On Two problems of Defective Choosability of Graphs

based on an article by Jie Ma, Rongxing Xu, Xuding Zhu

Katarzyna Kępińska

(k, d, p)-choosability

G is (k, d, p)-choosable if for every List assignment L, such that:

1. $L(v) \geqslant k$
2. $|\bigcup L(v)| \leqslant p$

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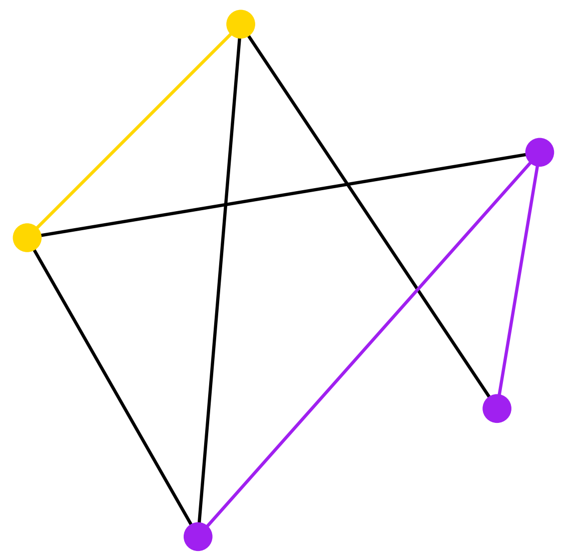

2-defective
2-coloring

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examples:

1. (k, 0, k)-choosable $=$ k-colorable
2. (k, 0, $+\infty$)-choosable $=$ k-choosable
3. (k, d, $+\infty$)-choosable $=$ d-defective k-choosable
4. (k, d, k)-choosable $=$ d-defective k-colorable

Previous results

1. Every outerplanar graph is 2-defective 2-colorable (Cowen and Woodall)
2. Every planar graph is 2-defective 3-colorable (Cowen and Woodall)
3. Every planar graph is 2-defective 3-choosable (Eaton and Hull; Škrekovski)
4. Every outerplanar graph is 2-defective 2-choosable (Eaton and Hull; Škrekovski)
5. There are 4-choosable planar graphs that are not 1-defective 3-colorable (Wang and Xu)
6. For each $l \geqslant k \geqslant 3$, there exists a $(k, 0, l)$-choosable graph which is not $(k, 0, l+1)$-choosable

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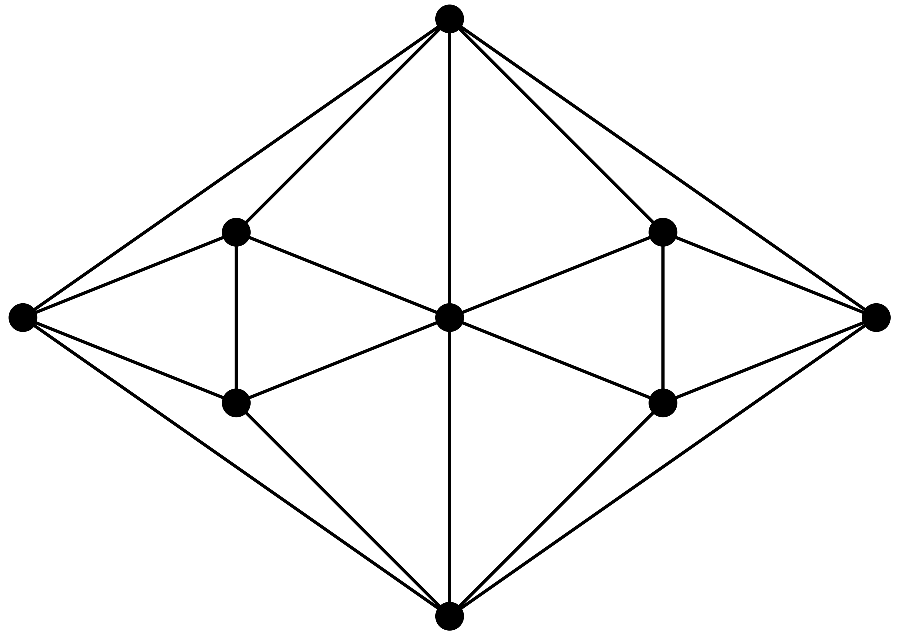
Construction For a positive integer k, let T (k) be the graph obtained from the disjoint union of k copies of T by identifying all the copies of top vertex and identifying all the copies of the bottom vertex.



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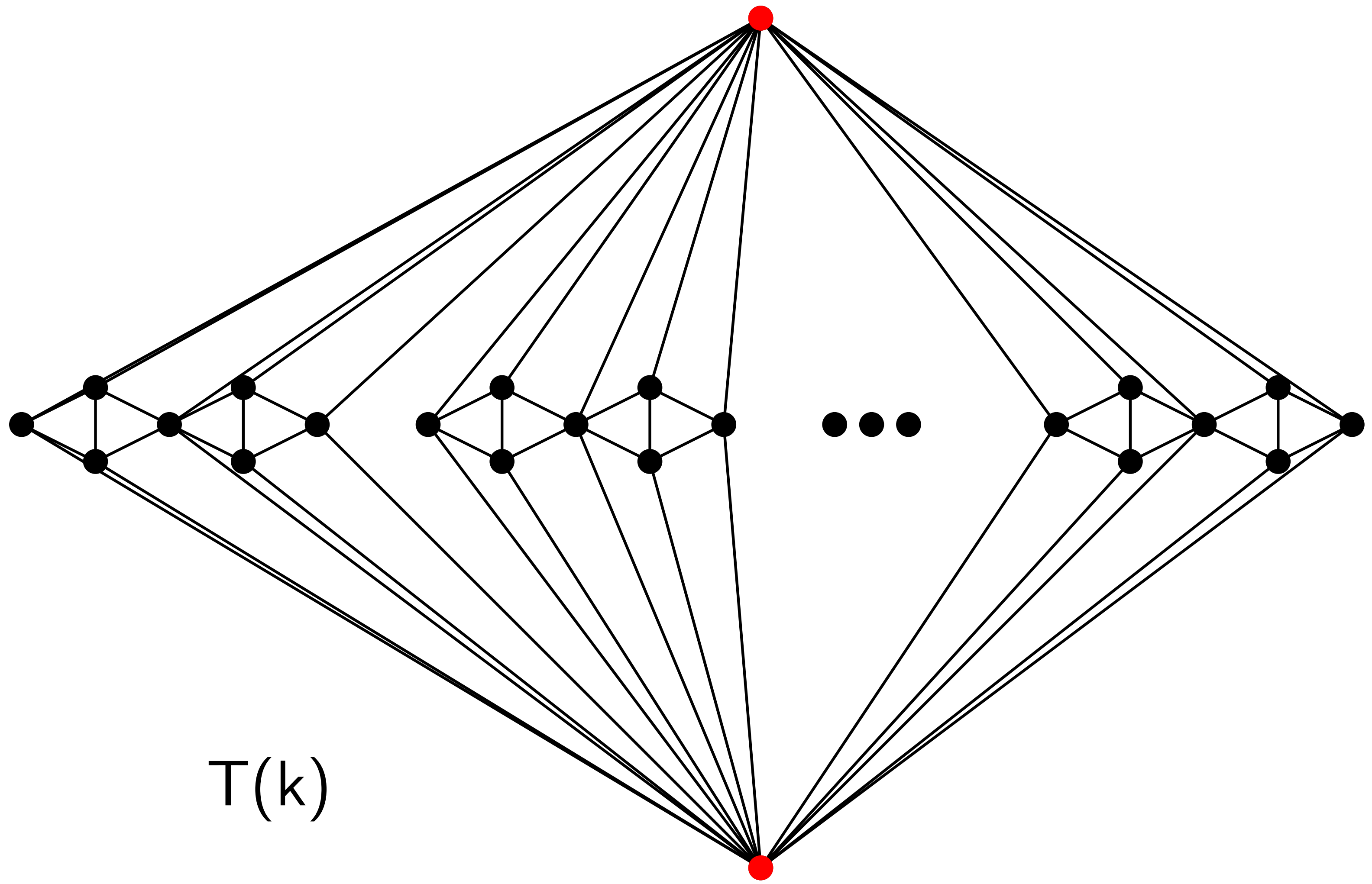

