A game generalizing Hall's Theorem

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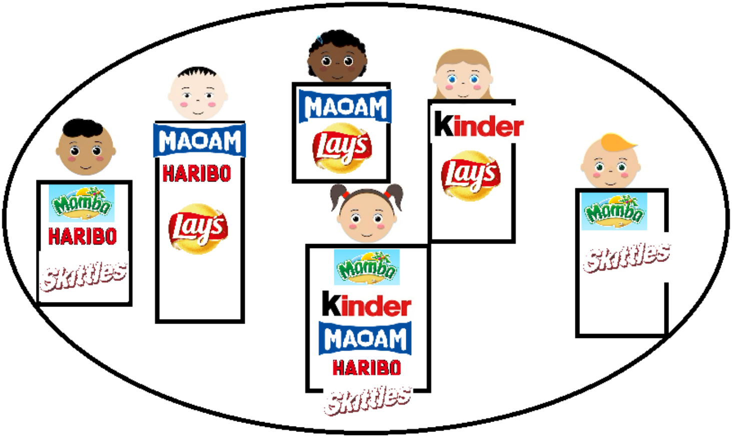
Definitions

Definition

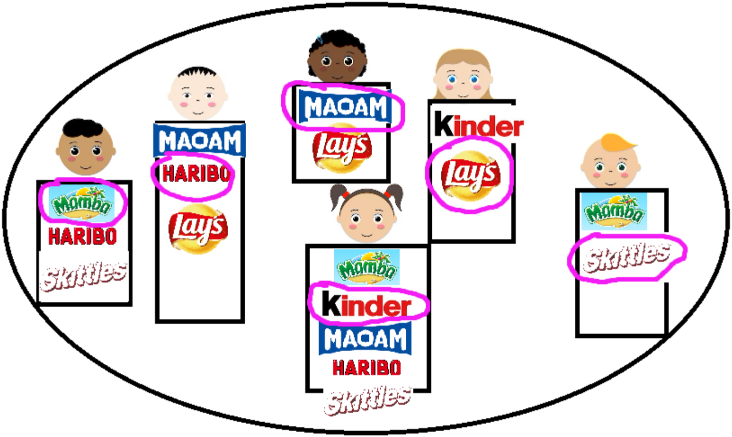
$\mathbb{S}$ - A set system is a finite family of finite sets.



Definitions

Definition

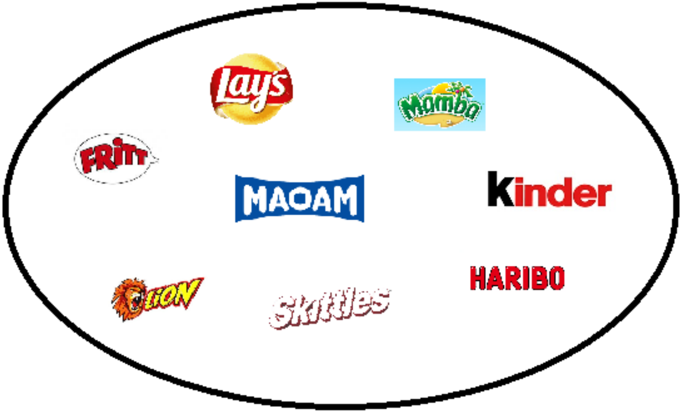
A transversal of a set system is injection $f : \mathbb{S} \to \bigcup \mathbb{S}$, such that $f(S) \in S$, for each $S \in \mathbb{S}$.



Definitions

Definition

A pot P is a set that: $\bigcup \mathbb{S} \subseteq P$.



Definitions

Hall's Theorem

A set system $\mathbb{S}$ has a transversal if and only if $|\bigcup W| \geqslant |W|$ for each $W \subseteq \mathbb{S}$.

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Game rules

We will analyze winning strategies for game for two-players: Fixer and Breaker. Fixer wins the game by eventually modifying the set system so that it has a transversal. If Breaker has a strategy to prevent this forever, then Breaker wins.

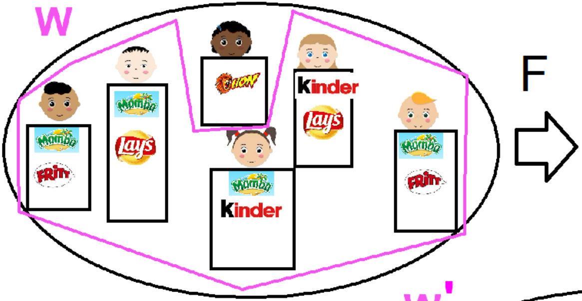
Fixer's turn

Pick $x \in P$ and $S \in \mathbb{S}$ with $x \notin S$ and replace S with $S \cup \{x\} \setminus \{y\}$ for some $y \in S$.

