Two-sided Online Bipartite Matching and Vertex Cover: Beating the Greedy Algorithm

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presented by Maciej Sanocki

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Ski rental problem

We know that renting skis costs a and buying them costs b. If we haven't bought skis yet, each time we are going skiing we can either decide to rent or to buy skis. Our goal is to minimize cost.

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competitiveness

optimal algorithm for this problem is $\frac{1}{(1-1/e)}$-competitive

Online Bipartite b-Matching problem (**OBM**)

Initially, we are given L and their capacity $b_u, u \in L$. At each step a vertex $v \in R$ is revealed with b_v and its incident edges. An algorithm for *OBM* maintains a *b-matching* x, and must irrevocably decide at each step on the values of x_e for each new edge e. The objective is to *maximize* the size of x.

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$$s.t. \forall_{v \in (L \cup R) \cap T} \sum_{u \in N(v) \cap T} x_{uv} \leq b_v$$

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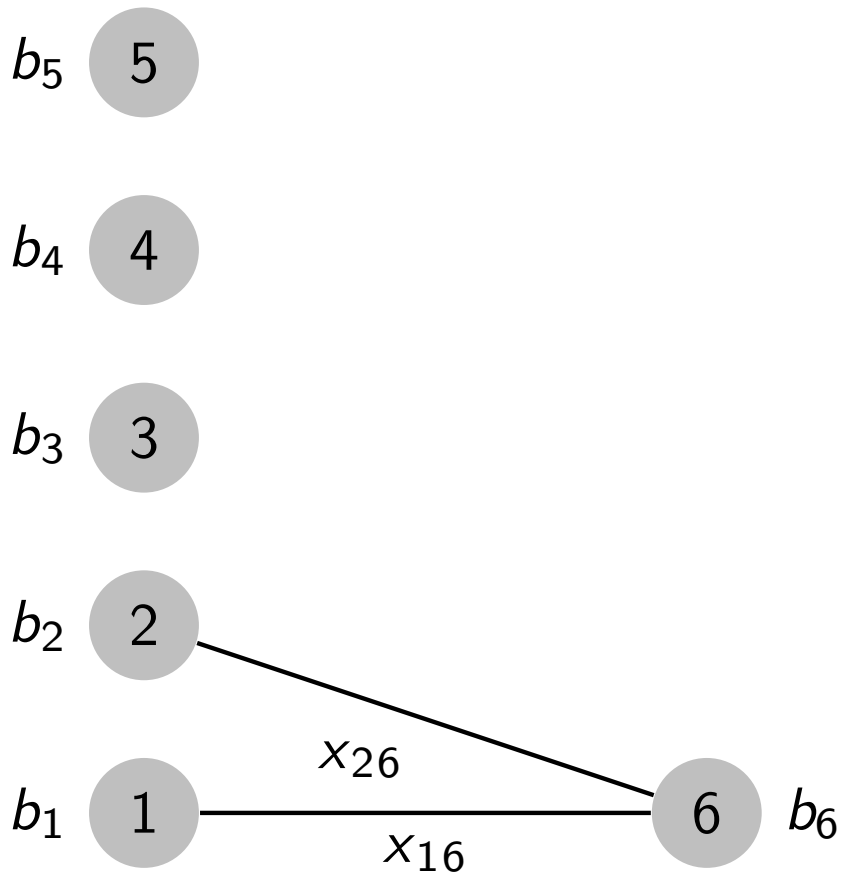

