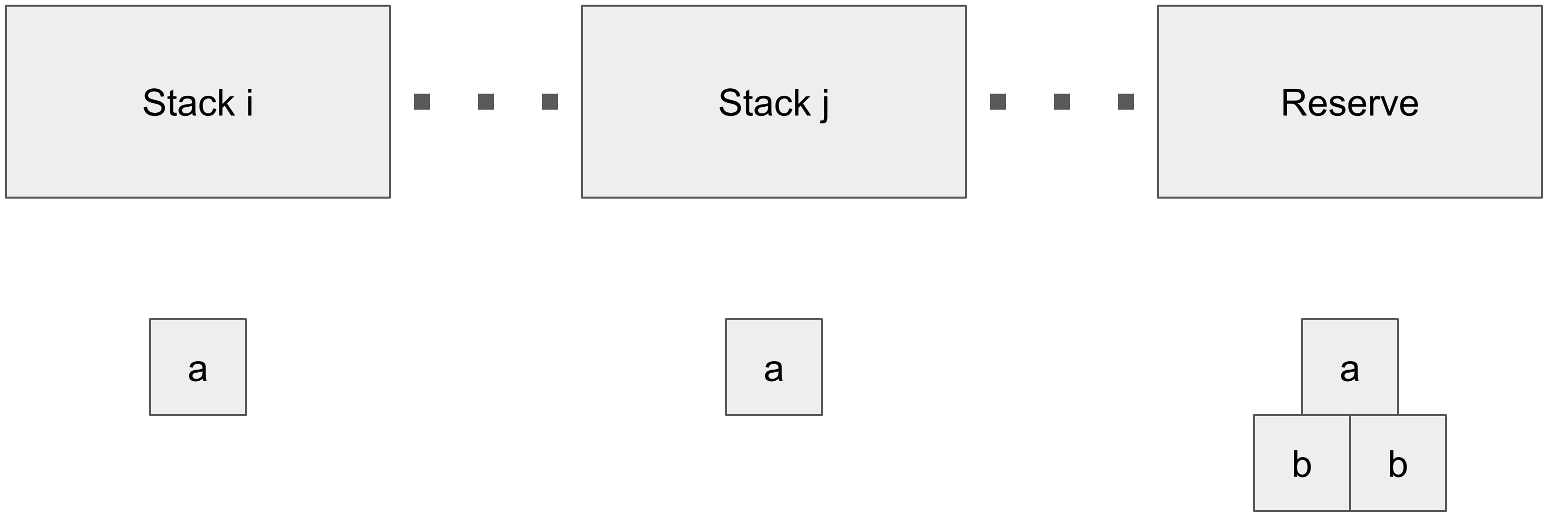
Player can swap an a-card with b-card and make a larger hand.

Since the hand uses a b-card from only the stack i, the demon swapping out a b-card in another stack cannot decrease the size of the player's hand.