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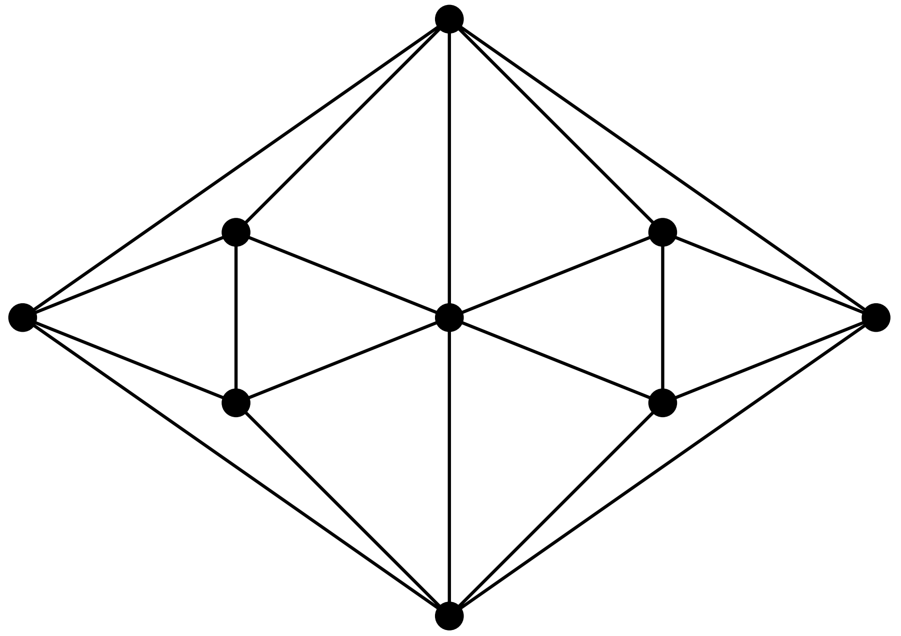

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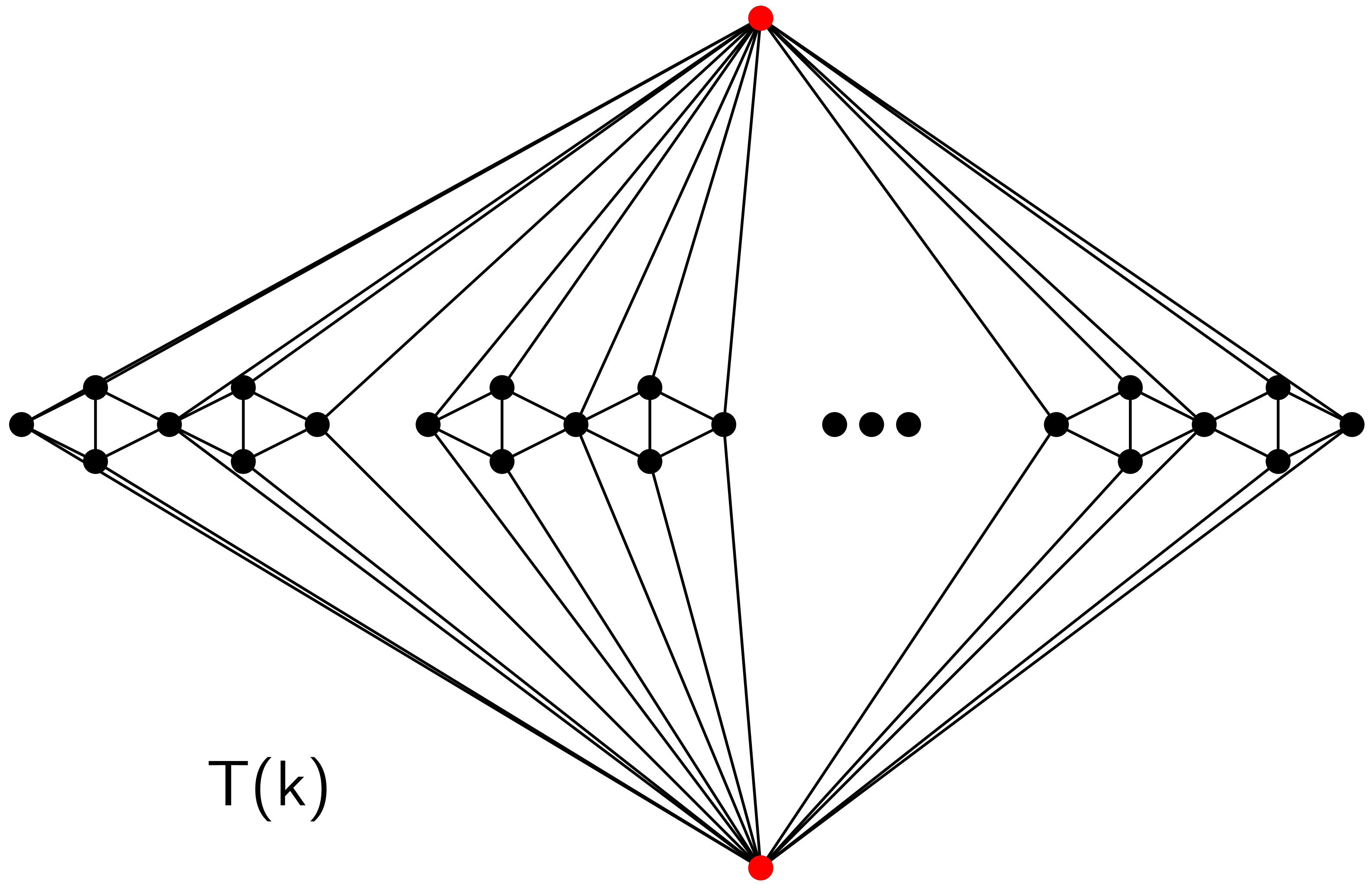

