Generalizations of the Alon–Tarsi polynomial method

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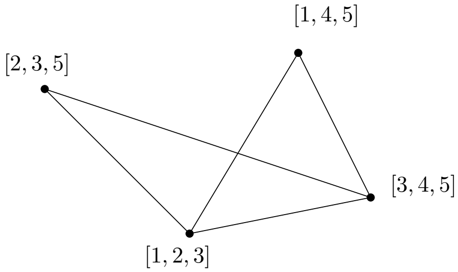
Choosability



Definition

ch(G) := *the smallest* k *such that for any assignment of lists of size* k *to the vertices of* G*, we can choose a color for each vertex to obtain a proper coloring.*

$$\chi(G) \leq \mathsf{ch}(G)$$

Graph polynomial

Definition

Let G be a graph and let $v_1, \dots, v_n$ be its vertices. Then, by the graph polynomial of G we denote the following polynomial $P_G \in \mathbb{R}[x_1, \dots, x_n]$:

$$P_G(x_1, \dots, x_n) := \prod_{\substack{v_i < v_j \\ (v_i, v_j) \in E[G]}} (x_i - x_j)$$

for some arbitrary ordering $<$ of the vertices.

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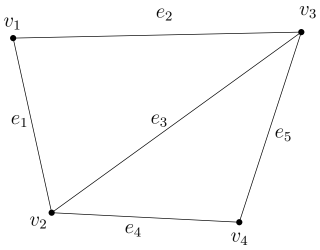

$$P_G(x_1, x_2, x_3, x_4) = \underbrace{(x_1 - x_2)}_{e_1}\underbrace{(x_1 - x_3)}_{e_2}\underbrace{(x_2 - x_3)}_{e_3}\underbrace{(x_2 - x_4)}_{e_4}\underbrace{(x_3 - x_4)}_{e_5}$$

$$= x_3^2 x_2^3 - x_1 x_3 x_2^3 + x_1 x_4 x_2^3 - x_3 x_4 x_2^3 - x_3^3 x_2^2 - x_1 x_4^2 x_2^2 + x_3 x_4^2 x_2^2 + x_1^2 x_3 x_2^2 - x_1^2 x_4 x_2^2 + x_1 x_3 x_4 x_2^2 + x_1 x_3^3 x_2 - x_1^2 x_3^2 x_2 + x_1^2 x_4^2 x_2 - x_3^2 x_4^2 x_2 + x_3^3 x_4 x_2 - x_1 x_3^2 x_4 x_2 + x_1 x_3^2 x_4^2 - x_1^2 x_3 x_4^2 - x_1 x_3^3 x_4 + x_1^2 x_3^2 x_4$$

Basic properties of the graph polynomial

$$P_G(x_1, \ldots, x_n) := \prod_{\substack{v_i < v_j \\ (v_i, v_j) \in E[G]}} (x_i - x_j)$$

Observation

For a graph G, function $f : V[G] \to \mathbb{R}$ is a proper coloring if and only if

$$P_G(f(v_1), \ldots, f(v_n)) \neq 0$$

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Definition

For $P \in \mathbb{R}[x_1, \ldots, x_n]$, we define:

- *mon(P) := set of all the monomials in the expansion of P having non-zero coefficient.*
- *mon$_{deg}(P)$:= $\left\{ x_1^{d_1} \cdot \ldots \cdot x_n^{d_n} \in \text{mon}(P) \mid \sum d_i = \deg(P) \right\}$.*

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$$P(x, y, z) = 48x^2y^2z^2 - 13x^5y + 18x^4z + y$$