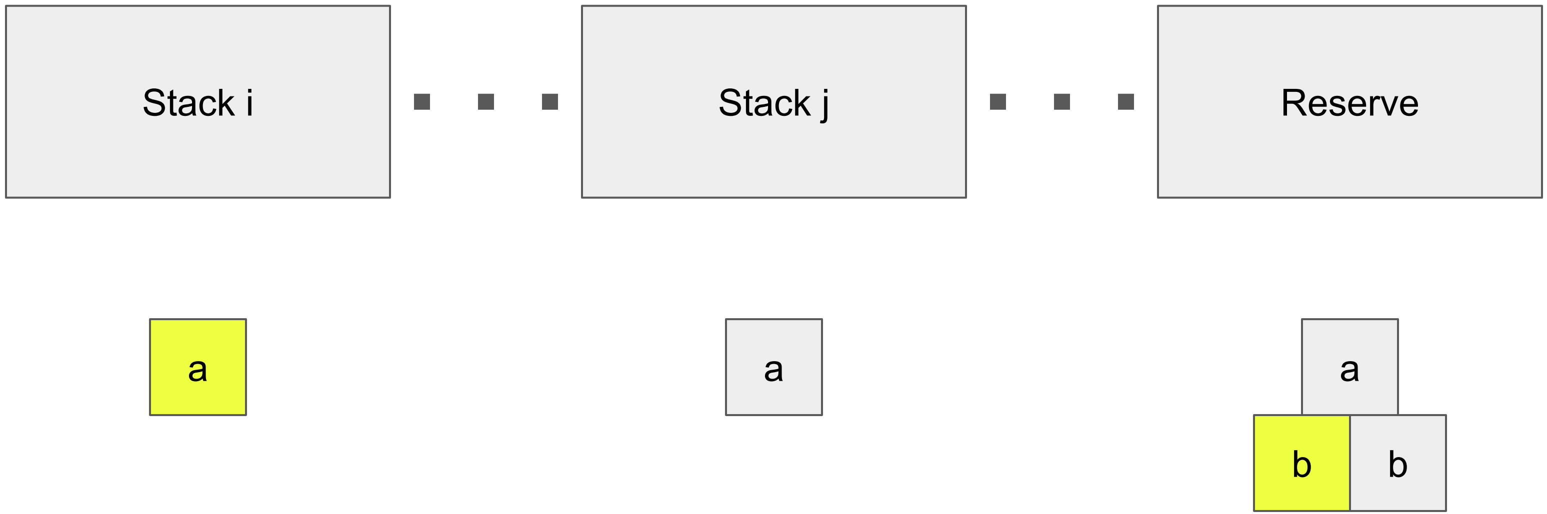
Repeat till the hand size equals k.

Vizing’s demon

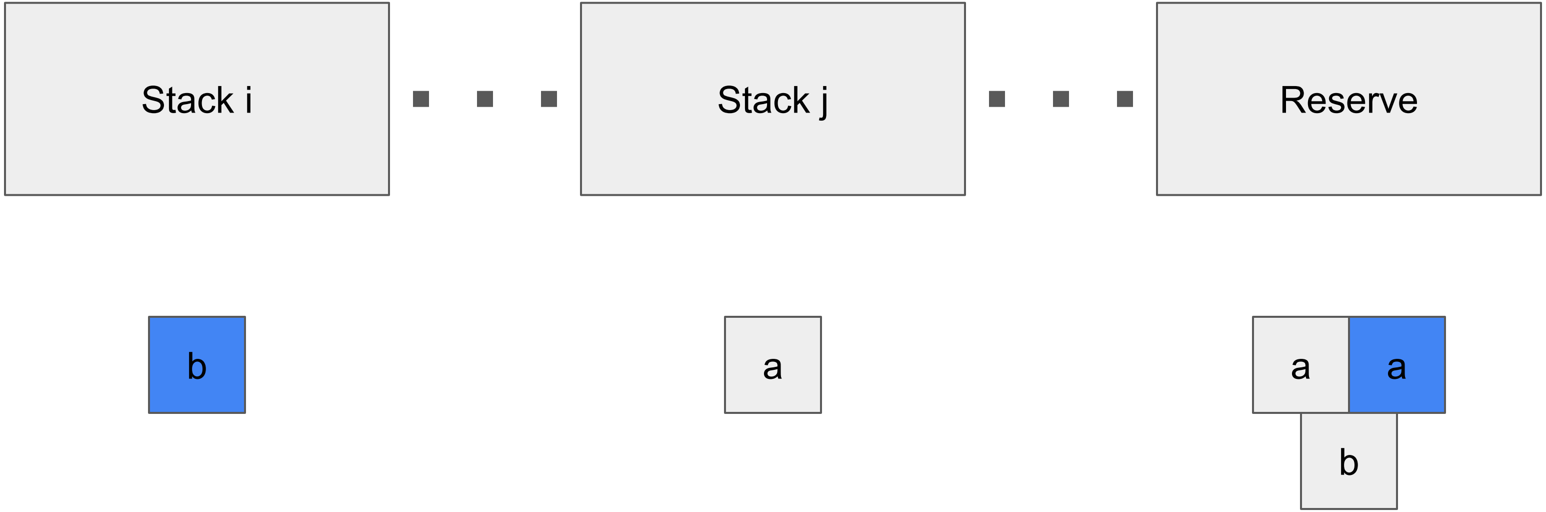
Vizing’s demon moves = Kőnig’s moves or move below



