Online Algorithms for Maximum Cardinality Matching with Edge Arrivals

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Rafał Pyzik

Introduction

Online maximum matching problem:

- bipartite graph, vertex arrival, one sided
- bipartite graph, vertex arrival
- edge arrival
- edge arrival with preemption

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Fractional matching – each edge is given weight $w_e \geq 0$ and for each v

$$\sum_{e \in \delta(v)} w_e \leq 1.$$

Algorithms can be **randomized**.

Vertex arrival, one sided

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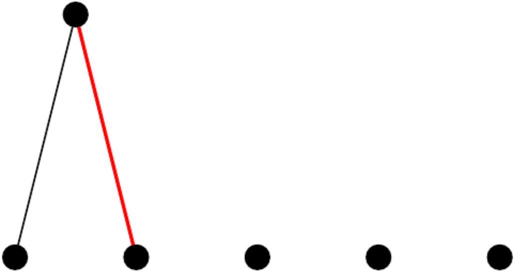

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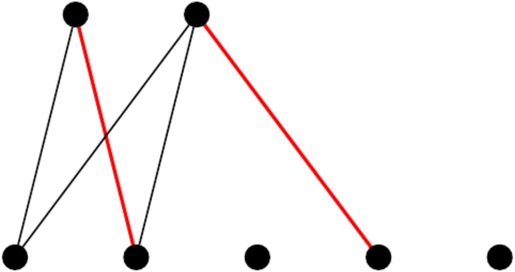

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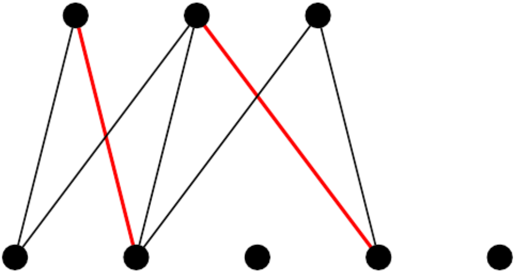

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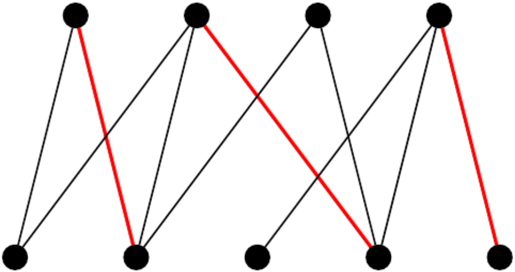

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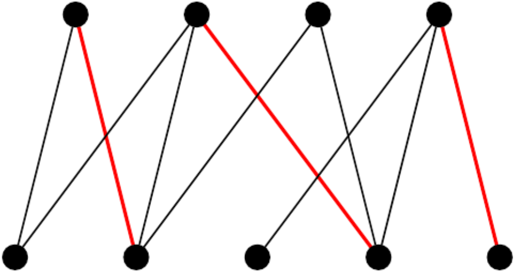

There exists a randomized $(1 - 1/e)$-competitive algorithm and it is optimal (Karp et al.).

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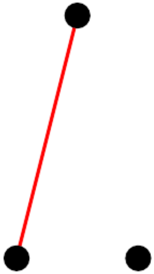

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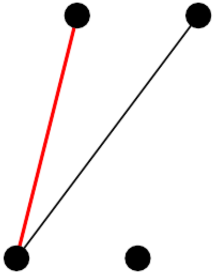

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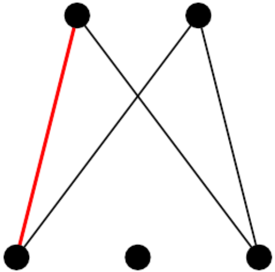

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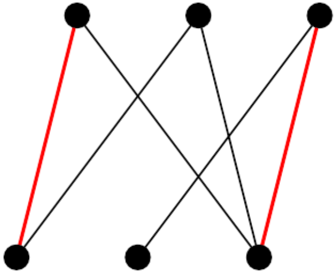