$$\text{mon}(P) = \{x^2y^2z^2, x^5y, x^4z, y\}$$

$$\text{mon}_{deg}(P) = \{x^2y^2z^2, x^5y\}$$

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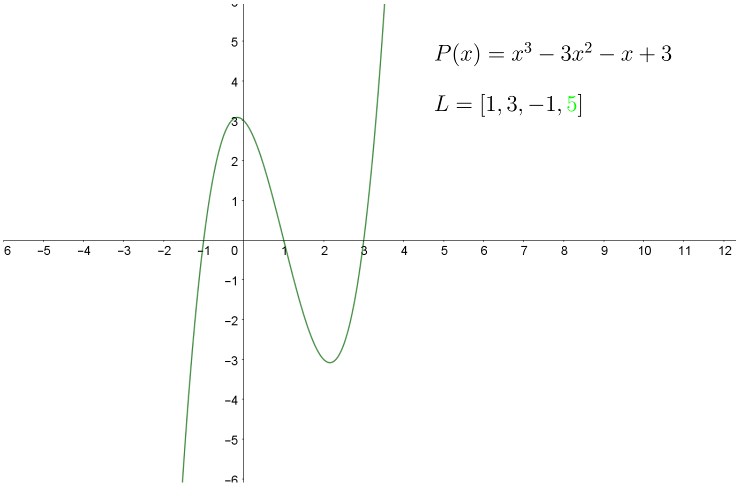
Observation (Homogeneity of P_G)

$$\textit{mon}(P_G) = \textit{mon}_{deg}(P_G)$$

Combinatorial Nullstellensatz

We know that for a polynomial $P \in \mathbb{R}[x]$ of degree d and a list $L \in \binom{\mathbb{R}}{d+1}$, there is a value q in L such that $P(q) \neq 0$.



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Similar property holds for multivariate polynomials:

Theorem (Combinatorial Nullstellensatz)

Let $P \in \mathbb{R}[x_1, \ldots, x_n]$ *and let* $x_1^{d_1} \cdot \ldots \cdot x_n^{d_n} \in mon_{deg}(P)$. *Let* $L_i \in \binom{\mathbb{R}}{d_i+1}$, *for* $i \in [n]$. *Then, there exists a tuple* $(s_1, \ldots, s_n) \in S_1 \times \ldots \times S_n$ *satisfying*

$$P(s_1, \ldots, s_n) \neq 0$$

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$$P(s_1, \ldots, s_n) \neq 0$$

$$P(x, y, z) = 48x^2y^2z^2 - 13x^5y + 18x^4z + y$$
$$\text{mon}_{deg}(P) = \{x^2y^2z^2, x^5y\}$$

$$L_1 = \{a_1, a_2, a_3\}, L_2 = \{b_1, b_2, b_3\}, L_3 = \{c_1, c_2, c_3\}$$

$$L_1 = \{a_1, a_2, a_3, a_4, a_5, a_6\}, L_2 = \{b_1, b_2\}, L_3 = \{c_1\}$$

Combinatorial Nullstellensatz for graph polynomials

Corollary

If $x_1^{d_1} \cdot \ldots \cdot x_n^{d_n} \in \mathit{mon}(P_G)$ $(= \mathit{mon}_{\mathit{deg}}(P_G))$, *and* $k = \max\{d_i + 1\}$, *then* $\mathit{ch}(G) \leq k$.

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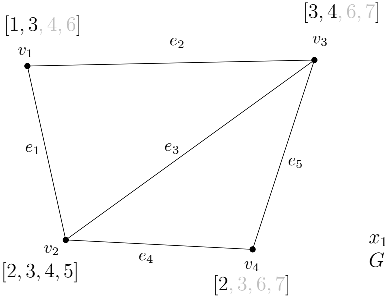
If $x_1^{d_1} \cdot \ldots \cdot x_n^{d_n} \in mon(P_G)$ $(= mon_{deg}(P_G))$, and $k = \max\{d_i + 1\}$, then $ch(G) \leq k$.



$$P_G(x_1, \ldots, x_4) = x_3^2x_2^3 - x_1x_3x_2^3 + x_1x_4x_2^3 - x_3x_4x_2^3 - x_3^3x_2^2 - x_1x_4^2x_2^2 + x_3x_4^2x_2^2 + x_1^2x_3x_2^2 - x_1^2x_4x_2^2 + x_1x_3x_4x_2^2 + x_1x_3^3x_2 - x_1^2x_3^2x_2 + x_1^2x_4^2x_2 - x_3^2x_4^2x_2 + x_3^3x_4x_2 - x_1x_3^2x_4x_2 + x_1x_3^2x_4^2 - x_1^2x_3x_4^2 - x_1x_3^3x_4 + x_1^2x_3^2x_4$$

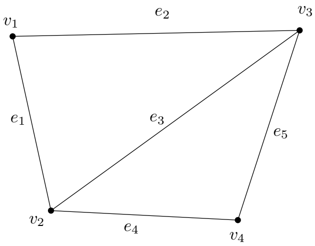
$x_1x_3x_2^3 \in \mathrm{mon}(P_G) \Longrightarrow$ for any list assignment L of size 4, G is L-colorable.