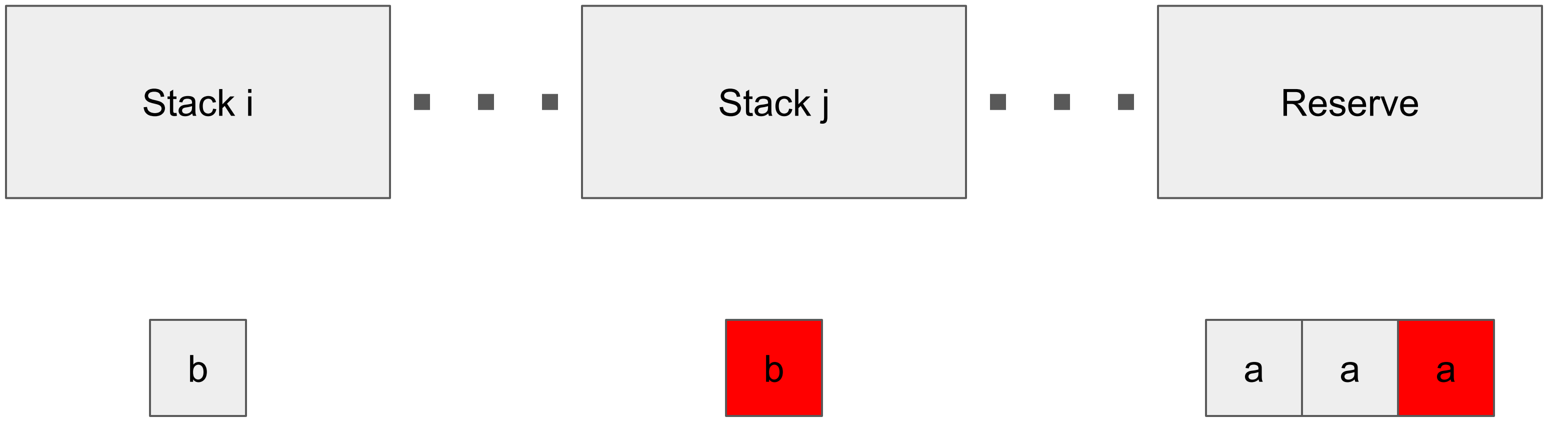
Vizing’s demon moves = Kőnig’s moves or move below



Vizing’s demon moves = Kőnig’s moves or move below



Vizing’s demon moves = Kőnig’s moves or move below



Sketch: Winning strategy against Vizing's demon

A position is reducible when for some nonempty subset S of at most k-1 stacks, there is a choice of differently numbered cards, one for each in S, so that the number on these cards appear in none of the stacks outside S. We may then play rest of the game only on stacks outside S.

Sketch: Winning strategy against Vizing's demon

A position is reducible when for some nonempty subset S of at most k-1 stacks, there is a choice of differently numbered cards, one for each in S, so that the number on these cards appear in none of the stacks outside S. We may then play rest of the game only on stacks outside S.

One may show that a player has the strategy to ensure that at least k differently numbered cards appear among the stacks (use Pigeonhole Principle).