Breaker's turn

If F (Fixer) modified $S \in \mathbb{S}$ by inserting x and removing y, B_t (Breaker t) can pick up to t sets in $\mathbb{S} \setminus S$ and modify them by swapping x for y or y for x.

Fixer move example

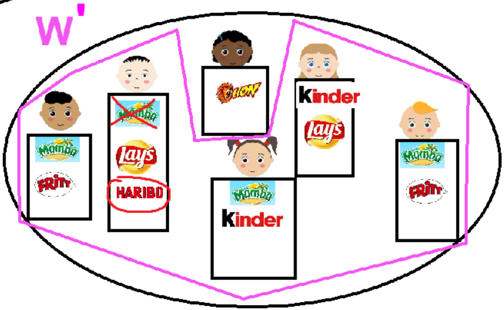


$|UW| < |W|$

The condition of Hall's theorem does not hold.

Fixer fixed Hall's theorem condition.

$|UW'| >= |W'|$



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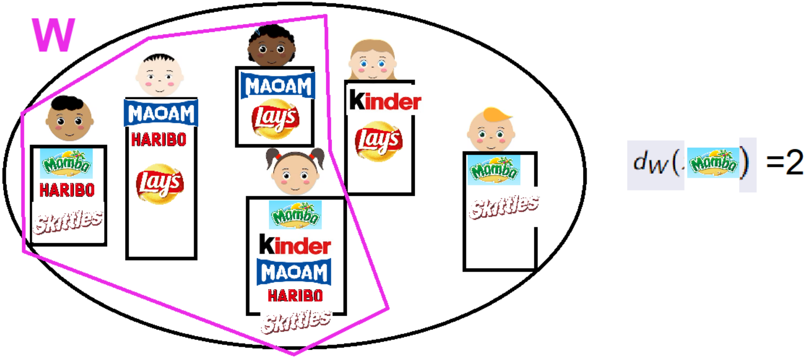
More definitions

Degree

We define the degree in W of x, written $d_W(x)$, by

$$d_W(x) = |S \in W : x \in S|,$$

for $W \subseteq \mathbb{S}$ and $x \in P$

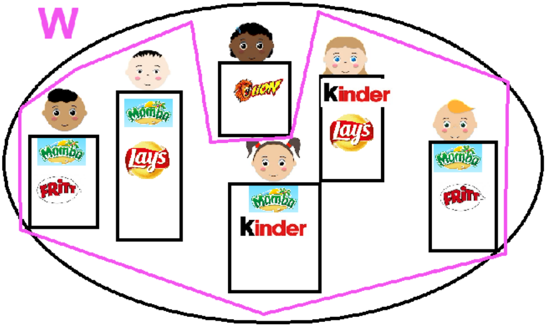


More definitions

t-value

The t-value of $W \subseteq \mathbb{S}$:

$$v_t(W) = \sum_{x \in \bigcup W} \left\lfloor \frac{d_W(x) - 1}{t+1} \right\rfloor$$



For t=2

$$\left\lfloor \frac{d_W(\text{Mamba})-1}{t+1} \right\rfloor = 1$$

$$v_t(W) = 1$$

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Main Theorem

Theorem

In a set system $\mathbb{S}$ with $\bigcup \mathbb{S} \subseteq P$ and $|P| \geqslant |\mathbb{S}|$, F has a winning strategy against $B_t \Leftrightarrow$
$|\bigcup W| \geqslant |W| - v_t(W)$ for each $W \subseteq \mathbb{S}$.

We can recover Hall's Theorem from $t = |\mathbb{S}| - 1$ case. B_t neutralizes Fixer's move by swapping all other y's to x's (when Fixer swapped one x for y), so this situation is the same as swapping names of x and y.

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Main Theorem Proof

Step 1 - necessity of the condition

Suppose we have $W \subseteq \mathbb{S}$ with $|\bigcup W| < |W| - v_t(W)$. We show that no matter what moves F makes, B_t can maintain this invariant. We then always have $|\bigcup W| < |W| - v_t(W) \leqslant |W|$ and hence W can never have a transversal.

Main Theorem Proof - Step 1 - necessity of the condition

$\longrightarrow$ Fixer modifies $S \in \mathbb{S}$ by removing y and inserting x to get S'.
1° $S \notin W$ Then B_t does nothing and we still have $|\bigcup W| < |W| - v_t(W)$.