Combinatorial Nullstellensatz for graph polynomials

Definition (Alon-Tarsi number)

Let G be a graph. Then the Alon-Tarsi number of G, denoted as $AT(G)$, is defined in the following way:

$$AT(G) := \min\left\{k \mid \exists_{x_1^{d_1}\cdot\ldots\cdot x_n^{d_n} \in mon(P_G)}\ k = \max\left\{d_i + 1\right\}\right\}$$



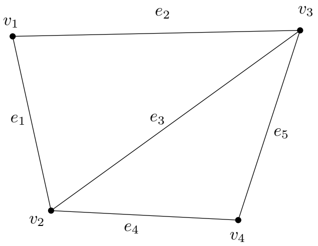
$$P_G(x_1,\ldots,x_4) = x_3^2x_2^3 - x_1x_3x_2^3 + x_1x_4x_2^3 - x_3x_4x_2^3 - x_3^3x_2^2 - x_1x_4^2x_2^2 + x_3x_4^2x_2^2 + x_1^2x_3x_2^2 - x_1^2x_4x_2^2 + x_1x_3x_4x_2^2 + x_1x_3^3x_2 - x_1^2x_3^2x_2 + x_1^2x_4^2x_2 - x_3^2x_4^2x_2 + x_3^3x_4x_2 - x_1x_3^2x_4x_2 + x_1x_3^2x_4^2 - x_1^2x_3x_4^2 - x_1x_3^3x_4 + x_1^2x_3^2x_4$$

$$\mathrm{AT}(G) = 3$$

Combinatorial Nullstellensatz for graph polynomials

Observation

$$\chi(G) \leq \mathit{ch}(G) \leq \mathit{AT}(G)$$

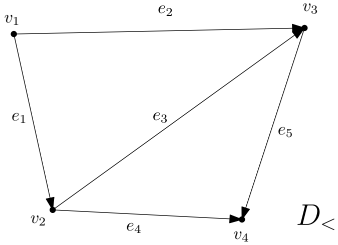


$P_G(x_1, \ldots, x_4) = x_3^2 x_2^3 - x_1 x_3 x_2^3 + x_1 x_4 x_2^3 - x_3 x_4 x_2^3 - x_3^3 x_2^2 - x_1 x_4^2 x_2^2 + x_3 x_4^2 x_2^2 + x_1^2 x_3 x_2^2 - x_1^2 x_4 x_2^2 + x_1 x_3 x_4 x_2^2 + x_1 x_3^3 x_2 - x_1^2 x_3^2 x_2 + x_1^2 x_4^2 x_2 - x_3^2 x_4^2 x_2 + x_3^3 x_4 x_2 - x_1 x_3^2 x_4 x_2 + x_1 x_3^2 x_4^2 - x_1^2 x_3 x_4^2 - x_1 x_3^3 x_4 + x_1^2 x_3^2 x_4$

$$\mathrm{AT}(G) = 3$$

Monomials in P_G vs orientations

The order $<$ can be interpreted as an orientation $D_<$ of the graph G.



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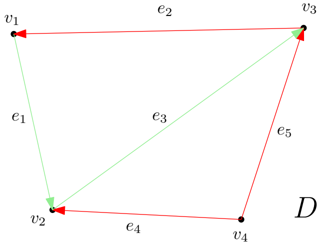
We calculate P_G by choosing variable x_i or $-x_j$ from each pair of brackets. Each such choice corresponds to some orientation D of G.