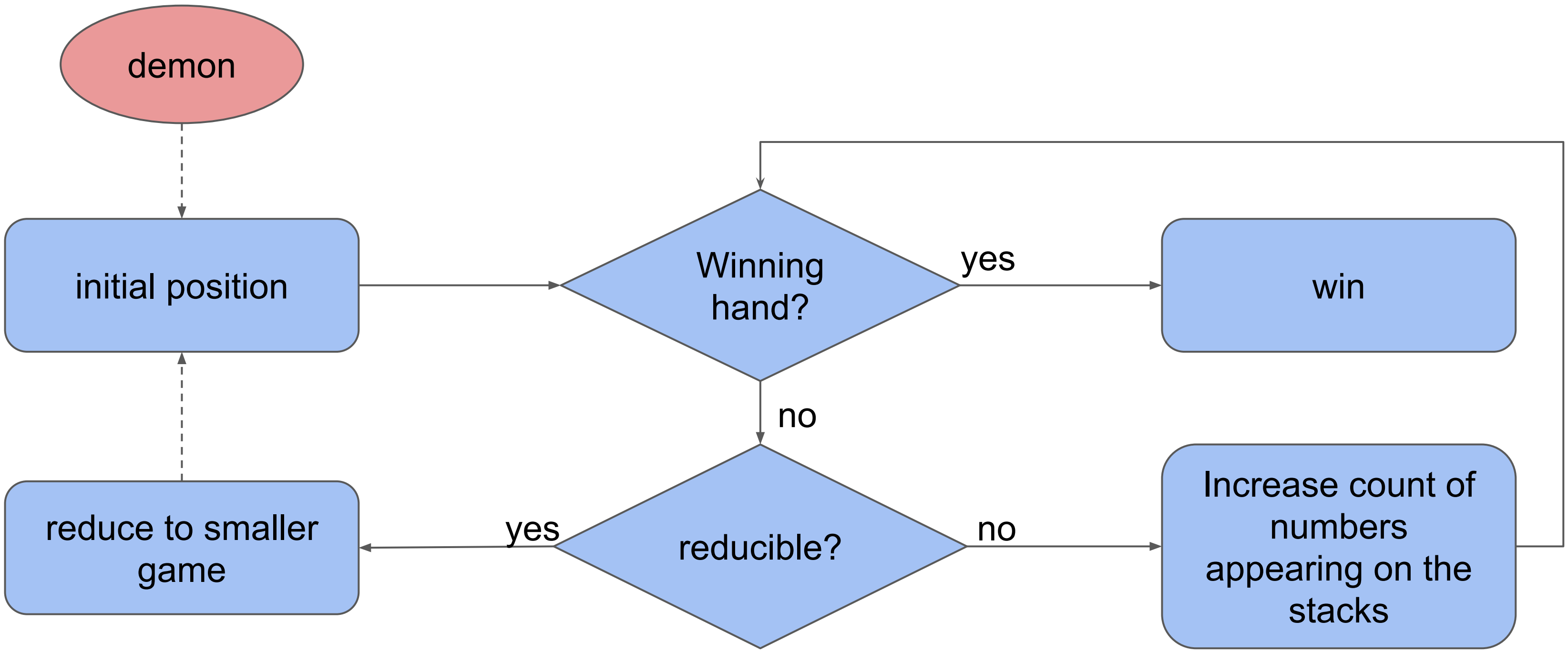
Sketch: Winning strategy against Vizing’s demon

A position is reducible when for some nonempty subset S of at most k-1 stacks, there is a choice of differently numbered cards, one for each in S, so that the number on these cards appear in none of the stacks outside S. We may then play rest of the game only on stacks outside S.

One may show that a player has the strategy to ensure that at least k differently numbered cards appear among the stacks (use Pigeonhole Principle).

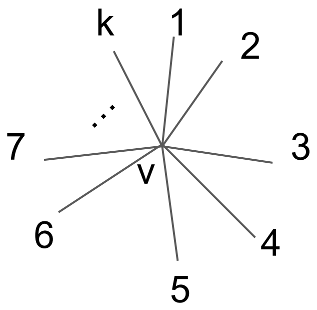
Such position is either reducible or the player can win (Hall’s Marriage Theorem).

Winning strategy against Vizing’s demon



Reduction

Let G be a graph with a vertex v with k neighbors. If we remove v, we can color the resulting graph using the colors 1,2,...,m where m is at least k.



Reduction

Let G be a graph with a vertex v with k neighbors. If we remove v, we can color the resulting graph using the colors 1,2,...,m where m is at least k.

We wish to extend this edge coloring to an edge coloring of G using only the colors 1,2,...,m. We will play a solitaire game with a game number k and a card number m.

Reduction

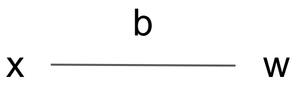
We wish to extend this edge coloring to an edge coloring of G using only the colors 1,2,...,m. We will play a solitaire game with a game number k and a card number m.