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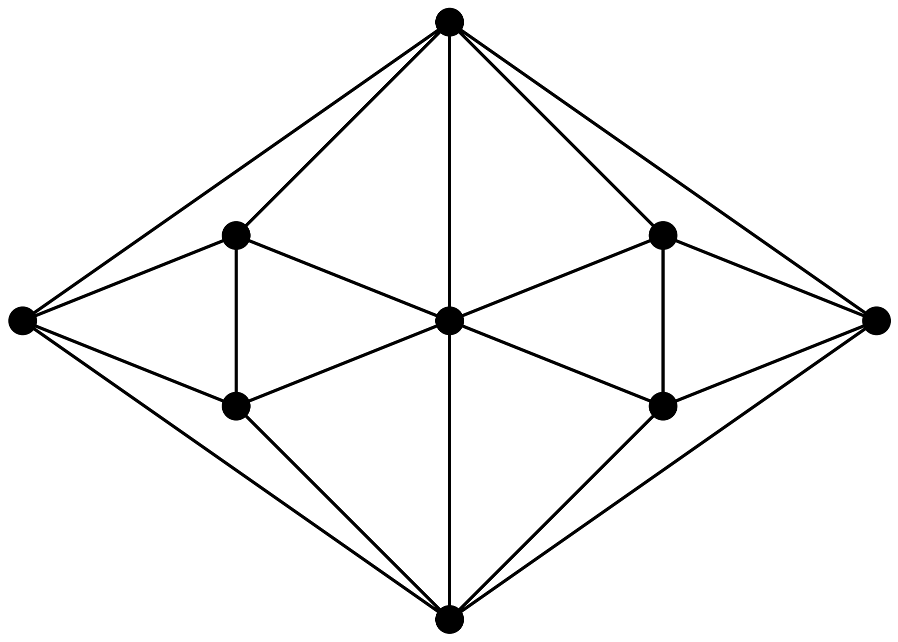
For $k \leqslant 26$ G is 1-defective 3-choosable.



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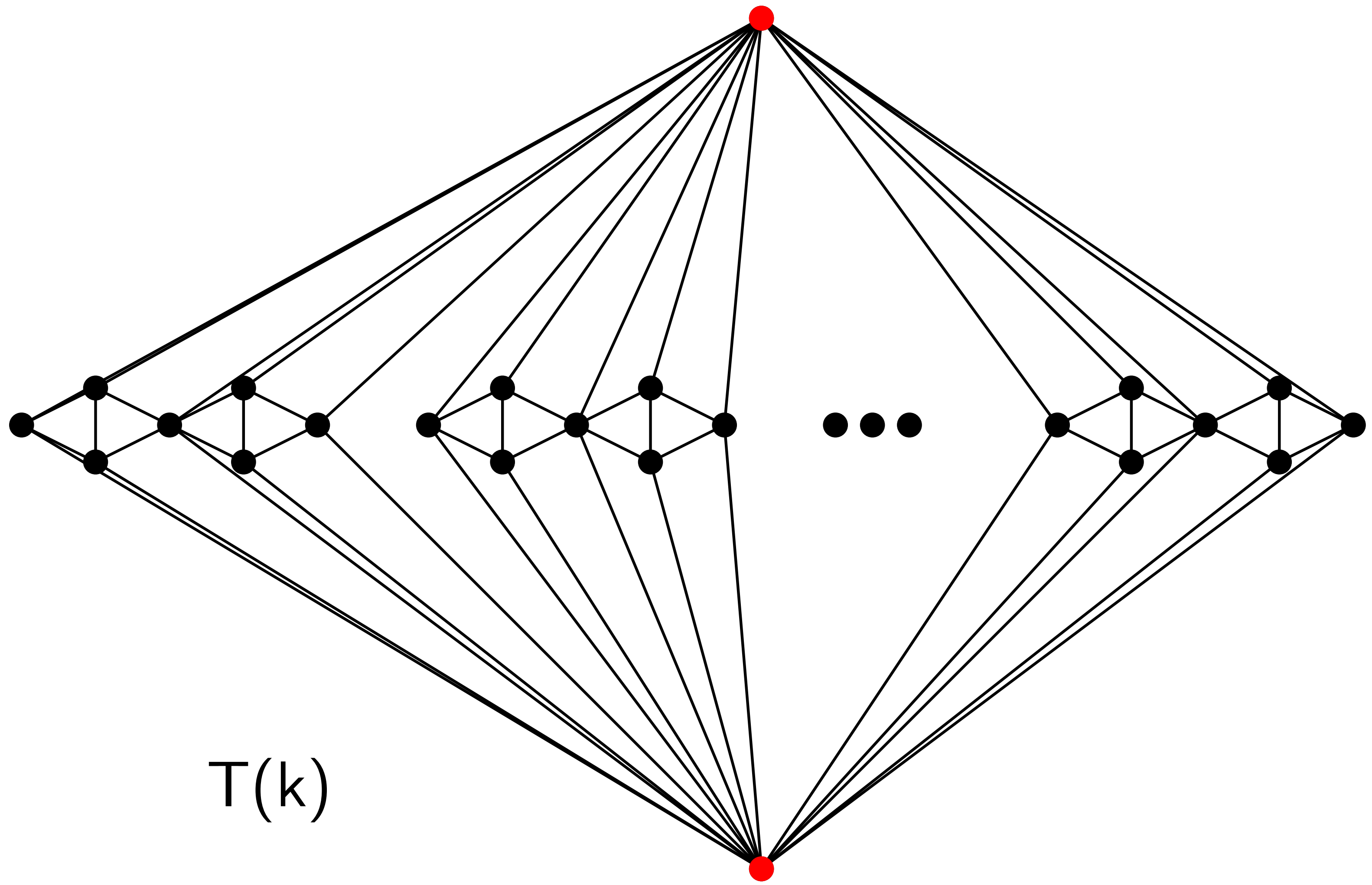
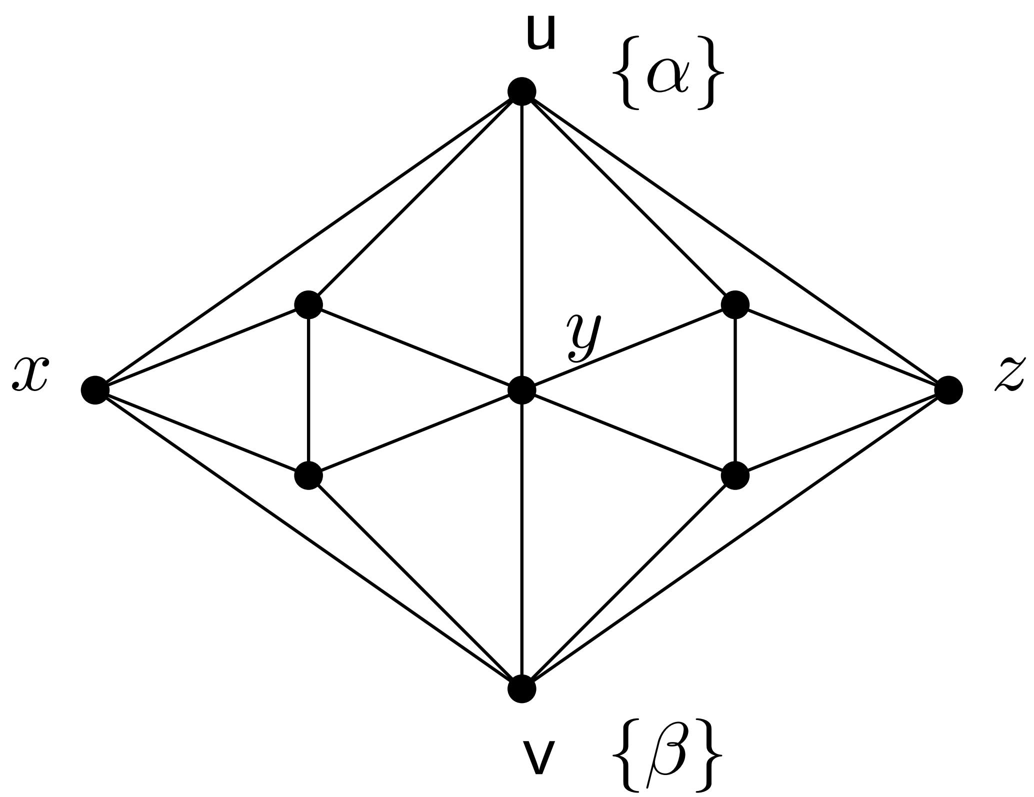


