Any 7-chromatic graph has K_7 or $K_{4,4}$ as a minor

based on an aricle by Ken-Ichi Kawarabayashi and Bjarne Toft

Izabela Tylek

November 23, 2023

The Hadwiger conjecture

Conjecture (Hugo Hadwiger, 1943)

Every k-chromatic graph has a K_k-minor

k-chromatic graph

A graph is called k-chromatic if its chromatic number is equal to k

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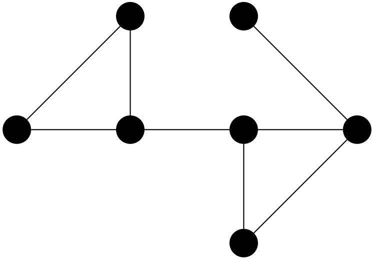

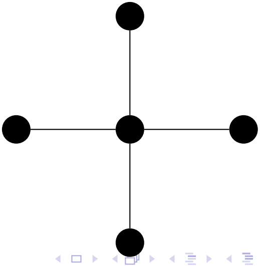

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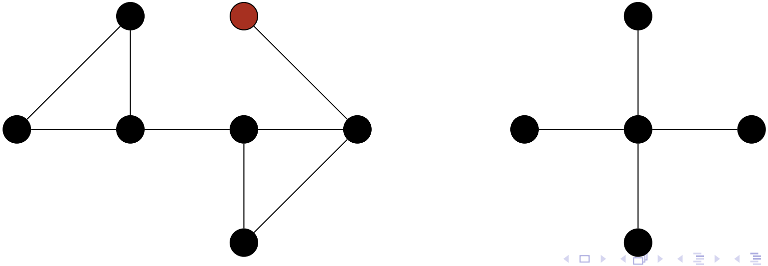

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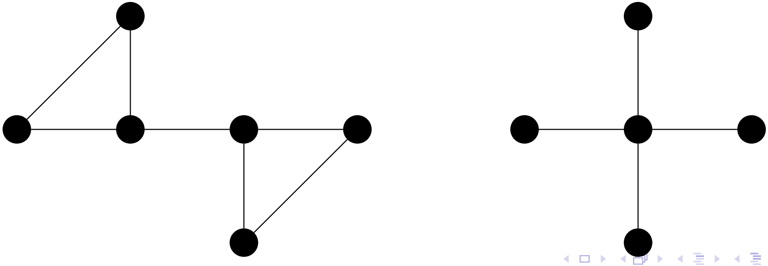

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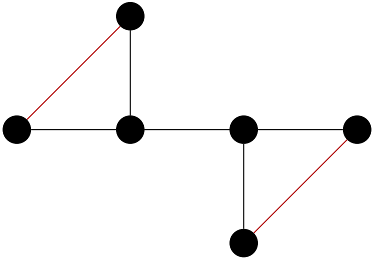

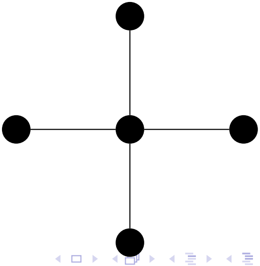

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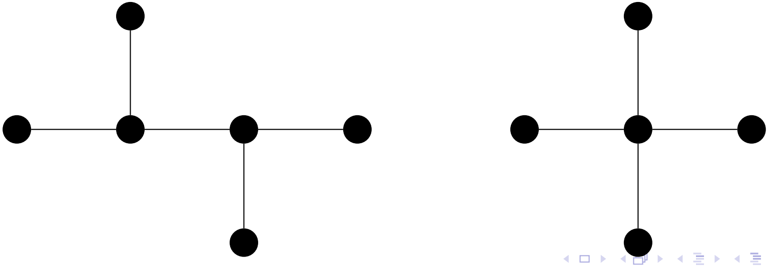

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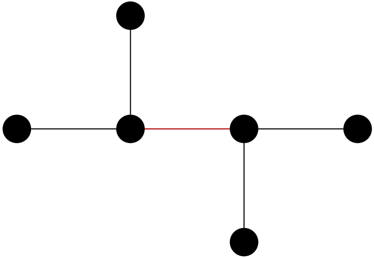

