A note on computer-assisted proofs in flag algebras

Ref. Ignacy Buczek

Graphons

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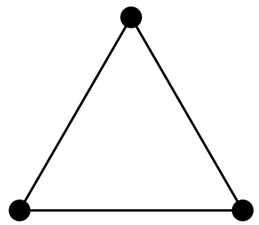

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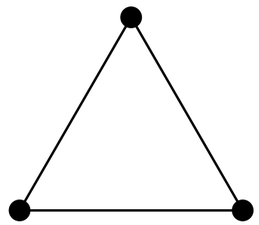

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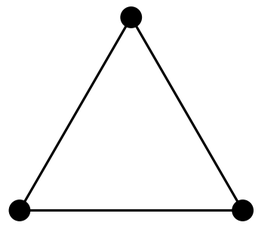

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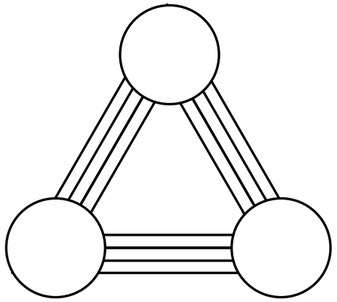

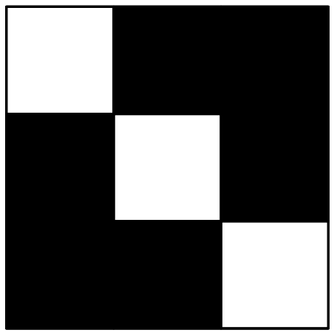

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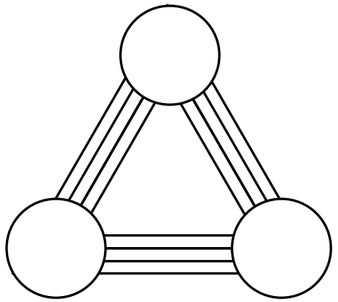

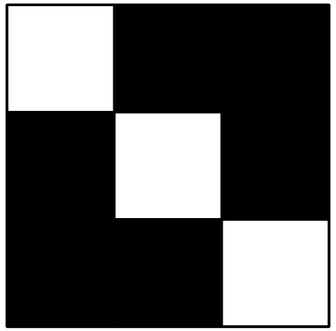

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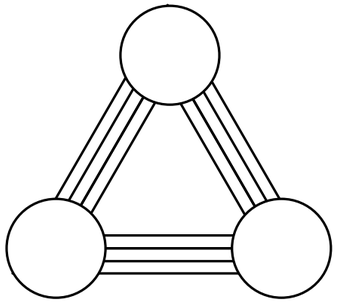

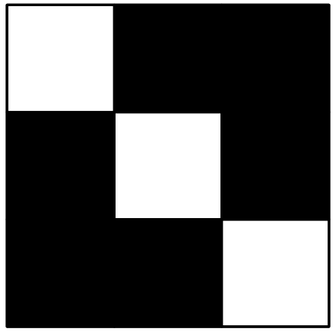

$$\text{△} = \frac{6}{27}$$

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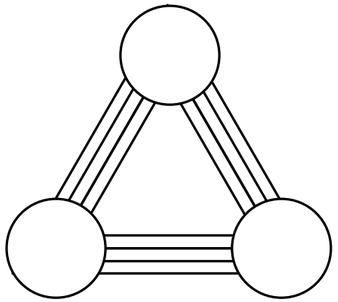

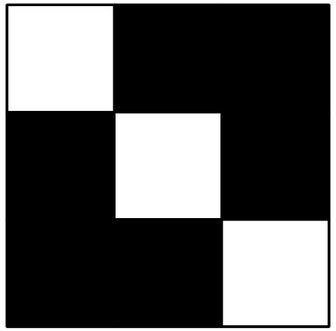

 $= \frac{6}{27}$

 $=$

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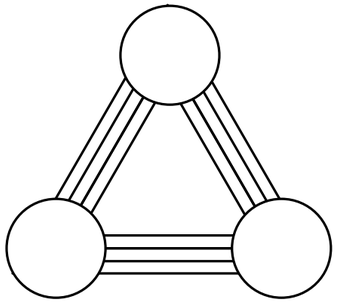