T(k)

Lemma 6 Let L be a list assignment of T with L(u) $= \alpha$, L(v) $= \beta$ and $|L(w)| \geqslant 3$ for $w \in V(T) \setminus \{u, v\}$. If
- $\alpha = \beta$, or
- $\alpha \neq \beta$ and $\{\alpha, \beta\} \nsubseteq L(x) \cap L(y) \cap L(z)$, or
- $\alpha \neq \beta$ and $L(x) \cap L(y) \cap L(z) - \{\alpha, \beta\} \neq \emptyset$,

then T has a 1-defective L-coloring ϕ such that $\lambda_T(u, \phi) = \lambda_T(v, \phi) = 0$.

Lemma 7 Let L be a list assignment of T with L(u) $= \alpha$, L(v) $= \beta$ and $|L(w)| \geqslant 3$ for w $\in V(T) \setminus \{u, v\}$. Then T has a 1-defective L-coloring ϕ such that $\lambda_T(u, \phi) =$ 0, and a 1-defective L-coloring ϕ such that $\lambda_T(v, \phi) =$ 0.



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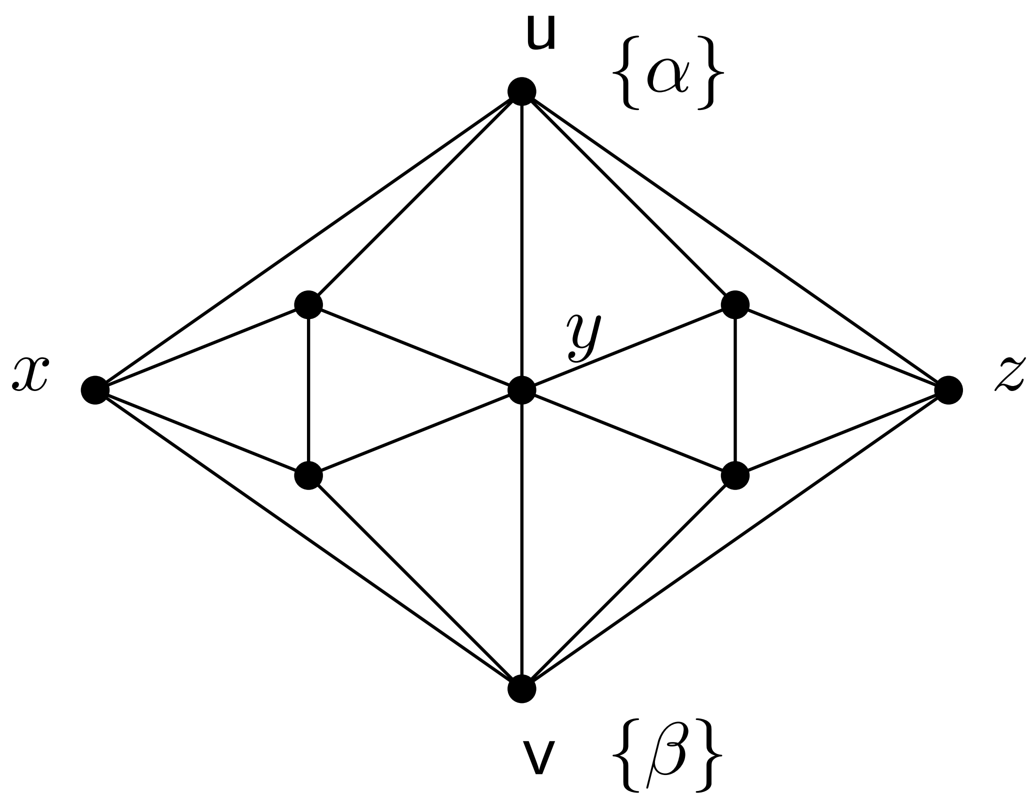
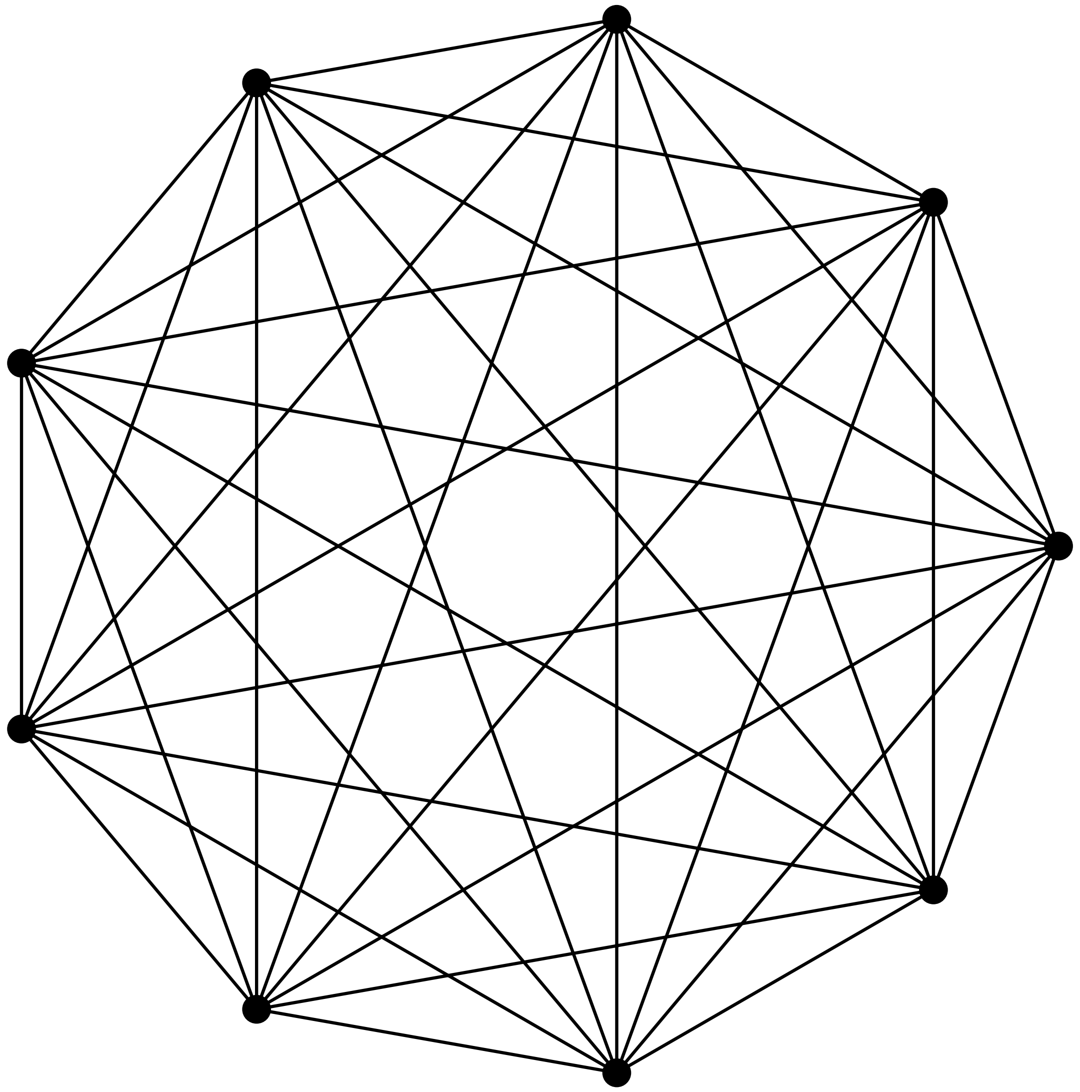