Main Theorem Proof - Step 1 - necessity of the condition

2° $S \in W$

Put $W' = W \cup \{S'\} \setminus \{S\}$ (W after Fixer's move) and put $W^* = (W'$ after Breaker's move).

Important observations:

· Only terms that may change between $v_t(W)$ and $v_t(W^*)$ are $\left\lfloor \frac{d_W(x)-1}{t+1} \right\rfloor$ and $\left\lfloor \frac{d_W(y)-1}{t+1} \right\rfloor$

· $d_W(x) + d_W(y) = d_{W^*}(x) + d_{W^*}(y)$, so

$$v_t(W) - 1 \leqslant v_t(W^*) \leqslant v_t(W) + 1$$

Main Theorem Proof - Step 1 - necessity of the condition

2.1° $d_W(x) = 0$
$|\bigcup W'| = |\bigcup W| + 1$ (because of new element - x)
$2.1.1^\circ$ $d_{W'}(y) \leqslant t$
$\rightarrow B_t$ swaps x in for y in $d_{W'}(y)$ sets of W'.
So Fixer and Breaker together replaced all y with x, so it's equivalent of renaming element, thus it doesn't change that $|\bigcup W| < |W| - v_t(W)$.

Main Theorem Proof - Step 1 - necessity of the condition

2.1.2° $d_{W'}(y) > t$
$\rightarrow B_t$ swaps x in for y in t sets of W'.
$|\bigcup W^*| = |\bigcup W'| = |\bigcup W| + 1$
$|W^*| = |W'| = |W|$

$$v_t(W^*) = v_t(W) + \left\lfloor \frac{d_{W^*}(x) - 1}{t+1} \right\rfloor + \left\lfloor \frac{d_{W^*}(y) - 1}{t+1} \right\rfloor - \left\lfloor \frac{d_W(y) - 1}{t+1} \right\rfloor$$

$$v_t(W^*) = v_t(W) + \left\lfloor \frac{(t+1) - 1}{t+1} \right\rfloor + \left\lfloor \frac{(d_W(y) - (t+1)) - 1}{t+1} \right\rfloor - \left\lfloor \frac{d_W(y) - 1}{t+1} \right\rfloor$$

$$v_t(W^*) = v_t(W) + 0 + \left\lfloor \frac{d_W(y) - 1}{t+1} \right\rfloor - 1 - \left\lfloor \frac{d_W(y) - 1}{t+1} \right\rfloor$$

$$v_t(W^*) = v_t(W) - 1$$

So:

$$|\bigcup W^*| < |W^*| - v_t(W^*) \Leftrightarrow |\bigcup W| + 1 < |W| - (v_t(W) - 1)$$

$$|\bigcup W^*| < |W^*| - v_t(W^*) \Leftrightarrow |\bigcup W| < |W| - v_t(W)$$

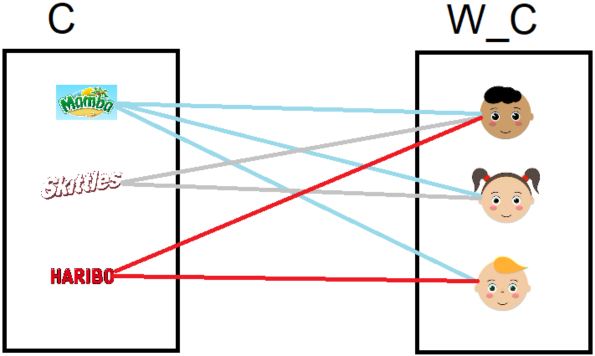
2.2° $d_W(x) > 0$
$\longrightarrow$ B_t swaps y in for x in one set in $W' \backslash S'$.
Doing so maintains the invariant, since now every element has the same degree in the new set system as in W.

Main Theorem Proof - Step 2 - sufficiency of the condition

Suppose the condition ($|\bigcup W| \geqslant |W| - v_t(W)$ for each $W \subseteq \mathbb{S}$) is not sufficient for F to have a winning strategy. Among all counterexamples having the **fewest sets** (minimal $|\mathbb{S}|$), choose $\mathbb{S}$ to maximize $|\bigcup \mathbb{S}|$.