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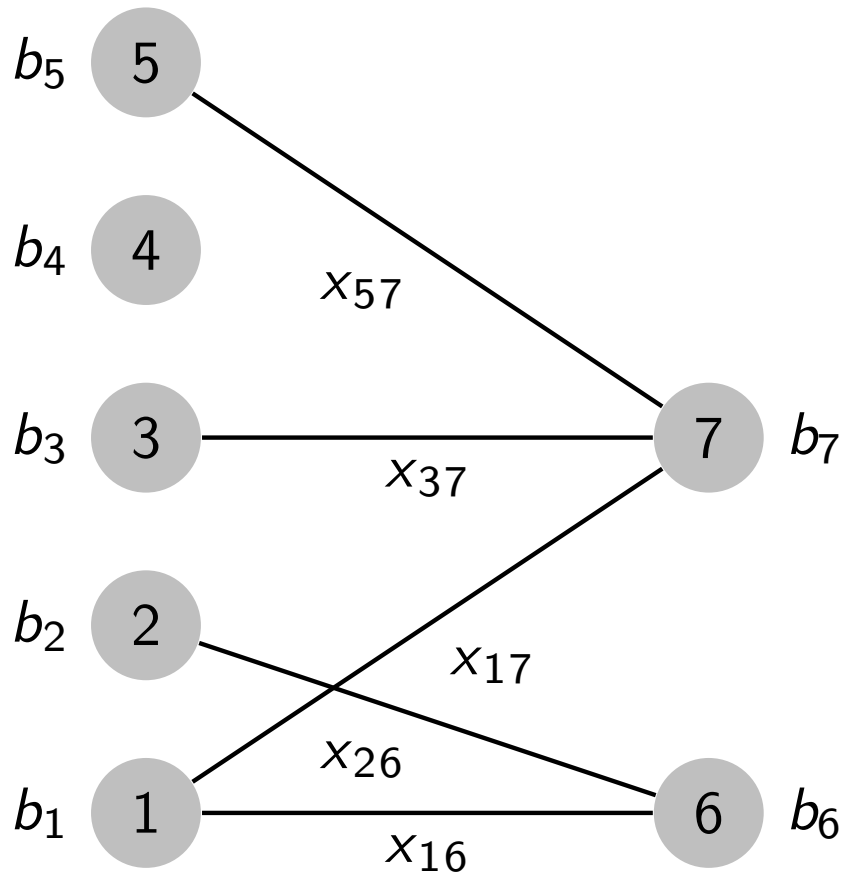

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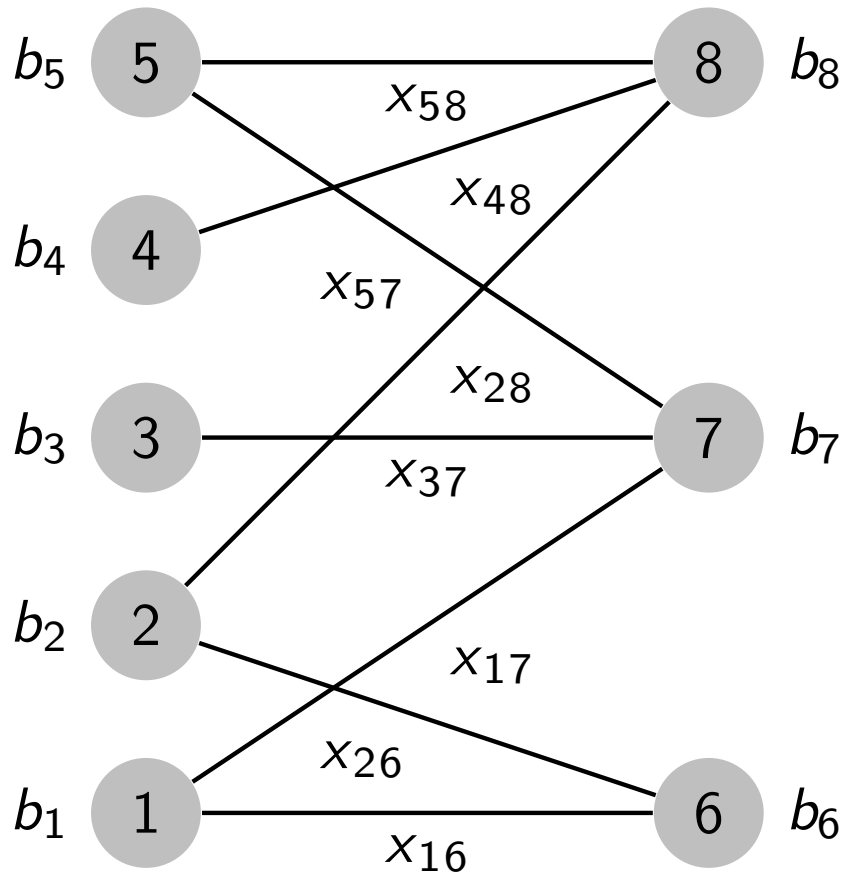