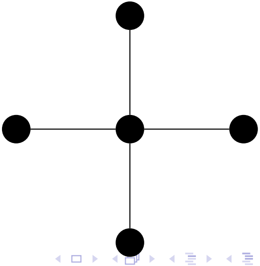

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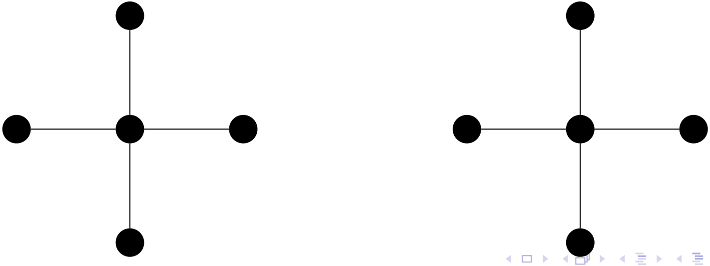

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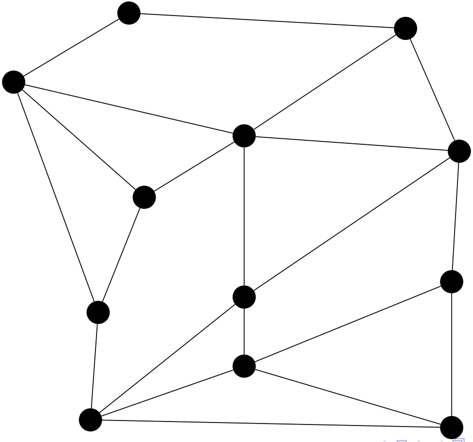

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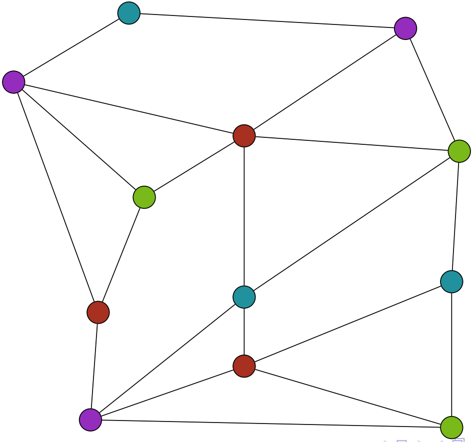

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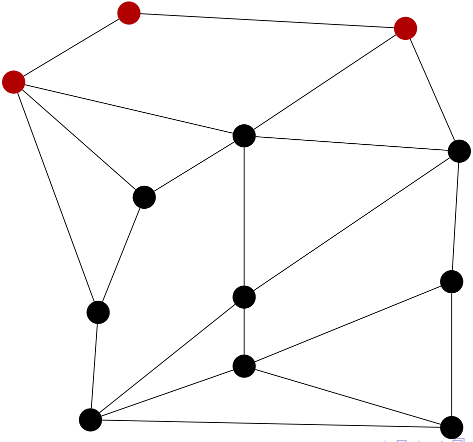

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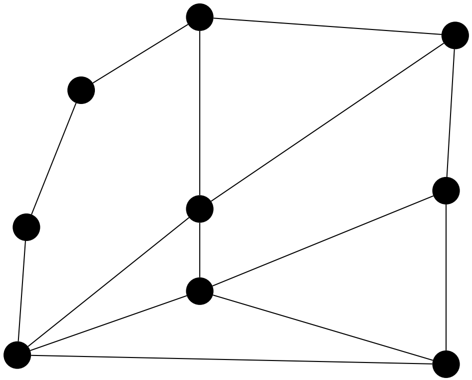

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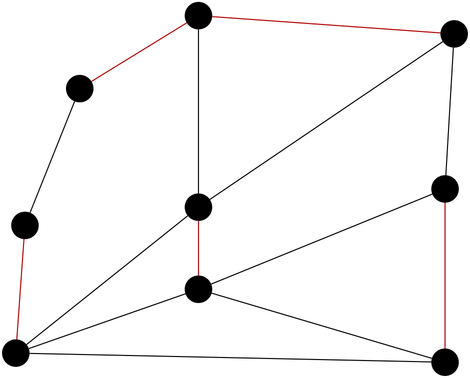