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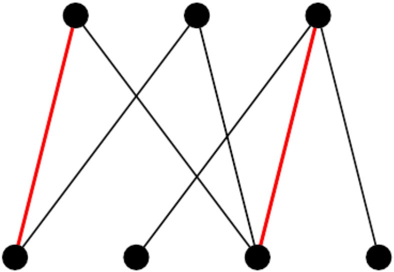

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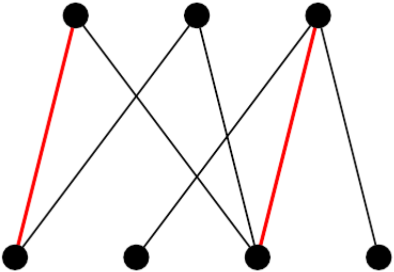

Upper bound: 0.6252 (Wang and Wong)
Lower bound: 0.526-competitive algorithm (Wang and Wong) and 0.535-competitive algorithm for trees with preemption (Chiplunkar, Tirodkar, and Vishwanathan)

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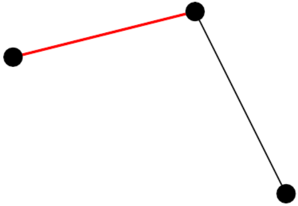

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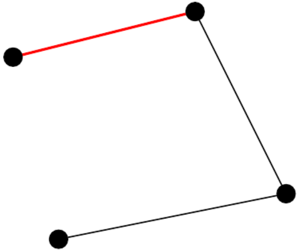

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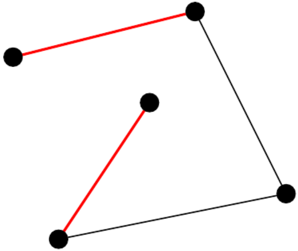

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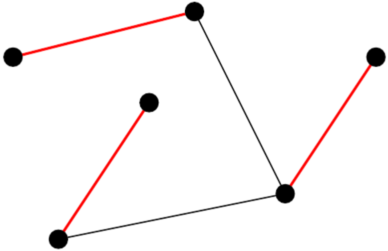

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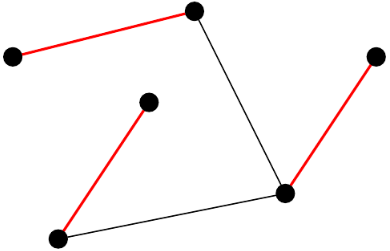

Greedy algorithm is $1/2$-competitive.
Copetitiveness better than $1/2$ is an open question for any graph class like trees or graphs with bounded degree.

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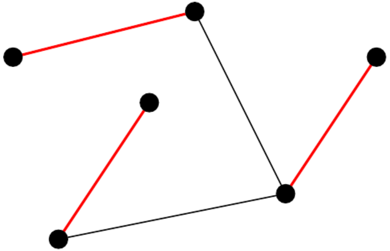

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Edge arrival with preemtion:
Upper bound: 0.591, even on bipartite graphs (Epstein et al.)
Lower bound: 0.515-competitive algorithm for trees (Tirodkar and Vishwanathan)

The Min-Index framework

Min-Index$(k, p_1, \ldots, p_k)$:

Initialization: $M_i \leftarrow \emptyset$, for every $i = 1, \ldots, k$
When edge e arrives:
If e cannot be added to any of the matchings $M_1, \ldots, M_k$, reject this edge. Otherwise, update $M_i \leftarrow M_i \cup \{e\}$, where i is the minimal index for which $M_i \cup \{e\}$ is a feasible matching.
Return M_i with probability p_i.

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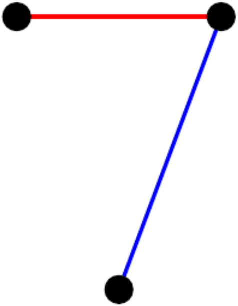

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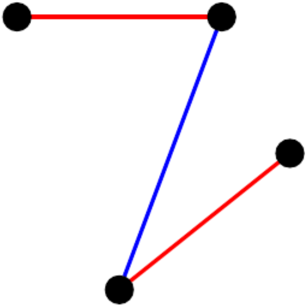