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Optimal algorithm for this problem is $(1 - 1/e)$-competitive

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Similarly, for *OBVC* we are required to maintain a vertex cover C at all time by only inserting vertices into C, i.e. no vertex removal is allowed. Thus at each step the task is essentially deciding if v or $N(v)$ should be inserted to C. The objective is to minimize the *weight* of C, $b(C) := \sum_{v \in C} b_v$

$$\min \sum_{v \in V} b_v \cdot y_v$$

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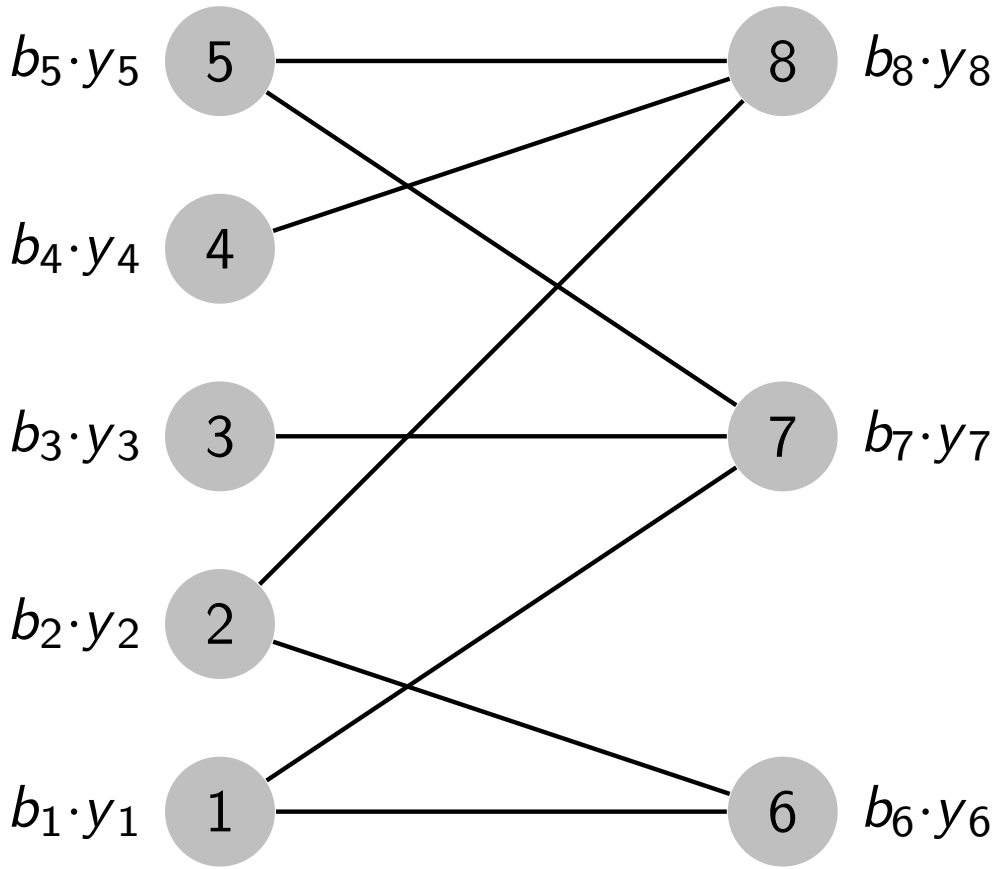

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Can we do better?

We know, that it is at most $(1 - 1/e)$, since it is a generalization of ski rental problem.