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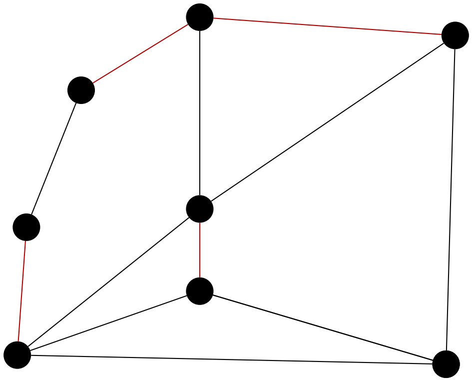

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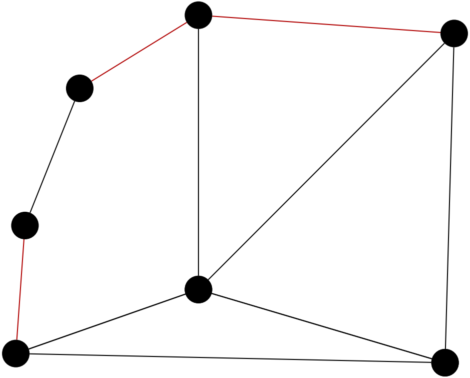

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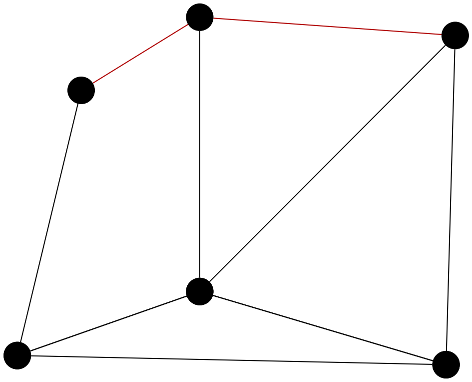

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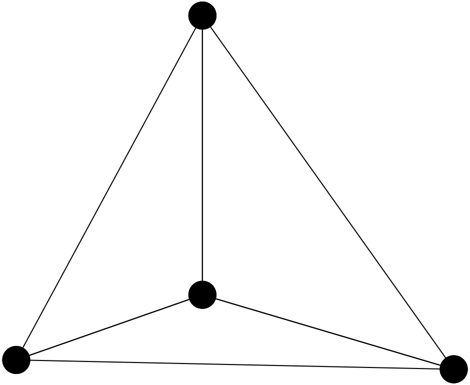

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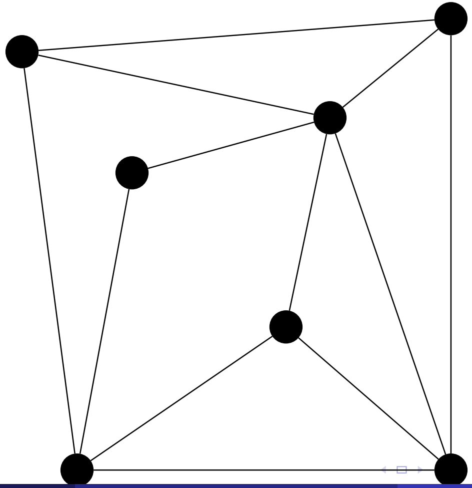

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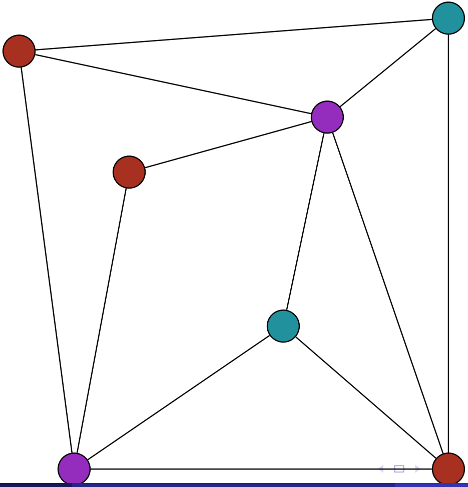