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$$-x_1x_2x_3x_4^2 = (-1)^3 \cdot x_1^{\deg_D^{\text{out}}(v_1)} \cdot \ldots \cdot x_4^{\deg_D^{\text{out}}(v_4)}$$

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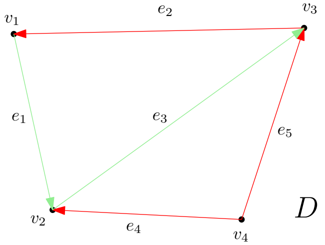

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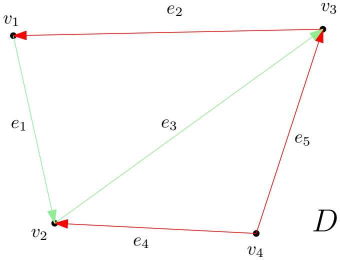
Theorem

$$P_G(x_1, \ldots, x_n) = \sum_{D \in \textit{orien}(G)} (-1)^{|D_< - D|} \cdot x_1^{\textit{deg}_D^{\textit{out}}(v_1)} \cdot \ldots \cdot x_n^{\textit{deg}_D^{\textit{out}}(v_n)}$$

How to determine whether $x_1^{d_1} \cdot \ldots \cdot x_n^{d_n} \in \mathrm{mon}(P_G)$?

For a monomial $x_1^{d_1} \cdot \ldots \cdot x_n^{d_n}$ to be in $\mathrm{mon}(P_G)$, it is necessary that $d_i = \deg_D^{\mathrm{out}}(v_i)$ for some orientation D. However, for a given orientation D, the corresponding monomial doesn't need to belong to $\mathrm{mon}(P_G)$.



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... where is $x_1x_2x_3x_4^2$?

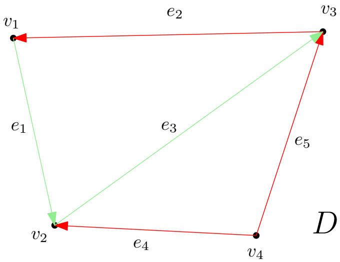
How to determine whether $x_1^{d_1} \cdot \ldots \cdot x_n^{d_n} \in \text{mon}(P_G)$?

Theorem (Alon, Tarsi)

Let D be some orientation of G. Then $x_1^{\text{deg}_D^{\text{out}}(v_1)} \cdot \ldots \cdot x_n^{\text{deg}_D^{\text{out}}(v_n)} \in \text{mon}(P_G)$ if and only if

$$|EE(D)| - |EO(D)| \neq 0$$

where $EE(D)$ and $EO(D)$ are the sets of all Eulerian sub-orientations of D having respectively even and odd size.



$$-x_1 x_2 x_3 x_4^2 = (-1)^3 \cdot x_1^{\deg_D^{\text{out}}(v_1)} \cdot \ldots \cdot x_4^{\deg_D^{\text{out}}(v_4)}$$

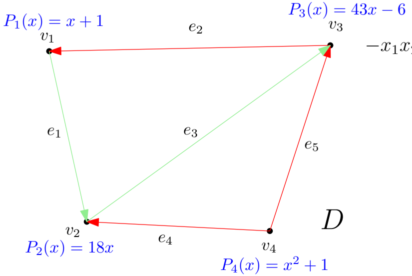
$\text{EO}(D) = \{v_1 v_2 v_3\}$
$\text{EE}(D) = \{\varepsilon\}$
$|\text{EE}(D)| - |\text{EO}(D)| = 0$

Generalization

Theorem (Dan Hefetz)

Let D be some orientation of G, and let $d_i = deg_D^{out}(v_i)$. Let $P_i \in \mathbb{R}[x]$, for $i \in [n]$, be any arbitrary sequence of polynomials, satisfying $\deg(P_i) = d_i$. Then, $x_1^{d_1} \cdot \ldots \cdot x_n^{d_n} \in mon(P_G)$ if and only if

$$\sum_{A \subseteq E[D]} (-1)^{|A|} \prod_{i=1}^{n} P_i \left(deg_A^{\Delta}(v_i) \right) \neq 0$$



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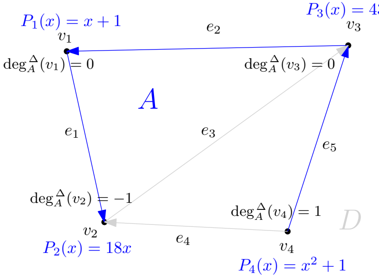


$$-x_1x_2x_3x_4^2 = (-1)^3 \cdot x_1^{\deg_D^{\text{out}}(v_1)} \cdot \ldots \cdot x_4^{\deg_D^{\text{out}}(v_4)}$$

$$(-1)^{|A|} \prod_{i=1}^{4} P_i(\deg_A^{\Delta}(v_i)) = -1 \cdot 1 \cdot (-18) \cdot (-6) \cdot 2$$