Two-sided Online Bipartite b-Matching and Vertex Cover (**TOBM & TOBVC**)

Two-sided version relaxes the constraint that only the right vertices are online.

1

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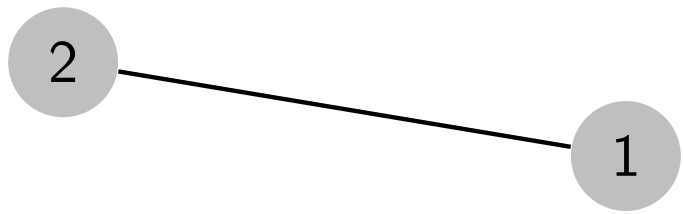

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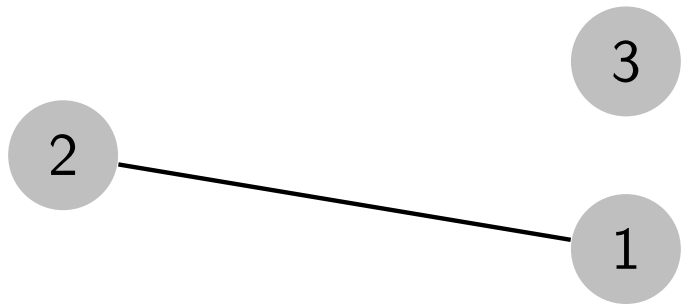

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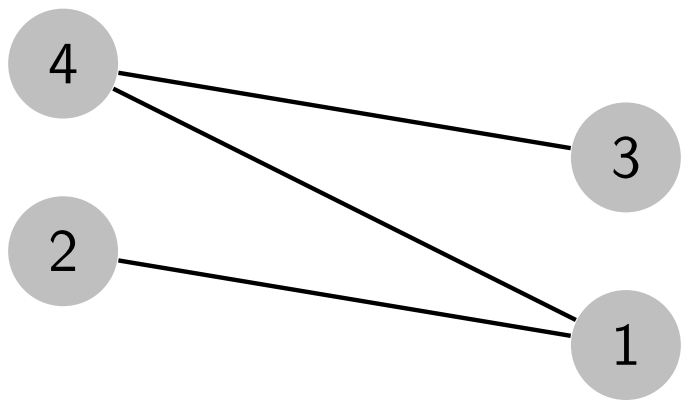

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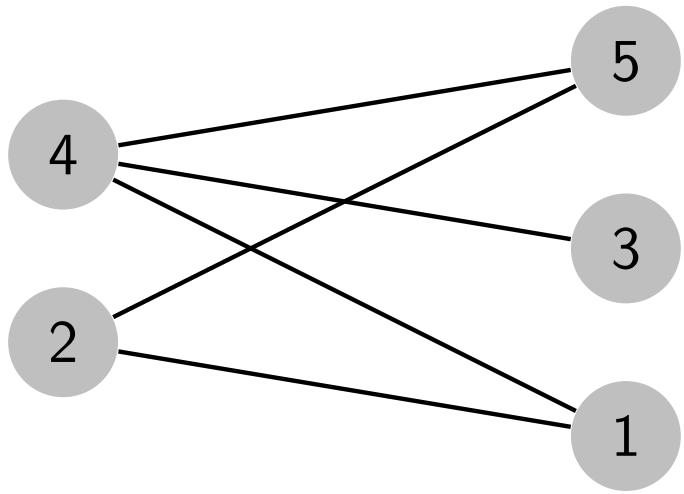