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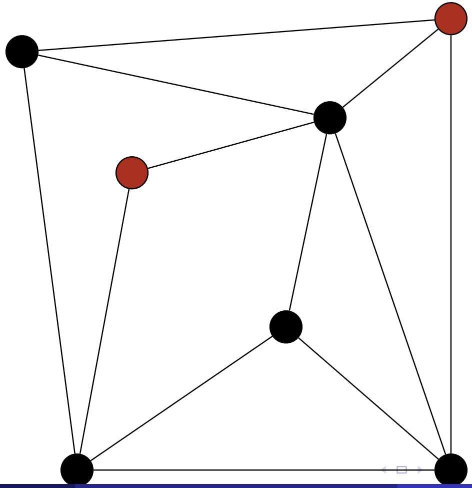

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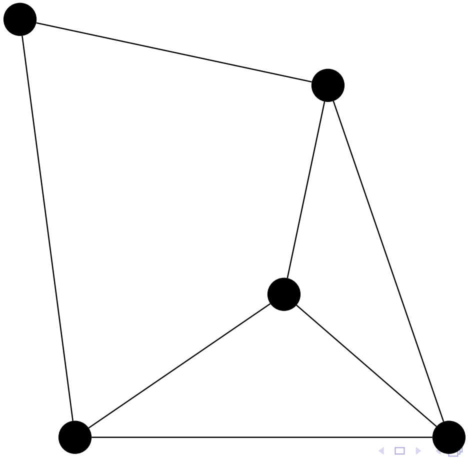

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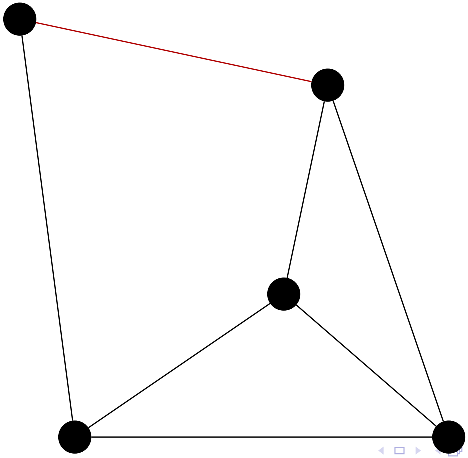

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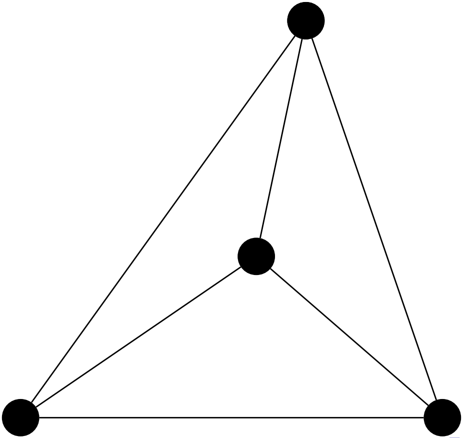

- $k = 2$
  A graph requires more than one colour if and only if it has an edge

- $k = 3$
  A graph requires more than two colours if and only if it is not bipartite. Every non-bipartite graph contains an odd cycle, which can be contracted to a 3-cycle

- $k = 4$

Theorem (Hugo Hadwiger, 1943)

Every 4-chromatic graph has a K_4 minor

Hadwiger conjecture

- $k = 5$

Theorem (Klaus Wagner, 1937)

A graph is planar if and only if its minors include neither K_5 nor $K_{3,3}$

So the Hadwiger conjecture for $k = 5$ implies the Four Colour Theorem (if all 5-chromatic have to contain K_5, they cannot be planar)

Theorem (Klaus Wagner, 1937)

Every graph that has no K_5 minor can be decomposed via clique-sums into pieces that are either planar or an 8-vertex Möbius ladder and each of the pieces can be 4-coloured independently of each other