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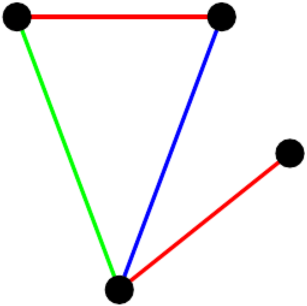

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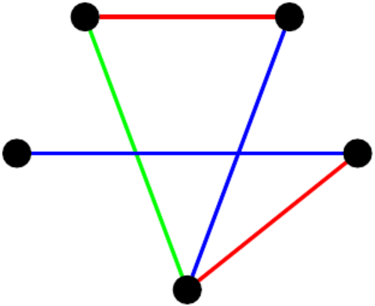

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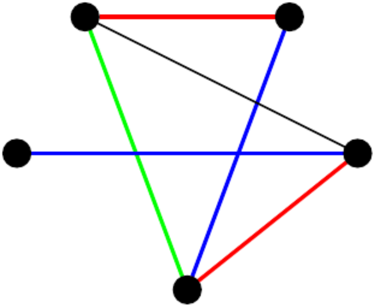

Results

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On trees, the Min-Index algorithm with $k = 3$ *matchings, picked with probabilities* $(p_1, p_2, p_3) = (5/9, 3/9, 1/9)$ *is* $5/9$*-competitive.*

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Theorem 3

There exists a $\frac{1}{2}\left(1 + \frac{1}{2^d - 1}\right)$*-competitive algorithm for fractional matching for graphs of maximum degree* d*.*

Primal-dual LP

Matching:

$$\text{maximize} \sum_{e \in E} y_e$$

subject to:

$$\sum_{e \in \delta(v)} y_e \leq 1 \quad \forall v \in V$$

$$y_e \geq 0 \quad \forall e \in E$$

Vertex cover:

$$\text{minimize} \sum_{v \in V} x_v$$

subject to:

$$x_u + x_v \geq 1 \quad \forall e \in E$$

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Proof of the Theorem 2

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Expected cardinality of the returned matching:

$$p_1 |M_1| + p_2 |M_2| + p_3 |M_3| = \frac{5}{9} |M_1| + \frac{3}{9} |M_2| + \frac{1}{9} |M_3| .$$

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Root T and orient edges downwards. Let $e = u \to v$.

- When e is accepted to M_1: $x_u \leftarrow x_u + 3/5$ and $x_v \leftarrow x_v + 2/5$.
- When e is accepted to M_2: $x_u \leftarrow x_u + 2/5$ and $x_v \leftarrow x_v + 1/5$.
- When e is accepted to M_3: $x_u \leftarrow x_u + 1/5$ and x_v is not updated.
- When e is rejected from M_1, M_2 and M_3: x_u and x_v are not updated.

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Cardinality of the matching is exactly $5/9$ of the value of produced vertex cover.

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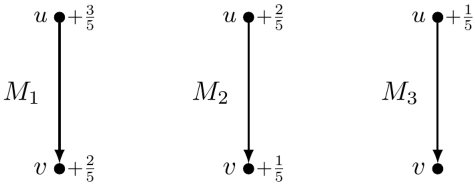

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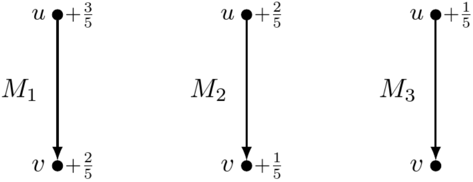
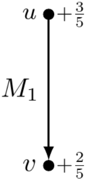


We need to show $x_u + x_v \geq 1$.

Case 1: The edge e is accepted to M_1.



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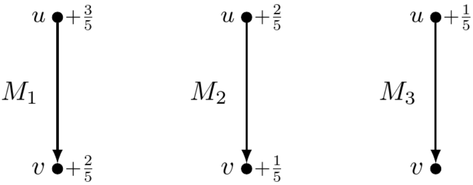

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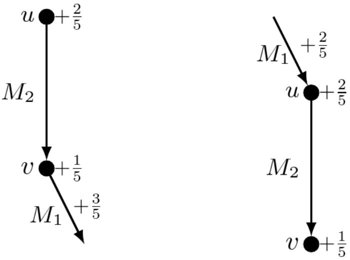
Case 2: The edge e is accepted to M_2.