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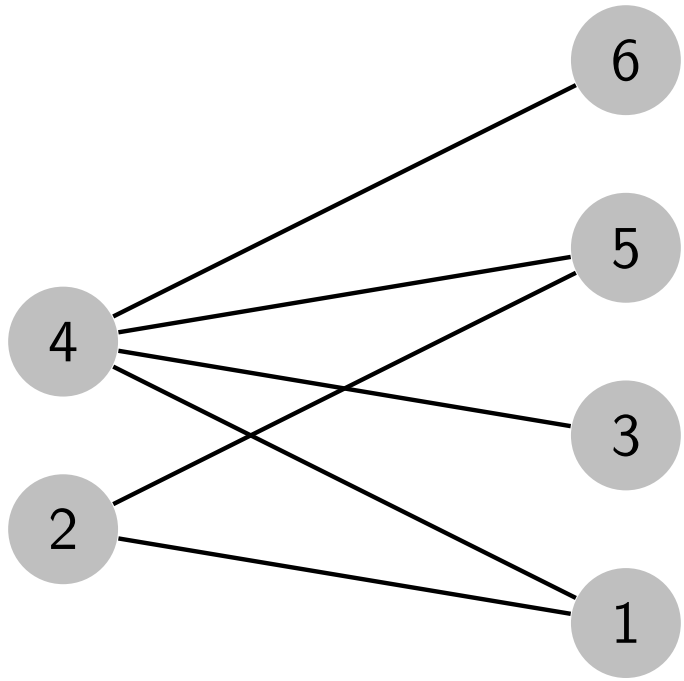

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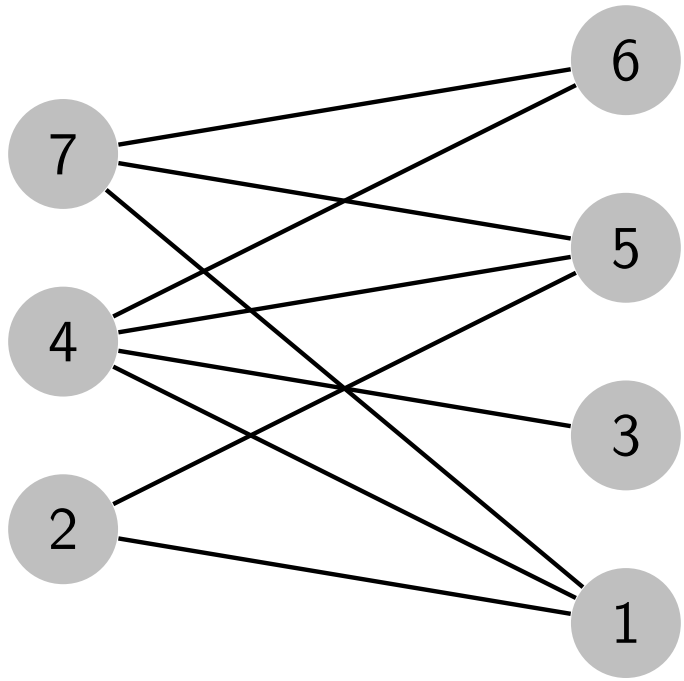

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Fractional vertex cover and b-matching

When analysing *fractional* versions, we enable y's and x's to be fractions rather then natural numbers.

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Connection between Bipartite Vertex Cover and Bipartite b-Matching