So the Four Colour Theorem implies the Hadwiger conjecture for $k = 5$ (K_5-minor-free graphs are 4-colourable)

- $k = 6$

Theorem (Robertson, Seymour & Thomas, 1993; 1994 Fulkerson Prize)

A minimal counterexample to the Hadwiger conjecture for the case k = 6 is a graph G which has a vertex v such that $G - v$ is planar (and therefore, assuming the Four Colour Theorem holds, there are no counterexamples)

Proof using linklessly embeddable graphs (three-dimensional analogue of planar graphs)

Theorem (Bollobás, Catlin & Erdős, 1980)

The Hadwiger conjecture in general is true for almost all graphs

Theorem (Zi-Xia Song, 2010)

The Hadwiger conjecture is true for all graphs with "claw-free" or $\overline{K_{1,3}}$-free degree sequences

A graph is a claw if it is isomorphic to $K_{1,3}$
A degree sequence is H-free if each realisation of the sequence is H-free

Theorem (Jakobsen, 1971)

Every 7-chromatic graph has a K_7 with two edges missing as a minor

Theorem(Kawarabayashi & Toft, 2005))

Every 7-chromatic graph has to contain a K_7-minor or a $K_{4,4}$-minor

Theorem(Kawarabayashi)

Every 7-chromatic graph has to contain a K_7-minor or both a $K_{4,4}$-minor and a $K_{3,5}$-minor

Outline of the paper

Let G be a graph satisfying the following conditions:

- G is 7-chromatic
- G is minimal with respect to the minor relation in the class of all 7-chromatic graphs
- G does not contain K_7 as a minor
- G does not contain $K_{4,4}$ as a minor

These conditions together lead to a contradiction

Outline of the paper

1. Contraction-criticality and general properties of the graph
2. Non-planarity of G minus two vertices
3. Forbidden relations between complete 5-graphs in G
4. Finding three "nearly disjoint" complete 5-graphs
5. Finding K_7 or $K_{4,4}$ using the "nearly disjoint" complete 5-graphs

Contraction-critical graphs

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A graph H is k-contraction-critical if it is k-chromatic and every proper minor of H has a proper $(k-1)$-colouring

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A graph H is k-contraction-critical if it is k-chromatic and every proper minor of H has a proper $(k-1)$-colouring

- G is 7-chromatic
- G is minimal with respect to the minor relation in the class of all 7-chromatic graphs
- G does not contain K_7 as a minor

G is a non-complete 7-contraction-critical graph

Properties of contraction-critical graphs

The following results apply to non-complete 7-contraction-critical graphs

Lemma 1 (Dirac)

$\delta(G) \geq 7$ and no three neighbors of a degree 7 vertex are independent

Lemma 2 (Dirac)

G does not contain a K_6

Lemma 3 (Mader)

G is 7-connected

Lemma 4 (Stiebitz, Toft)

G has at least three vertices of degree at least 8

Properties of G

Theorem (Jørgensen)

Every 4-connected graph G with $|E(G)| \geq 4|V(G)| - 7$ has a $K_{4,4}$-minor or is a K_7

Lemma 5 (from lemma 3)
$|E(G)| \leq 4|V(G)| - 8$

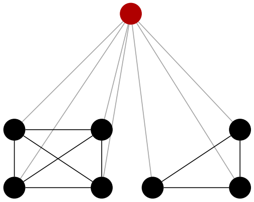
Lemma 6 (from lemmas 1 and 5)
G has at least 16 vertices of degree 7

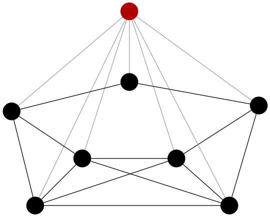
Lemma 7 (from lemmas 4 and 6)
$|V(G)| \geq 19$

Lemma 8 (from lemma 3 and the fact that we have no $K_{4,4}$-minor)
G does not contain a $K_{3,4}$

Properties of G

Lemma 9 For any vertex x of degree 7, $[N_G(x)]$ is a graph containing either disjoint complete graphs K_3 and K_4 or a 7-vertex inflation of a 5-cycle where two neighboring vertices are replaced by complete 2-graphs