We have 26 copies of graph T and there are 9 combinatons of α and β, so from pigeonhole principle we can find such α_1 and β_1 that they don't satisfy assumtions of lemma 6 for at most 2 graphs T.

Theorem 4 For any integers $d \geqslant 0$ and $l \geqslant k \geqslant 3$, there exists a (k, d, l)-choosable graph which is not (k, d, l + 1)-choosable.

$K_{(d+1)k}$



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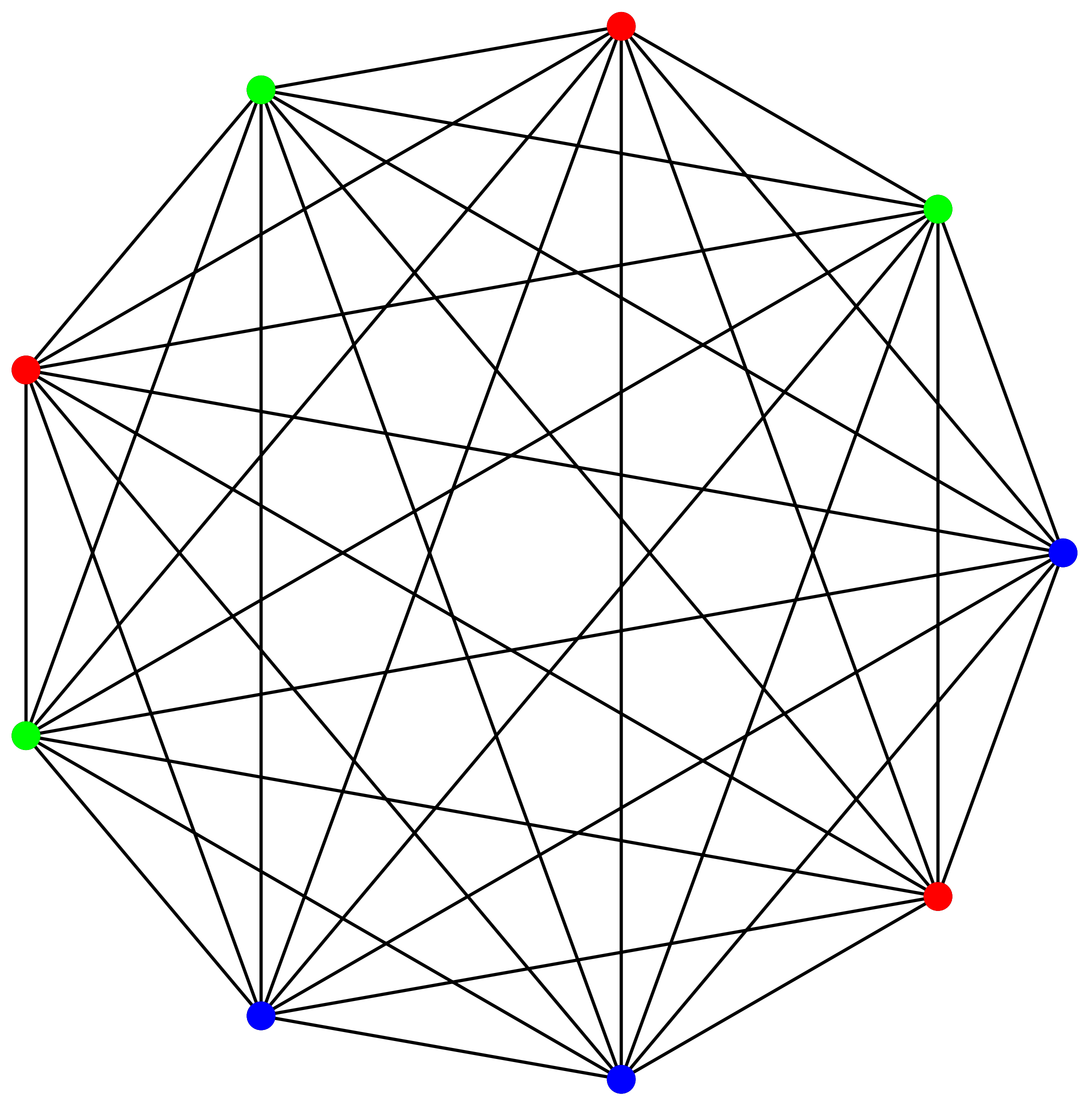