Algebraic tool – permament

Definition

Let $A = (a_{ij})$ be a $n \times n$ matrix. Then the permanent of A is defined by

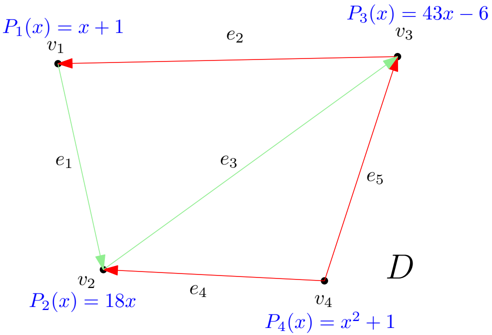
$$\textit{perm}(A) := \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}$$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \qquad \mathrm{perm}(A) = a_1 b_2 c_3 + a_1 b_3 c_2 + a_2 b_1 c_3 + \dots$$

Special matrix

Consider the following $m \times m$ matrix M:

- Each row corresponds to exactly one edge of D.
- For each vertex v_i, we assign exactly d_i columns to it (note that $\sum d_i = m$).



	v_1	v_2	v_3	v_4	v_4
e_1					
e_2					
e_3					
e_4					
e_5					

Special matrix

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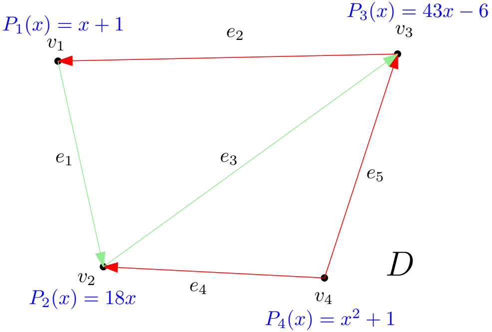
- For each vertex v_i, we assign exactly d_i columns to it (note that $\sum d_i = m$).
- We fill M like an adjacency matrix (some columns will be duplicated)

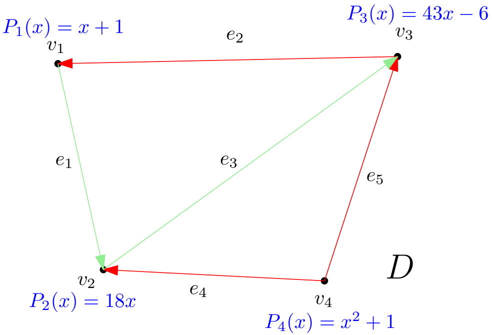


	v_1	v_2	v_3	v_4	v_4
e_1	1	−1	0	0	0
e_2	−1	0	1	0	0
e_3	0	1	−1	0	0
e_4	0	−1	0	1	1
e_5	0	0	−1	1	1

Special matrix

Fact

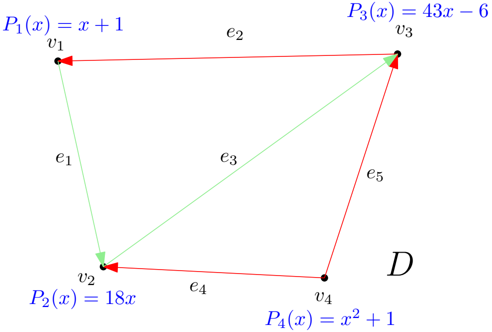
$x_1^{d_1} \cdot \ldots \cdot x_n^{d_n} \in$ *mon*(P_G) *if and only if* *perm*$(M) \neq 0$.



	v_1	v_2	v_3	v_4	v_4
e_1	1	−1	0	0	0
e_2	−1	0	1	0	0
e_3	0	1	−1	0	0
e_4	0	−1	0	1	1
e_5	0	0	−1	1	1

Sketch of the proof

- Take M and turn it to a $m+1 \times m+1$ matrix by adding the complex roots of polynomials P_i to the bottom row and adding column $0, 0, \ldots, 1$ on the right.