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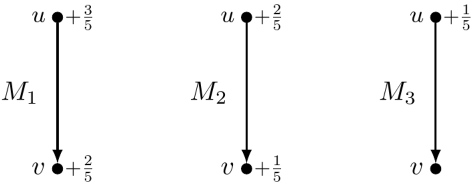

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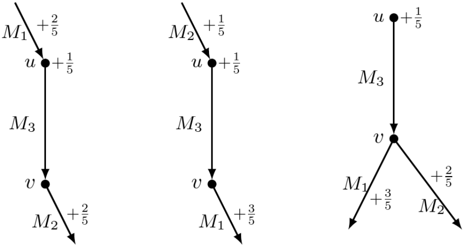
Case 3: The edge e is accepted to M_3.



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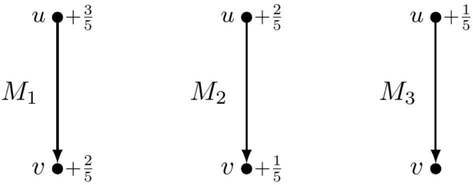


We need to show $x_u + x_v \geq 1$.

Case 4: The edge e is rejected from M_1, M_2 and M_3.

Argument similar to the one in case 3.

Upper bounds for Min-Index

Theorem 4

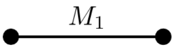
For any number of matchings $k \geq 1$ and probabilities $p_1, \ldots, p_k$, the Min-Index algorithm is:

1. *At most $2/3$-competitive on graphs of maximum degree at most 2.*
2. *At most $5/9$-competitive on forest graphs.*
3. *At most $\frac{1}{2}\left(1 + \frac{1}{2^d - 1}\right)$-competitive for bipartite graphs of maximum degree at most d.*

Proof of the upper bounds

A 2/3 upper bound for graphs of maximum degree at most 2

Sequence 1:



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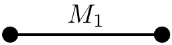

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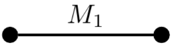

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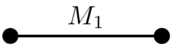

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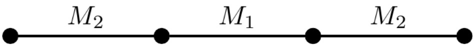

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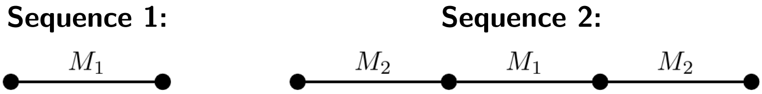

Sequence 1: $c \leq p_1$
Sequence 2: Expected matching cardinality is $p_1 + 2p_2$. The optimal cardinality is 2. Thus $c \leq \frac{p_1}{2} + p_2$.

Also $p_1 + p_2 \leq 1$ and $p_1, p_2 \geq 0$.

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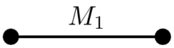

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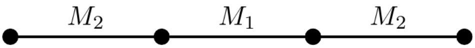

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If we want to maximize c, the optimal solution is $p_1 = \frac{2}{3}, p_2 = \frac{1}{3}, c = \frac{2}{3}$.

Proof of the upper bounds

A $\frac{1}{2}\left(1+\frac{1}{2^d-1}\right)$ upper bound for bipartite graphs

Sequence 1: a single edge
Sequences $\boldsymbol{\ell = 2, \ldots,}$ d:
An $(\ell - 1)$-regular bipartite graph with n vertices on each side. Later, to each of the $2n$ vertices, add a new edge to a new vertex.