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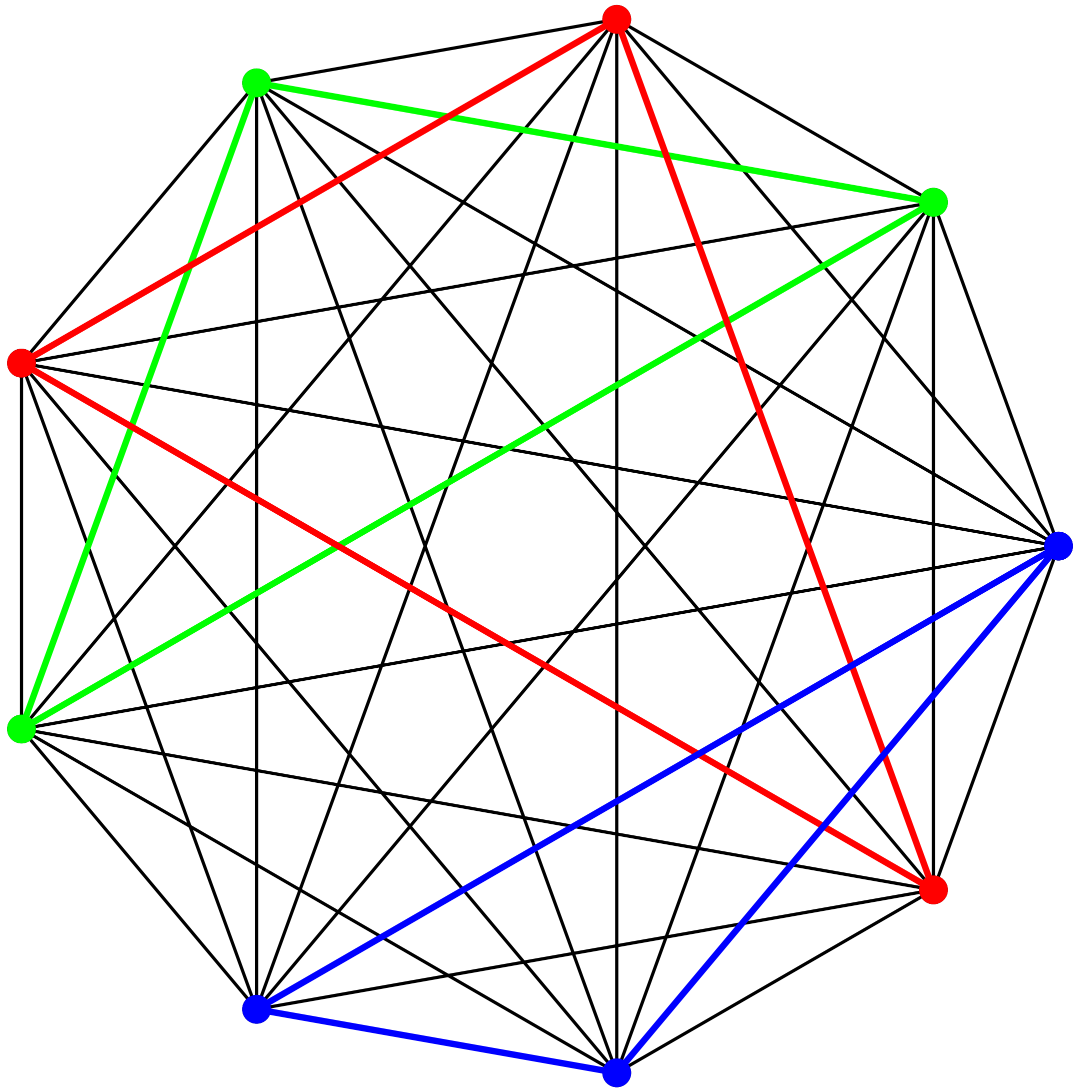

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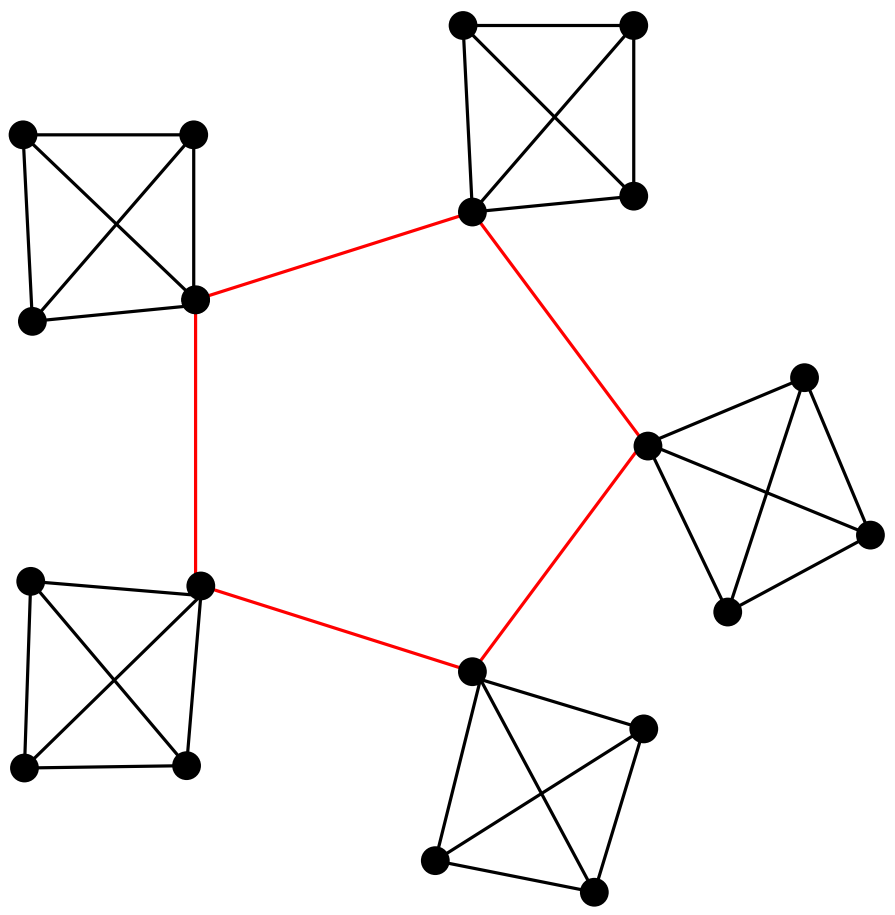
Definition G * d is a graph obtained from the disjoint union of G and $|V(G)|$ copies of the complete graph K_d , denoted as $\{B_v : v \in V(G)\}$, by identifying v with one vertex of B_v.