$$\max \sum_{e \in E} x_e$$

$$s.t. \sum_{u \in N(v)} x_{uv} \leq b_v, \forall v \in V$$

$$x \geq 0$$

$$\min \sum_{v \in V} b_v \cdot y_v$$

$$s.t. y_u + y_v \geq 1, \forall (u, v) \in E$$

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$$\max c^T x$$

$$\text{s.t.} Ax \leq b, x \geq 0.$$

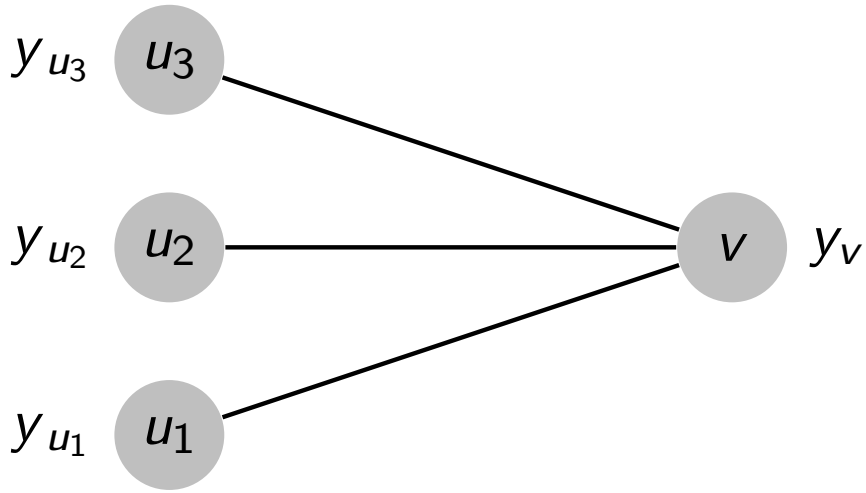
$$\min b^T y$$

$$\text{s.t.} A^T y \geq c, y \geq 0.$$

Lemma 1. *(Lossless rounding) Any deterministic algorithm for fractional* **OBVC** *can be converted to a randomized algorithm for integral* **OBVC** *with the same competitive ratio.*

(One-sided) Online Bipartite Vertex Cover

For each vertex v, algorithm should maintain a non-decreasing cover potential y_v.



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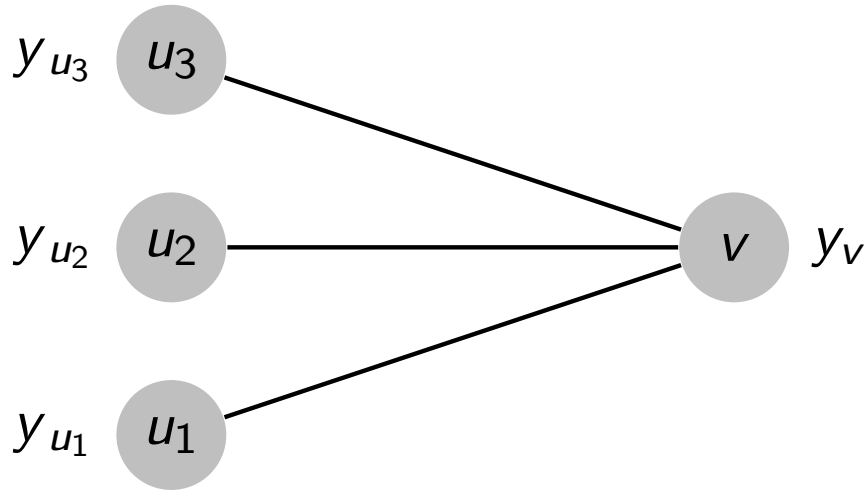

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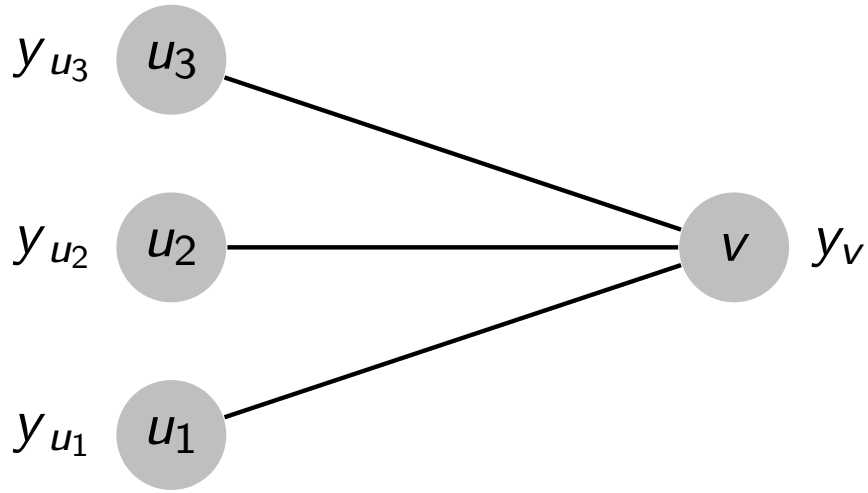