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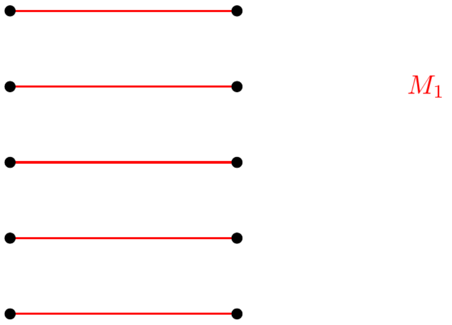

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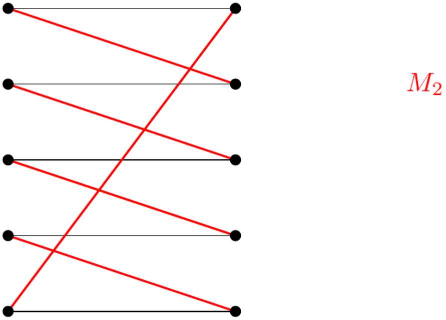

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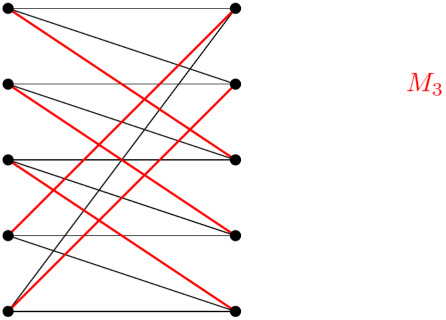

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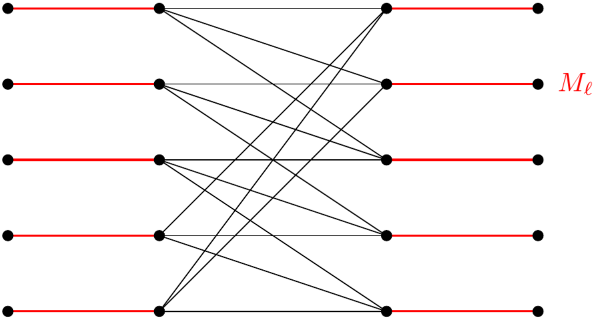

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$$c \leq \frac{1}{2} \sum_{i=1}^{\ell-1} p_i + p_\ell$$

Upper bound for any algorithm

Theorem 5

The competitive ratio of any fractional (or randomized) online algorithm for maximum matching in the vertex arrival model even for subcubic trees is at most $\frac{2}{3+1/\phi^2} \approx 0.5914$*, where* $\phi = \frac{1+\sqrt{5}}{2}$ *is the golden ratio.*

Proof of the Theorem 5

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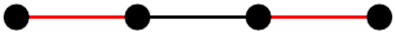

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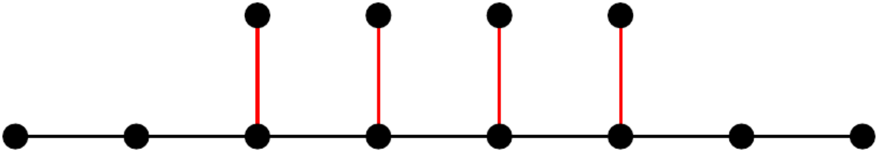

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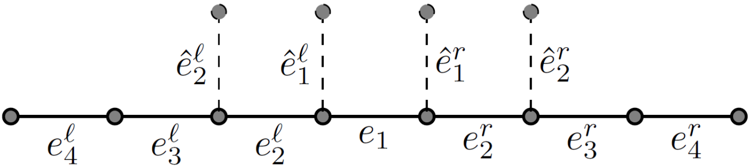

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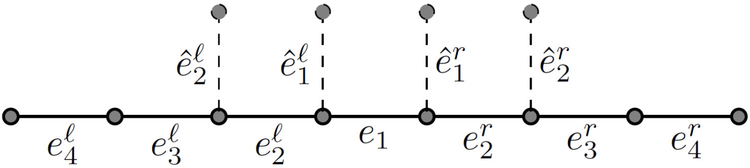

- $e_1 = (v_1^\ell, v_1^r)$, $e_i^\ell = (v_i^\ell, v_{i-1}^\ell)$, $e_i^r = (v_{i-1}^r, v_i^r)$,