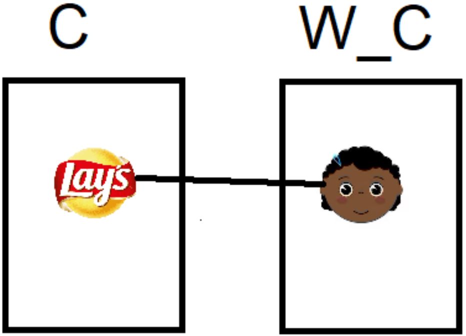
Main Theorem Proof - Step 2 - sufficiency of the condition

1° $|\bigcup \mathbb{S}| \geqslant |\mathbb{S}|$
For $C \subseteq \bigcup \mathbb{S}$ let's define $W_C = \{S \in \mathbb{S} | C \cap S \neq \emptyset\}$.
Let C be a minimal nonempty subset of $\bigcup \mathbb{S}$ such that $|W_C| \leqslant |C|$ (we can make this choice because $\bigcup \mathbb{S}$ is such a subset).
Create a bipartite graph with parts C and W_C and an edge from $x \in C$ to $S \in W_C \Leftrightarrow x \in S$.



Main Theorem Proof - Step 2 - sufficiency of the condition

1.1° $|C| = 1$
$|W_C| \leqslant |C| \Rightarrow |W_C| = 1$, so we have matching between C and W_C.



Main Theorem Proof - Step 2 - sufficiency of the condition

1.2° $|C| > 1$

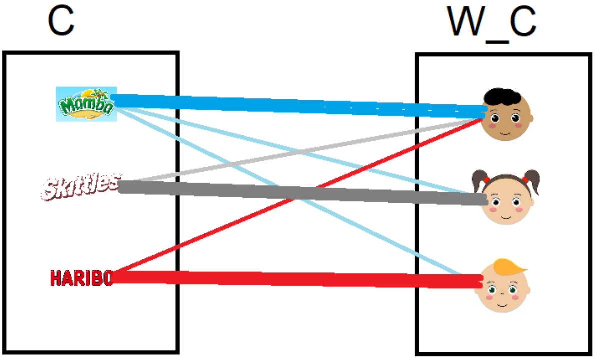
Let's take a set D, which will be the set C without one element.
We have conditions

$$\emptyset \neq D \subset C, |D| + 1 = |C|, |W_D| > |D|.$$

So $|C| \geqslant |W_C| \geqslant |W_D| \geqslant |D| + 1 = |C|$. Since the same expression appears on both the left and right sides of the inequality, it means that $|C| = |W_C|$.

Main Theorem Proof - Step 2 - sufficiency of the condition

Now using Hall's Theorem for bipartite graphs we have matching between C and W_C.



So in both cases (1.1° and 1.2°) we have transversal $f : W_C \to \bigcup W_C$ with image C.

Main Theorem Proof - Step 2 - sufficiency of the condition

$\mathbb{S}' = \mathbb{S} \backslash W_C$ and $P' = P \backslash C$. If Fixer continues to play only using $\mathbb{S}'$ and P', then Breaker cannot destroy the transversal of W_C. So we have generated counterexample with set system $\mathbb{S}'$, that contradicts minimality of $|\mathbb{S}|$.

Main Theorem Proof - Step 2 - sufficiency of the condition

2° $|\bigcup \mathbb{S}| < |\mathbb{S}|$
Proof will show how Fixer can win, which will lead us to a contradiction.
$|\bigcup W| \geqslant |W| - v_t(W)$ for each $W \subseteq \mathbb{S}$, so taking $W = \mathbb{S}$, we get:
$|\bigcup \mathbb{S}| \geqslant |\mathbb{S}| - v_t(\mathbb{S})$.
So

$$|\bigcup \mathbb{S}| < |\mathbb{S}| \leqslant |\bigcup \mathbb{S}| + v_t(\mathbb{S})$$

Hence $v_t(\mathbb{S}) \geqslant 1$, which implies $\exists y : d_{\mathbb{S}}(y) \geqslant t + 2$.
$|P| \geqslant |\mathbb{S}| > |\bigcup \mathbb{S}|$, so $\exists x : d_{\mathbb{S}}(x) = 0$.

Main Theorem Proof - Step 2 - sufficiency of the condition

$\rightarrow$ Now Fixer should swap x in for y in some $S \in \mathbb{S}$ to form $\mathbb{S}'$. $d_{\mathbb{S}}(x) = 0$, so we have $|\bigcup \mathbb{S}'| > |\bigcup \mathbb{S}|$. We also have $d_{\mathbb{S}'}(y) \geqslant t + 1$. Now B_t moves and creates $\mathbb{S}^*$. Since $d_{\mathbb{S}^*}(y) \geqslant d_{\mathbb{S}'}(y) - t > 0$, we have $|\bigcup \mathbb{S}^*| > |\bigcup \mathbb{S}|$.