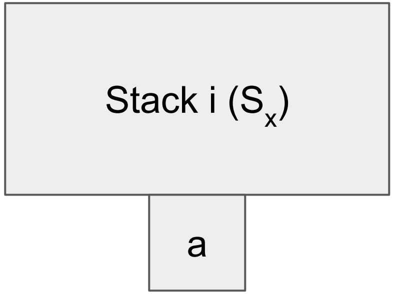
Reduction

We wish to extend this edge coloring to an edge coloring of G using only the colors 1,2,...,m. We will play a solitaire game with a game number k and a card number m.

For each $x \in N(v)$ the demon creates a stack S_x with one card for each number in 1,2,...,m that does not appear on an edge incident to x. If x has d neighbors in G, then S_x has m+1-d cards since all of the edges incident to x get different colors, except the edge to v gets no color.

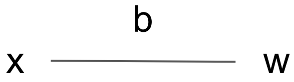
Reduction

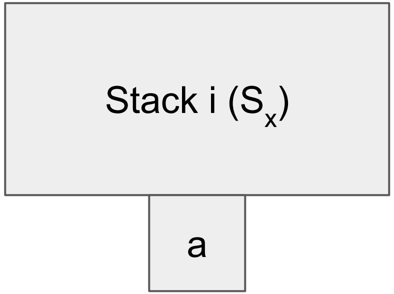




Suppose S_x contains an a-card but no b-card. Then there is an edge incident to x colored with b but none colored a.

Reduction





Suppose S_x contains an a-card but no b-card. Then there is an edge incident to x colored with b but none colored a.

Consider a path starting at x and alternating between edges colored b and edges colored a (such longest path is unique). If we swap colors a and b along this path we get another edge coloring of G.

Reduction

Moreover S_x has been changed by swapping its a-card for a b-card from the reserve.

Reduction

Moreover S_x has been changed by swapping its a-card for a b-card from the reserve.

If the path end $y \notin N(v)$, then no other stack is changed (the demon passed). So suppose $y \in N(v)$. $y \neq x$ because x has no edge colored a and only one colored b.

Reduction

Moreover S_x has been changed by swapping its a-card for a b-card from the reserve.

If the path end y ∉ N(v), then no other stack is changed (the demon passed). So suppose y ∈ N(v). y ≠ x because x has no edge colored a and only one colored b.

If G is bipartite then the path must be of an even length (otherwise y–v–x–...–y is an odd cycle). Therefore, since the path started with color b, it must end with color a. A swap along this path corresponds to changing S_y by swapping its b-card for an a-card (so it's a player's move followed by a Kőnig's Demon move).

Reduction

Suppose one of the theorems were false. Let’s pick a graph smallest G for which it fails. G has at least two vertices. Let $v \in V(G)$ $d(v) = k$ be a vertex with maximal degree.

Reduction

Suppose one of the theorems were false. Let’s pick a graph smallest G for which it fails. G has at least two vertices. Let $v \in V(G)$ $d(v) = k$ be a vertex with maximal degree.

By minimality of G, removing v gives a graph that can be colored using m colors where m is k (+1) for Konig’s (Vizing’s) theorem. For any $x \in N(v)$ $d(x) = d$, the stack S_x has $m+1-d \geqslant m+1-k$ cards, since $d \leqslant k$. That is $\geqslant 1$ ($\geqslant 2$) card(s) for Konig’s (Vizing’s) theorem.

Reduction

Suppose one of the theorems were false. Let's pick a graph smallest G for which it fails. G has at least two vertices. Let $v \in V(G)$ $d(v) = k$ be a vertex with maximal degree.

By minimality of G, removing v gives a graph that can be colored using m colors where m is k (+1) for Konig's (Vizing's) theorem. For any $x \in N(v)$ $d(x) = d$, the stack S_x has $m+1-d \geqslant m+1-k$ cards, since $d \leqslant k$. That is $\geqslant 1$ ($\geqslant 2$) card(s) for Konig's (Vizing's) theorem.

But then a player can make a hand of k cards. Coloring the edges incident to v with the numbers on these cards gives a coloring of G.