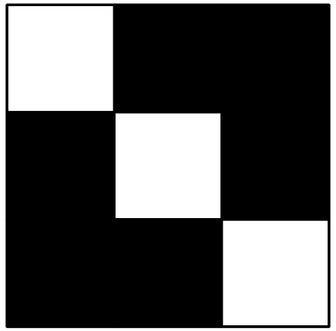

 $= \frac{6}{27}$  $= \frac{2}{3}$

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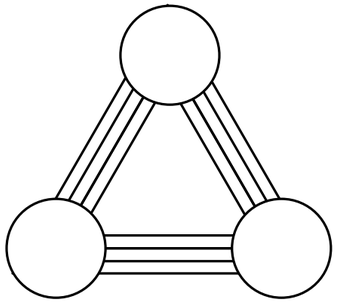

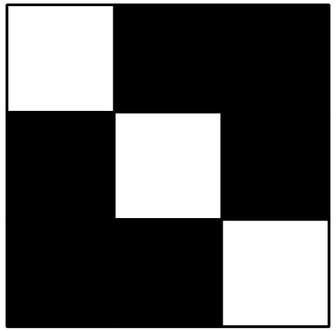

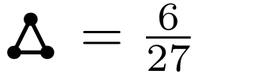

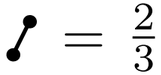

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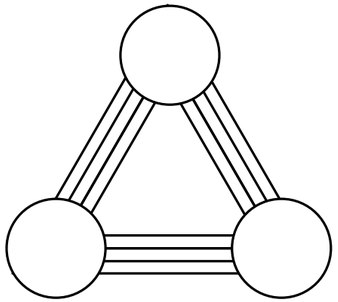

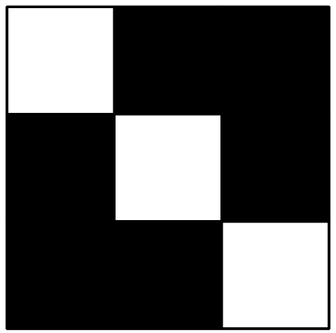

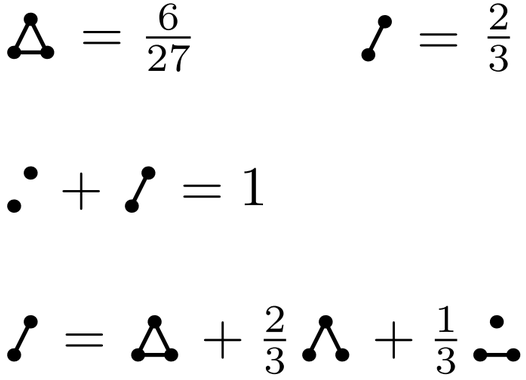

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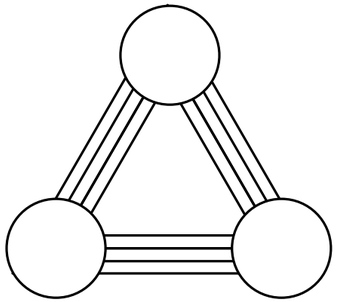

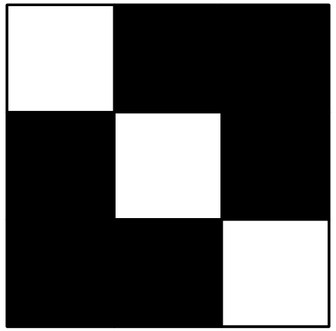

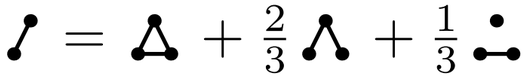

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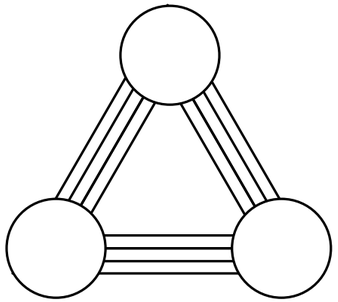