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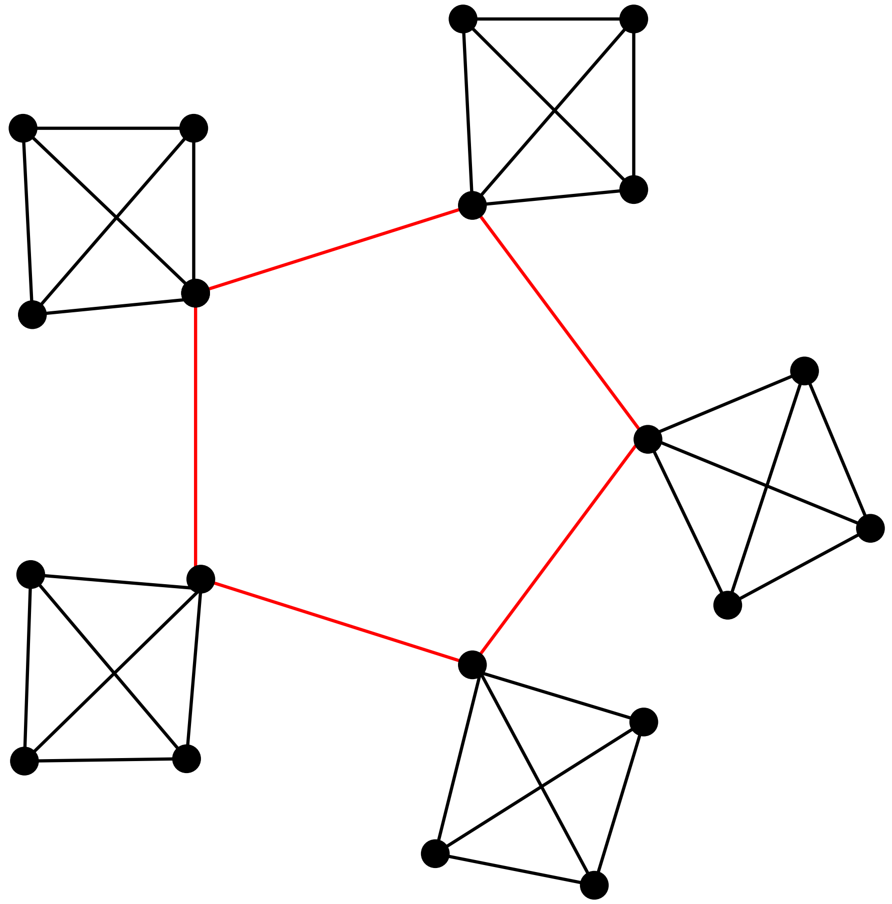

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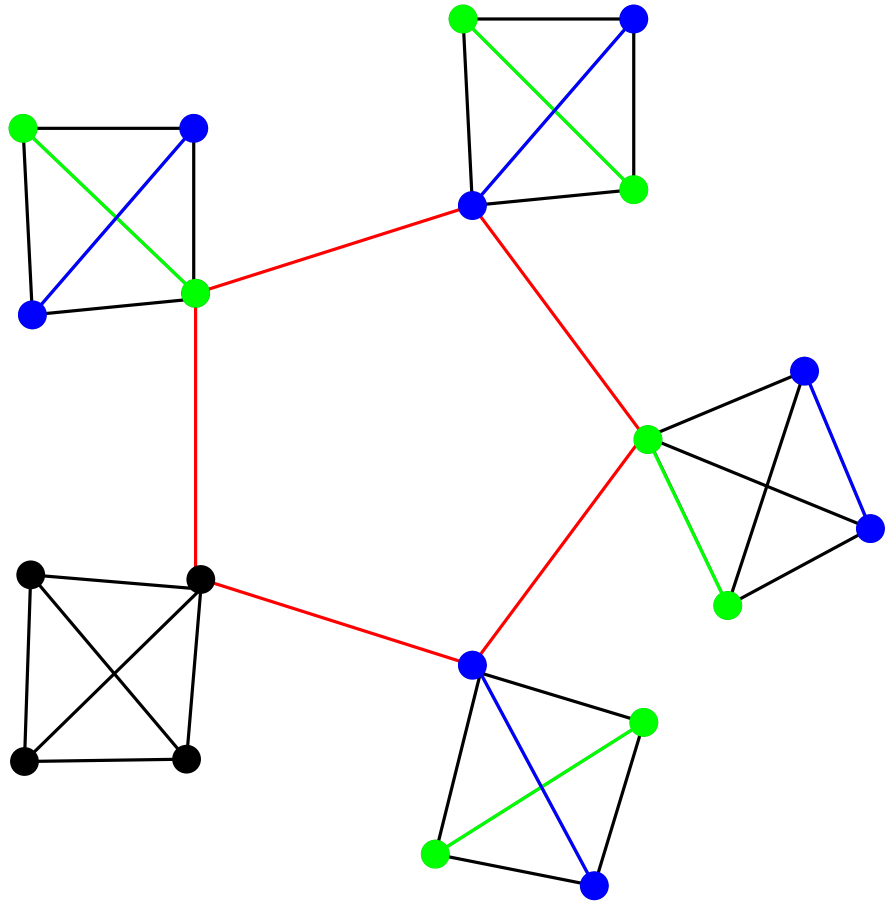

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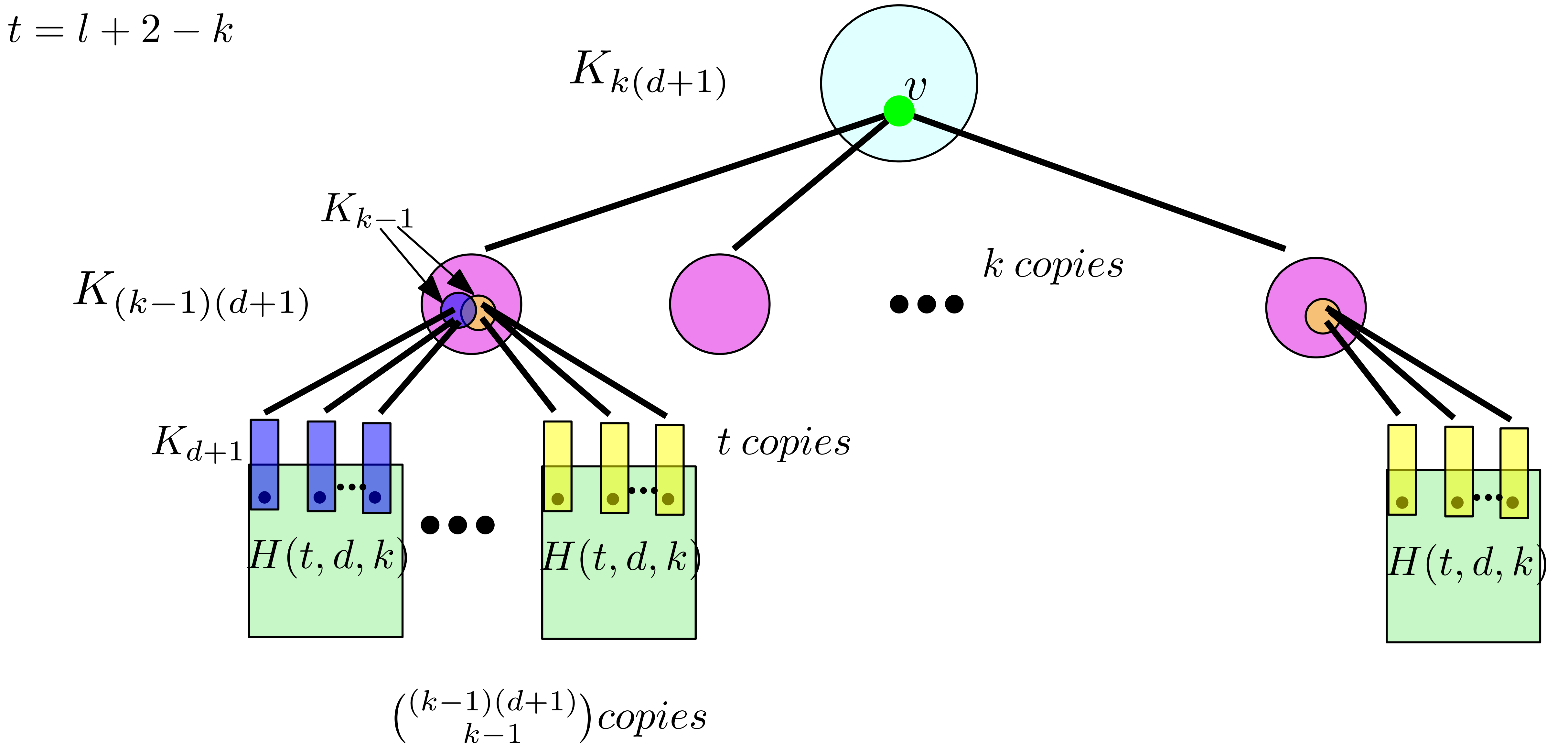