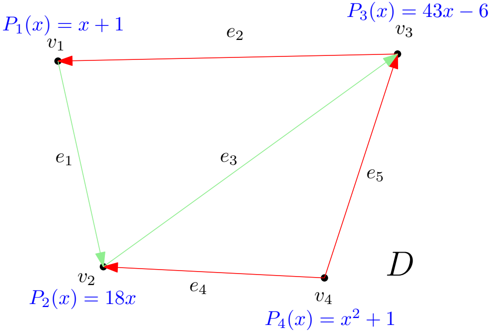


	v_1	v_2	v_3	v_4	v_4	
e_1	1	-1	0	0	0	0
e_2	-1	0	1	0	0	0
e_3	0	1	-1	0	0	0
e_4	0	-1	0	1	1	0
e_5	0	0	-1	1	1	0
	1	0	$-\frac{6}{43}$	i	$-i$	1

Sketch of the proof

- Transpose the matrix and use Ryser's formula to calculate the permanent.

$$\text{perm}(M) = (-1)^{m+1} \sum_{A \subseteq \text{columns}} (-1)^{|A|} \prod_{i \in \text{rows}} \sum_{j \in A} a_{ij}$$

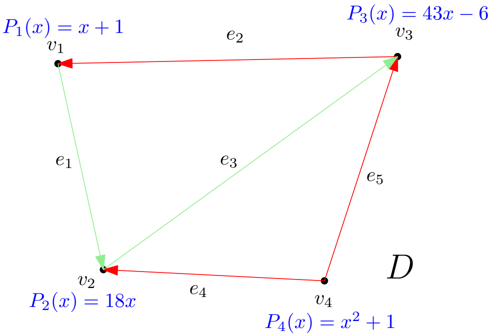


$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_4 \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \color{red}{1} \\ -1 & 0 & 1 & -1 & 0 & \color{red}{0} \\ 0 & 1 & -1 & 0 & -1 & \color{red}{-\frac{6}{43}} \\ 0 & 0 & 0 & 1 & 1 & \color{red}{i} \\ 0 & 0 & 0 & 1 & 1 & \color{red}{-i} \\ \color{blue}{0} & \color{blue}{0} & \color{blue}{0} & \color{blue}{0} & \color{blue}{0} & \color{blue}{1} \end{bmatrix} \end{array}$$

Sketch of the proof

- Transpose the matrix and use Ryser's formula to calculate the permanent.

$$\text{perm}(M) = (-1)^{m+1} \sum_{A \subseteq \text{columns}} (-1)^{|A|} \prod_{i \in \text{rows}} \sum_{j \in A} a_{ij}$$

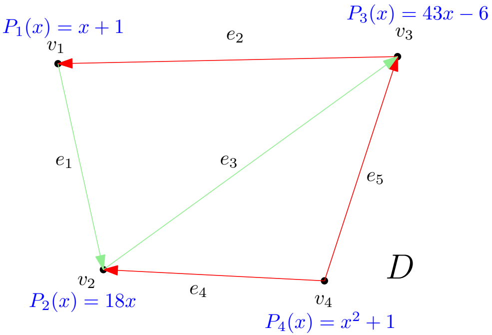


	e_1	e_2	e_3	e_4	e_5		A
v_1	1	−1	0	0	0	1	0
v_2	−1	0	1	−1	0	0	1
v_3	0	1	−1	0	−1	$-\frac{6}{43}$	$-1\frac{6}{43}$
v_4	0	0	0	1	1	i	i
v_4	0	0	0	1	1	$-i$	$-i$
	0	0	0	0	0	1	1

Sketch of the proof

- Transpose the matrix and use Ryser's formula to calculate the permanent.

$$\text{perm}(M) = (-1)^{m+1} \sum_{A \subseteq \text{columns}} (-1)^{|A|} \prod_{i \in \text{rows}} \sum_{j \in A} a_{ij}$$