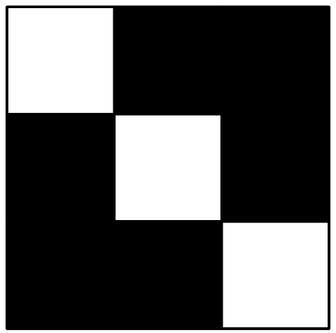

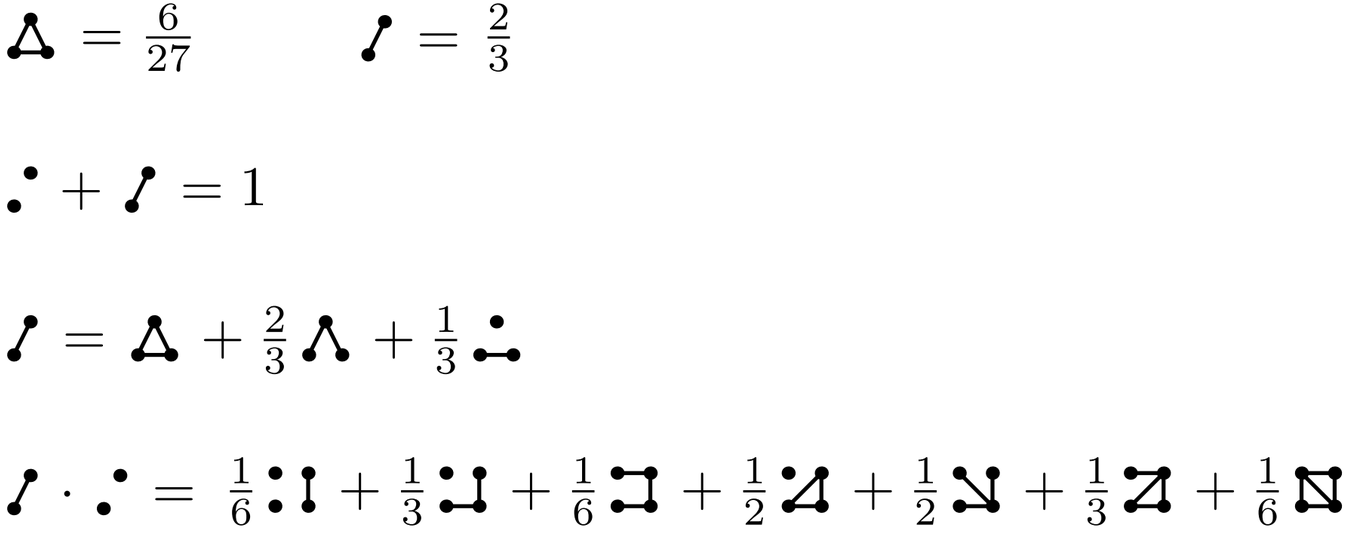

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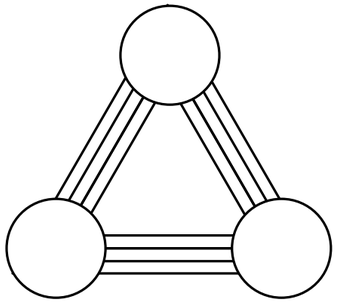
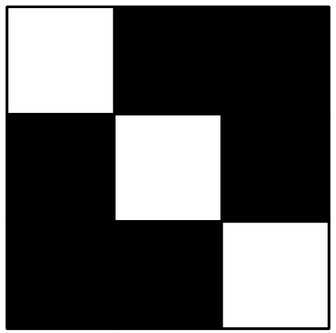
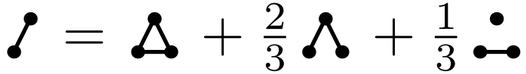
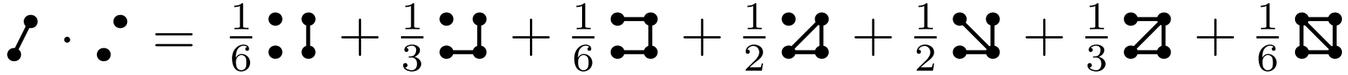


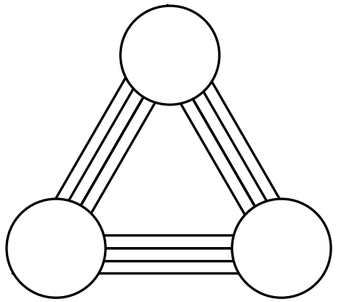


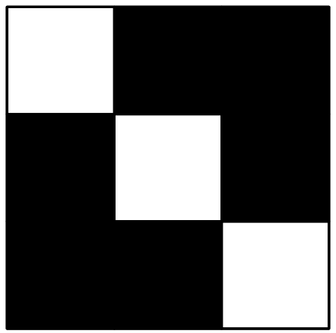


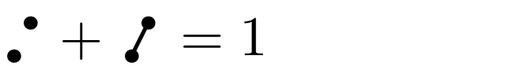


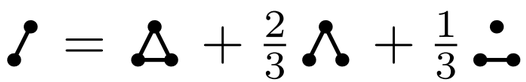
△-free graphs have at most $\frac{n^2}{4}$ edges.