Construction of graph H(t, d, k)

Assume $k \geqslant 3$, $t \geqslant 2$, $d \geqslant 0$ and $l = k - 2 + t$. There exists a graph H(t, d, k) = (V, E) with a precolored independent set $T = \{u_1, u_2, ..., u_t\}$ for which the following hold:

Assume the precoloring ϕ of T uses t distinct colors in [l + 1]. Then there is a k-list assignment L of H(t, d, k) with $L(v) \subseteq [l+1]$ for each vertex v such that ϕ cannot be extended to a d-defective coloring ψ of H(t, d, k) with $\lambda_{H(t,d,k)}(u_i, \psi) = 0$ for each $u_i \in T$. On the other hand, if $d \geqslant 1$, then for any k-list assignment L of H(t, d, k)-T , ϕ can be extended to a d-defective L-coloring ψ of H(t, d, k) such that $\lambda_{H(t,d,k)}(u_i, \psi) = 0$ for $i = 1, 2, ..., t-1$ and $\lambda_{H(t,d,k)}(u_t, \psi) \leqslant 1$.

Assume the precoloring ϕ of T uses with at most t-1 colors. Then for any k-list assignment L of H(t, d, k)-T , ψ can be extended to a d-defective L-coloring ϕ of H(t, d, k) such that $\lambda_{H(t,d,k)}(u_i, \psi) = 0$ for each $u_i \in T$.

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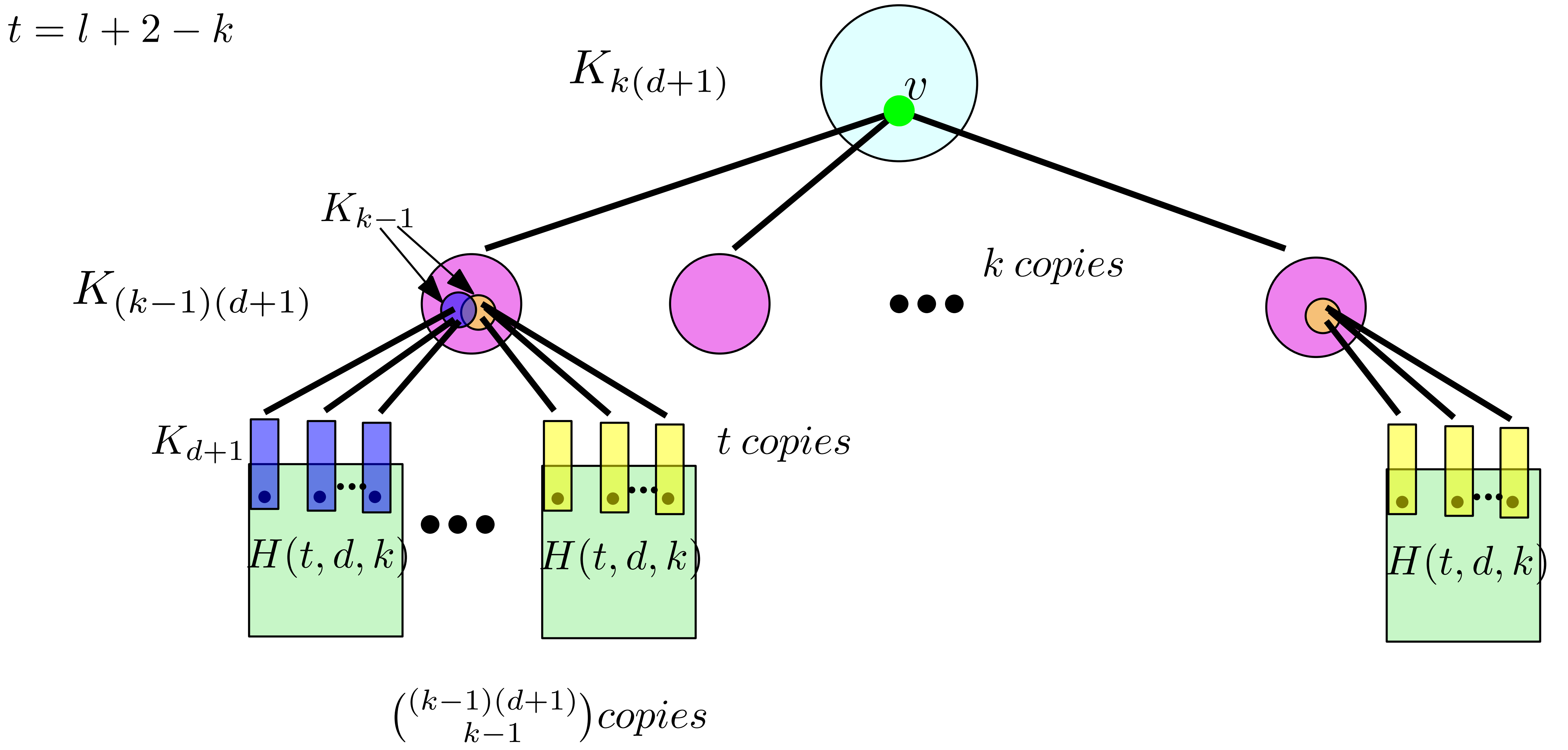

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Is it true that for each $k \geqslant 3$, there exists a number l such that each $(k, 0, l)$- choosable graph is $(k + 1)$-choosable?

References:

1. Jie Ma, Rongxing Xu, Xuding Zhu. On Two problems of Defective Choosability of Graphs.
2. L. J. Cowen, R. H. Cowen, and D. R. Woodall. Defective colorings of graphs in surfaces: partitions into subgraphs of bounded valency.
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