Lemma 10 (from lemma 9)
Any vertex of degree 7 in G is contained in a K_5 in G

Lemma 11 (from lemmas 6 and 9)
G contains at least four different complete graphs on five vertices

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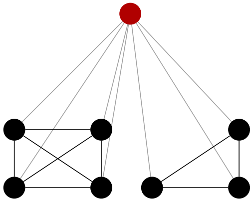

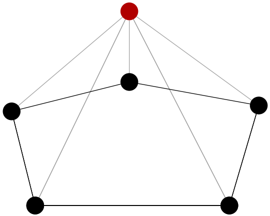

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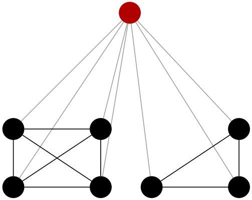

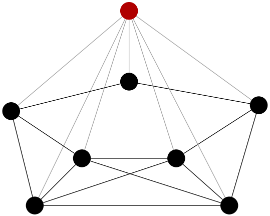

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Non-planarity of G minus two vertices

Let us take some two distinct vertices $x, y \in V(G)$ and assume that $G' = G - x - y$ is planar

- G' has to have at least 12 vertices of degree 5, and these vertices have degree 7 in G

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- G' has to have at least 12 vertices of degree 5, and these vertices have degree 7 in G

Lemma 1 $\delta(G) \geq 7$ and no three neighbors of a degree 7 vertex are independent

Lemma 3 G is 7-connected

Since G' is 5-connected and $\delta(G) \geq 5$, there at least 12 vertices of degree 5 in G' (by Euler's formula) that have degree 7 in G

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- There is no K_4 in G'

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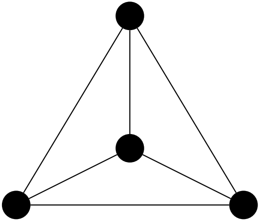

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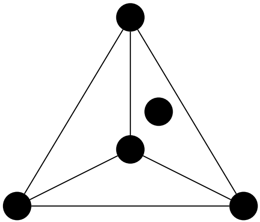

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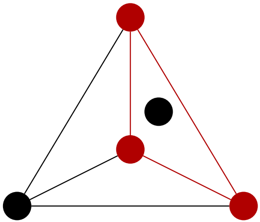
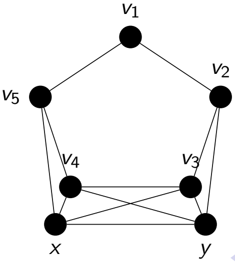


Non-planarity of G minus two vertices

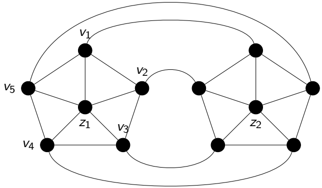
Let us take some two distinct vertices $x, y \in V(G)$ and assume that $G' = G - x - y$ is planar

- G' has to have at least 12 vertices of degree 5, and these vertices have degree 7 in G
- There is no K_4 in G'
- Any K_5 contains both x and y, every vertex of degree 7 is connected to x and y and has no triangle in its neighborhood



Non-planarity of G minus two vertices

- We can find two non-neighboring vertices z_1 and z_2 of degree 7 in G that form a following structure in G':



Where the arcs signify some paths (possibly of length 0)

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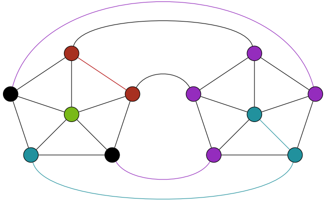

Where the arcs signify some paths (possibly of length 0)

- If the highlighted subgraphs are disjoint, the structure contains a $K_{4,4}$ minor

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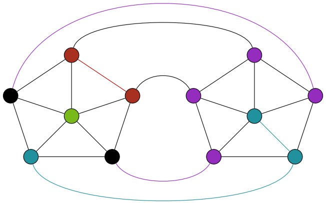