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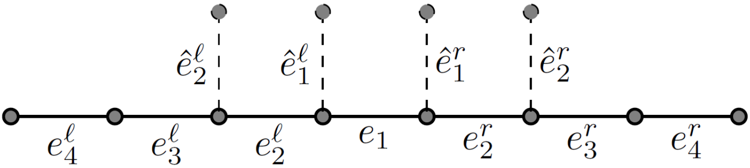
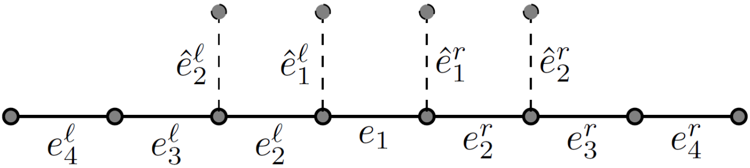


- $e_1 = (v_1^\ell, v_1^r), e_i^\ell = (v_i^\ell, v_{i-1}^\ell), e_i^r = (v_{i-1}^r, v_i^r),$
- y_e – fraction given to edge e
- $y_1 = y_{e_1}, \; y_i = y_{e_i^\ell} + y_{e_i^r}$



- $e_1 = (v_1^\ell, v_1^r), e_i^\ell = (v_i^\ell, v_{i-1}^\ell), e_i^r = (v_{i-1}^r, v_i^r),$
- y_e – fraction given to edge e
- $y_1 = y_{e_1}, \ y_i = y_{e_i^\ell} + y_{e_i^r}$
- $y_{e_1} + y_{e_2^\ell} \leq 1, \ y_{e_1} + y_{e_2^r} \leq 1$ so $2y_1 + y_2 \leq 2$
- $y_{e_i^\ell} + y_{e_{i+1}^\ell} \leq 1, \ y_{e_i^r} + y_{e_{i+1}^r} \leq 1$ so $y_i + y_{i+1} \leq 2$

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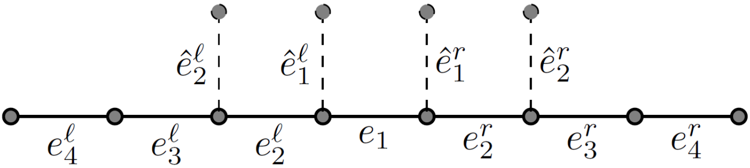

- for $n = 1$: $c \leq y_1$
- for $n = 2$: $c \leq \frac{1}{2}(y_1 + y_2)$

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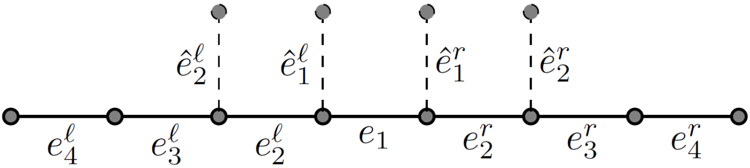
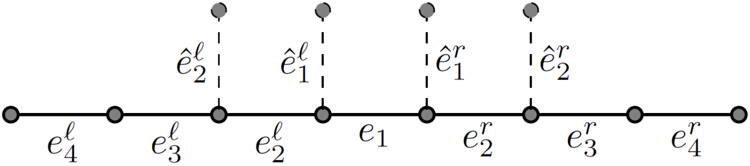


- for $n = 1$: $c \leq y_1$
- for $n = 2$: $c \leq \frac{1}{2}(y_1 + y_2)$
- optimal matching has $2(n-1)$ edges
- summing up the fractions from vertices $v_1^\ell, \ldots, v_{n-2}^\ell$ and $v_1^r, \ldots, v_{n-2}^r$ we get

$$c \leq \frac{1}{2(n-1)} \cdot \left(2(n-2) + y_n - \sum_{i=1}^{n-2} y_i \right)$$

Proof of the Theorem 5



c_m – solution for this LP when we maximize c, the value of n is up to m

$$c_m = \frac{2F_{m+1} - 2}{3F_{m+1} + F_{m-1} - 4},$$

where F_m is the m-th Fibonacci number.

$$\lim_{m \to \infty} c_m = \frac{2}{3 + {}^1\!/\!_{\phi^2}}$$

References

1. Buchbinder, N., Segev, D. Tkach, Y. Online Algorithms for Maximum Cardinality Matching with Edge Arrivals. Algorithmica 81, 1781–1799 (2019). https://doi.org/10.1007/s00453-018-0505-7