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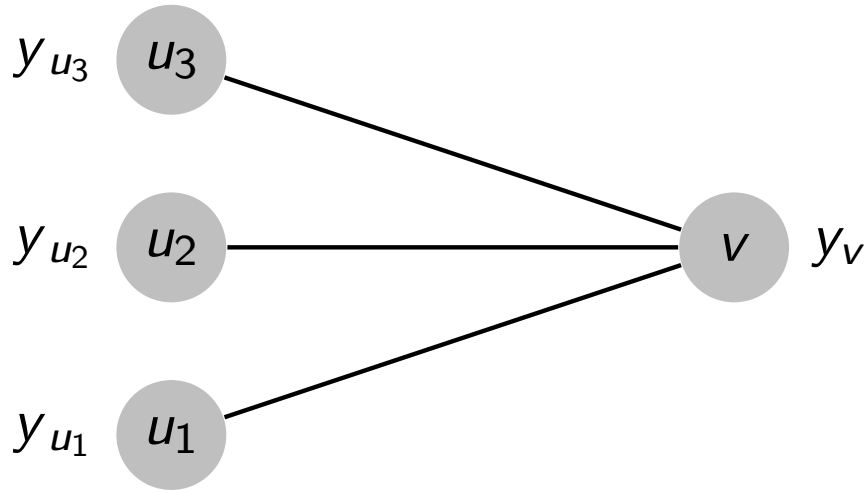

To maintain a feasible vertex cover: $\forall_{u \in N(V)} y_v + y_u \geq 1$

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To maintain a feasible vertex cover, any $y_u < y$ for $u \in N(v)$ should be increased to y.

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Total potential increment is equal to:

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Choosing total potential increment to be at most $\frac{b_v}{1-1/e}$ seems to be a reasonable option.

The water-filling algorithm for **OBVC**

Algorithm 1 Water-filling algorithm for OBVC

Input. L *and* b_u, $u \in L$

Initialize for each $u \in L, y_u = 0$;

for *each online vertex* $v \in R$ **do**

  Maximize $y \leq 1$,

    $s.t.\ b_v(1-y) + \sum_{u \in N(v)} b_u \max\{y - y_u, 0\} \leq b_v/(1-1/e)$;

  For each $u \in N(v)$, $y_u \leftarrow \max\{y_u, y\}$;

  $y_u \leftarrow 1 - y$;

end

Short analysis

Let C^* be a minimum vertex cover of G.

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Our strategy is to charge the potential increment to vertices of C^* in such a way that each vertex of $v \in C^*$ is charged at most $\frac{b_v}{1-1/e}$.

Let v be the current online vertex. Our algorithm sets $y_v = 1 - y$ for some y. Let y_u be the potential of $u \in N(v)$. We consider two cases.

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Case 1. $v \in C^*$: trivial

Case 2. $v \notin C^*$: We must have $N(v) \subseteq C^*$. Also we must find a way to charge $b_v(1 - y)$ to $N(v)$

Let $g(y) = \frac{1}{e-1} + y$. Rewrite
$b_v(1-y) + \sum_{u \in N(v)} b_u \max\{y - y_u, 0\} \leq \frac{b_v}{1-1/e}$ as:

$$\sum_{u \in N(v)} b_u \max\{y - y_u, 0\} \leq b_v(\frac{1}{e-1} + y) = b_v g(v)$$

If equality holds, we want to additionally charge
$\frac{1-y}{g(y)} b_u \max\{y - y_u, 0\}$ each $u \in N(v)$ for potential of v.

Since $\frac{1-t}{g(y)}$ is decreasing, $\frac{1-y}{g(y)} b_u(y - y_u)$ can be upper bounded by $b_u \int_{y_u}^{y} \frac{1-t}{g(t)}\, dt$.

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Lemma 2. Let $f : [0,1] \longrightarrow R^+$ be *continuous s.t.* $\frac{1-t}{f(t)}$ is decreasing, and $F(x) = \int_0^x \frac{1-t}{f(t)}\, dt$. If $\sum_{u \in X} b_u(y - y_u) = b_v f(y)$ for some set X and $y > y_u$ for $u \in X$, then

$$b_v(1 - y) \leq \sum_{u \in X} b_u(F(y) - F(y_u))$$

Theorem 1. *Algorithm 1* is $1 - 1/e$-competitive and hence optimal for **OBVC**.