Main Theorem Proof - Step 2 - sufficiency of the condition

Let's suppose that after Breaker's move there exists W that:
$|\bigcup W^*| < |W^*| - v_t(W^*)$, so $W \neq W^*$, so Fixer or Breaker swapped in x for y in some $S \in W$, thus $|\bigcup W^*| > |\bigcup W|$. Now:

$$|W| - v_t(W) \leqslant |\bigcup W| < |\bigcup W^*| < |W^*| - v_t(W^*)$$

So:

$$v_t(W) \geqslant v_t(W^*) + 2$$

Which is impossible according to "Important observation" in 2° of step 1 of proof. So after Fixer and Breaker moves we still have $|\bigcup W| \geqslant |W| - v_t(W)$ for each $W \subseteq \mathbb{S}^*$, since we choose counterexample with maximum $|\bigcup \mathbb{S}|$ and $|\bigcup \mathbb{S}^*| > |\bigcup \mathbb{S}|$ this means this is no longer counterexample, so Fixer has winning strategy and that leads to contradiction.

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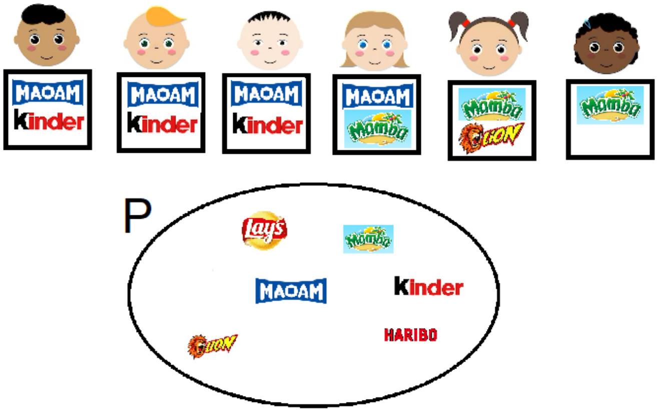
Vizing Theorem - Lemma

Lemma

If $\mathbb{S} = \{S_1, ..., S_k\}$, with $|S_k| \geqslant 1$, $|S_i| \geqslant 2$ for all $i \leqslant k-1$, and $|P| \geqslant k$, then F has winning strategy against B_1.



Vizing Theorem - Lemma

Lemma Proof

For $W \subseteq \mathbb{S}$, we have:

$$v_1(W) \geqslant \sum_{x \in \bigcup W} \frac{d_W(x) - 2}{2} = \frac{1}{2} \sum_{S \in W} |S| - \left|\bigcup W\right| \geqslant \frac{1}{2}(2|W| - 1) - \left|\bigcup W\right|$$

Hence $v_1(W) \geqslant |W| - |\bigcup W|$, as desired.

Vizing Theorem

Vizing Theorem

For a graph G, let $\chi'(G)$ be the edge-chromatic number of G and let $\Delta(G)$ be the maximum degree of G.
If G is a simple graph, then $\chi'(G) \leqslant \Delta(G) + 1$.

Vizing Theorem - Proof

Let G be a counterexample with fewest vertices. Let $v_1, ..., v_k$ be the neighbors of a vertex v of maximum degree in G. By minimality of $|G|$, we have a $(k+1)$-edge-coloring of $G \backslash v$. Among the $k+1$ colors used in this coloring, let S_i be the set of those not appearing on edges incident to v_i. Each v_i has degree at most $k-1$ in $G \backslash v$ and hence $S_i \geqslant 2$.

Vizing Theorem - Proof

Also, if $a \in S_i$ and $b \notin S_i$ we may exchange colors on a maximum length path M starting v_i and alternating between colors b and a. After the exchange, S_i has lost a and gained b. Consider S_j for some $j \in \{1, ..., i-1, i+1, ..., k$. If v_j is not in M or is an internal vertex in M, then S_j is maintained. If v_j is the endpoint of M then S_j has changed by either swapping a in for b or by swapping b in for a.

Vizing Theorem - Proof

Therefore, performing the exchange translates into an F move followed by a B_1 move on the set system $\{S_1, ..., S_k\}$. Since F can always make any of his legal moves this way, we may apply Lemma to get a transversal of the S_i. Now we can extend the $(k+1)$-edge-coloring to all of G by using the corresponding element of the transversal on vv_i for each $i \in \{1, ..., k\}$.

End

Thank you!

References

[1] Landon Rabern - A game generalizing Hall's Theorem
https://arxiv.org/pdf/1204.0139.pdf