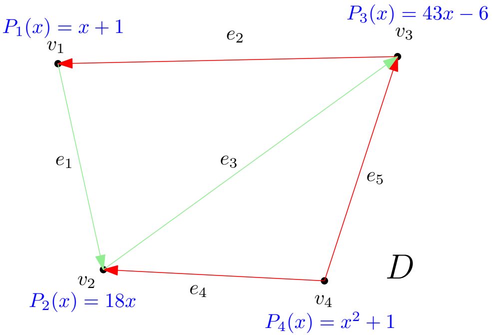


$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_4 \\ \\ \end{array}
\begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & \\ 1 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & -\frac{6}{43} \\ 0 & 0 & 0 & 1 & 1 & i \\ 0 & 0 & 0 & 1 & 1 & -i \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}
\quad
\begin{array}{c} A \\ 0 \\ 1 \\ -1\frac{6}{43} \\ i \\ -i \\ 1 \end{array}$$

$\prod \cdot (-1)^{|A|}$

Sketch of the proof

- If $m+1 \notin A$ (we didn't choose the last column), the sum in the last row is 0, and thus the entire product is zero.



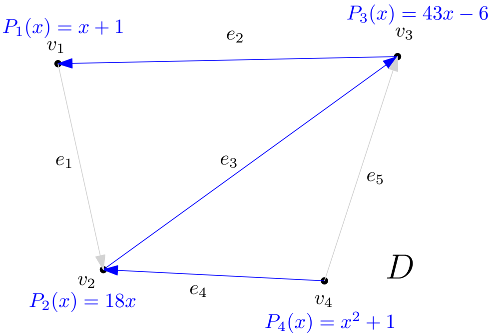
$$
\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_4 \\ \\ \end{array}
\begin{array}{c}
\begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & \end{array} \\
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 1 \\
-1 & 0 & 1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 & -1 & -\frac{6}{43} \\
0 & 0 & 0 & 1 & 1 & i \\
0 & 0 & 0 & 1 & 1 & -i \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

$m+1 \notin A$: $\cdots$, 0

$\prod \cdot (-1)^{|A|}$

Sketch of the proof

- Otherwise, we obtain precisely $\prod_{i=1}^{n} P_i(\deg_A^{\Delta}(v_i))$, where we interpret A as a subset of the edges of D.



	e_1	e_2	e_3	e_4	e_5	$m+1 \in A$	
v_1	1	-1	0	0	0	1	$P_1(-1)$
v_2	-1	0	1	-1	0	0	$\frac{1}{18}P_2(0)$
v_3	0	1	-1	0	-1	$-\frac{6}{43}$	$\frac{1}{43}P_3(0)$
v_4	0	0	0	1	1	i	$P_4(1)$
v_4	0	0	0	1	1	$-i$	
	0	0	0	0	0	1	1

$\prod \cdot (-1)^{|A|}$

Back to Alon-Tarsi

$$\boxed{\mathsf{S}=\sum_{A\subseteq E[D]}(-1)^{|A|}\underbrace{\prod_{i=1}^{n}P_i\left(\deg_A^{\Delta}(v_i)\right)}_{V_A}\neq 0}$$

Choose $P_i \equiv (x-1)(x-2)\dots(x-d_i)$.

Observation

For all $A \subseteq E[D]$*:*

1. $\sum deg_A^{\Delta}(v_i) = 0$
2. $deg_A^{\Delta}(v_i) \leq d_i = deg_A^{out}(v_i)$
3. A *is Eulerian* $\iff$ $deg_A^{\Delta}(v_i) = 0$ *for all* i.

Back to Alon-Tarsi

$$\text{S}=\sum_{A\subseteq E[D]}(-1)^{|A|}\underbrace{\prod_{i=1}^{n} P_i\left(\text{deg}_A^{\Delta}(v_i)\right)}_{V_A} \neq 0$$

Choose $P_i \equiv (x-1)(x-2)\dots(x-d_i)$.

- If A is not Eulerian, then there must exist vertex v_i such that $0 < \text{deg}_A^{\Delta}(v_i) \leq d_i$. Then $P_i(\text{deg}_A^{\Delta}(v_i)) = 0$ and $V_A = 0$.
- If A is Eulerian, then for all i

$$P_i\left(\text{deg}_A^{\Delta}(v_i)\right) = \prod_{i=1}^{d_i} -i = \pm d_i! \implies V_A = C \text{ for some constant } C$$

By combining these two facts, we get

$$|S| = \left|\sum_{\substack{A\subseteq E[D] \\ A \text{ Eulerian}}} (-1)^{|A|} \cdot C\right| = |C| \cdot ||EE(D)| - |EO(D)||$$

Thank you!