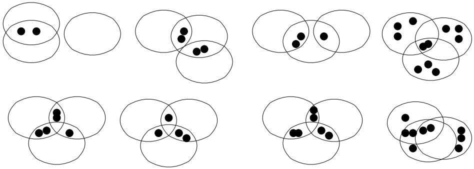
Where the arcs signify some paths (possibly of length 0)

- We can always find two vertices z_1 and z_2 such that the graphs are in fact disjoint

Forbidden relations between complete 5-graphs in G

Lemma 11 G contains at least four different complete graphs on five vertices

Let L_1, L_2, L_3 be three K_5, not necessarily disjoint, but not same
It is possible to prove that the following configurations are not possible:



(We rely on the fact that the graph is non-planar and some previous results from Robertson, Seymour and Thomas)

Finding three "nearly disjoint" complete 5-graphs

We want to prove that L_1, L_2, L_3 can be selected such that $|L_1 \cup L_2 \cup L_3| \geq 12$

Let L_1, L_2 be two K_5 that maximise $|L_1 \cup L_2|$

- Claim 1 $|L_1 \cup L_2| \geq 9$
- Claim 2 $|L_1 \cup L_2| = 10$ (and so $L_1 \cap L_2 = \varnothing$)
- Lemma L_1, L_2, L_3 can be selected such that $|L_1 \cup L_2 \cup L_3| \geq 12$

Finding K_7 or $K_{4,4}$ using the "nearly disjoint" complete 5-graphs

Good paths

Let $Z_1, ..., Z_h$ be subsets of $V(G)$. A path P of G with ends u, v is said to be good if there exist distinct i, j with $1 \leq i, j \leq h$ such that $u \in Z_i$ and $v \in Z_j$

Finding K_7 or $K_{4,4}$ using the "nearly disjoint" complete 5-graphs

Theorem (Robertson, Seymour and Thomas; based on Mader's "H-Wedge" theorem)

Let G be a graph, let $Z_1, ..., Z_h$ be subsets of $V(G)$, and let $K \leq 0$ be an integer. Then exactly one of the following two statements holds:

1. There are k mutually disjoint good paths of G
2. There exists a vertex set $W \subseteq V(G)$ and a partition $Y_1, ..., Y_n$ of $V(G) - W$, and for $1 \leq i \leq n$ a subset $X_i \subseteq Y_i$ such that
   1. $|W| + \sum_{1 \leq i \leq n} \lfloor \frac{1}{2} |X_i| \rfloor < k$
   2. for any i with $1 \leq i \leq n$, no vertex in $Y_i - X_i$ has a neighbor in $V(G) - (W \cup Y_i)$ and $Y_i \cap (\cup_{j=1}^{h} Z_j) \subseteq X_i$, and
   3. every good path P in $G - W$ has an edge with both ends in Y_i for some i

Finding K_7 or $K_{4,4}$ using the "nearly disjoint" complete 5-graphs

Let us take $Z_1, ..., Z_3 = L_1, ..., L_3$

Claim 1 There do not exist seven mutually disjoint good paths in G

For any possible i, j: $N_L(P_i) \cap V(P_j) \neq \varnothing$
Therefore $(V(P_1), V(P_2), V(P_3), V(P_5), V(P_6), V(P_7))$ is a K_7-minor, contradiction

Finding K_7 or $K_{4,4}$ using the "nearly disjoint" complete 5-graphs

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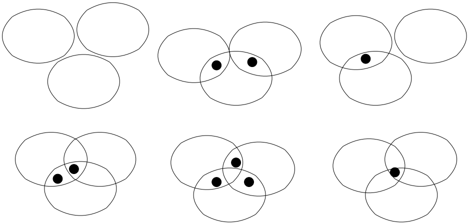

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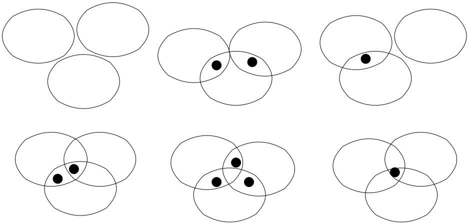

...and therefore get a contradiction

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