The water-filling algorithm for **TOBVC**

We replace $g(t) = 1/(e-1) + t$ with any continuous function $f : [0,1] \longrightarrow R_+$ for which $\frac{1-t}{f(t)}$ is decreasing.

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Algorithm 3 Water-filling algorithm for TOBVC

Let T be the set of arrived vertices. Initially $T = \emptyset$

for *each online vertex* v **do**

  Maximize $y \leq 1$, *s.t.* $\sum_{u \in N(v) \cap T} b_u \max\{y - y_u, 0\} \leq b_v f(y)$;

  For each $u \in N(v) \cap T$, $y_u \leftarrow \max\{y_u, y\}$;

  $y_v \leftarrow 1 - y$;

  $T \leftarrow T \cup \{v\}$

end

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Setting $z = 1 - y$, the total charges to each $v \in C^*$ are at most:

$$b_v \beta(f), \text{where } \beta(f) := \max_{z \in [0,1]} 1 + f(1-z) + \int_z^1 \frac{1-t}{f(t)}\, dt$$

Theorem 2. Suppose $f : [0,1] \longrightarrow R_+$ is continuous and $\frac{1-t}{f(t)}$ is decreasing. Let $\beta(f) := \max_{z\in[0,1]} 1 + f(1-z) + \int_z^1 \frac{1-t}{f(t)}\, dt$. Then *Algorithm 2* is $1/\beta(f)$-competitive for **TOBVC**.

Computing the Optimal f

In essence, finding the f(y) to get a small β is to solve the following optimization problem:

$$\inf_{f \in F} \max_{z \in [0,1]} 1 + f(1-z) + \int_z^1 \frac{1-t}{f(t)}\, dt$$

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Theorem 3. Let $f(z) = (\frac{1+k}{2} - z)^{\frac{1+k}{2k}} (z + \frac{k-1}{2})^{\frac{k-1}{2k}}$, where $k \approx 1.997$. *Algorithm 2.* is then 0.526-competitive for **TOBVC**.

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Modifying the Algorithm 2. to compute **TOBM**

let $f(z)$ be the same as in Theorem 3. and $\beta \approx 1.901$

Algorithm 4 Water-filling algorithm for TOBM

Let T be the set of arrived vertices. Initially $T = \emptyset$

for *each online vertex* v **do**

  Maximize $y \leq 1$, *s.t.* $\sum_{u \in N(v) \cap T} b_u \max\{y - y_u, 0\} \leq b_v f(y)$;

  let $X = \{u \in N(v) \cap T | y_u < y\}$

  for *each* $u \in X$ **do**

    $y_u \leftarrow y$;

    $x_{uv} \leftarrow \frac{b_u(y-y_u)}{\beta}(1 + \frac{1-y}{f(y)})$

  end

  For each $u \in (N(v) \cap T) \backslash X$, $x_{uv} \leftarrow 0$;

  $y_v \leftarrow 1 - y$, $T \leftarrow T \cup \{v\}$

end

Invariants

Invariant 1. $b_u \cdot \frac{y_u + f(1-z_u) + \int_{z_u}^{y_u} \frac{1-t}{f(t)}\, dt}{\beta} \geq x_u$

where z_u is the potential of u set upon it's arrival, y_u is current potential of u and $x_u = \sum_{v \in N(u)} x_{uv}$

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Invariant 2. $\sum_{u \in T} b_u y_u = \beta \sum_{(u,v) \in E \cap T^2} x_{uv}$

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Invariant 2. $\sum_{u \in T} b_u y_u = \beta \sum_{(u,v) \in E \cap T^2} x_{uv}$

Theorem 4. Our algorithm is $1/\beta \approx 0.526$-competitive for **TOBM** and **TOBVC**.

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