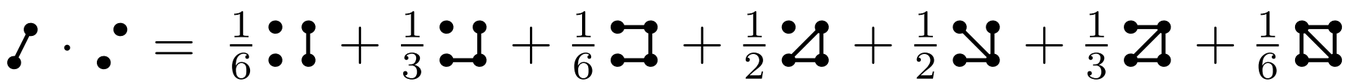
Flags











△-free graphs have at most $\frac{n^2}{4}$ edges.

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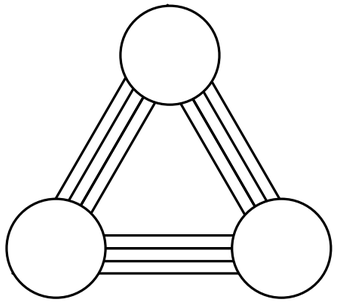
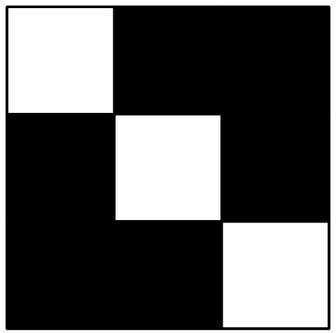
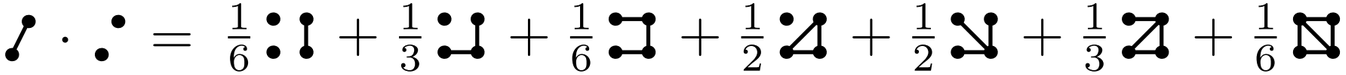
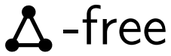
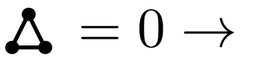




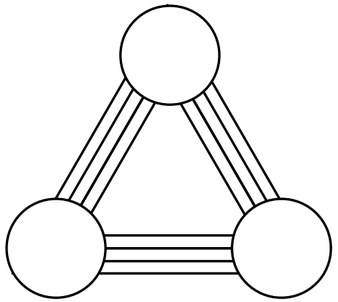


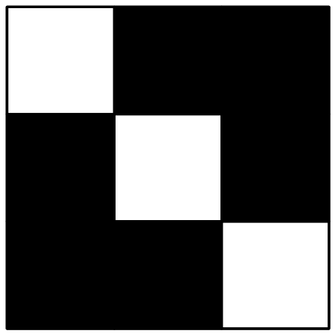
 -free graphs have at most $\frac{n^2}{4}$ edges.

 $= 0 \rightarrow$  $\leq \frac{1}{2}$

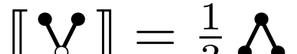
$\Delta(G) \leq \frac{n}{2}$  $\leq \frac{1}{2}$

Flags





△-free graphs have at most $\frac{n^2}{4}$ edges.



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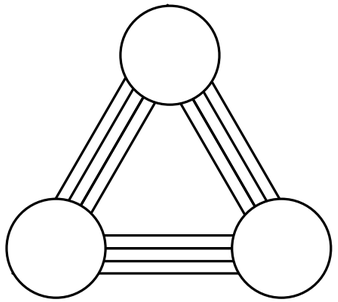
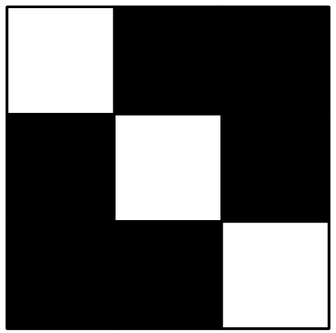




Input:

$\triangle \leq 0$

$\triangle = \frac{6}{27}$ $\quad$ $\diagup = \frac{2}{3}$

$\therefore + \diagup = 1$

$\diagup = \triangle + \frac{2}{3}\wedge + \frac{1}{3}\therefore$

$\diagup \cdot \therefore = \frac{1}{6}\colon\colon + \frac{1}{3}\colon\sqcup + \frac{1}{6}\sqcup + \frac{1}{2}\diagup\diagup + \frac{1}{2}\diagdown + \frac{1}{3}\diagup + \frac{1}{6}\boxtimes$

$\triangle$-free graphs have at most $\frac{n^2}{4}$ edges.

$\triangle = 0 \rightarrow \diagup \leq \frac{1}{2}$

$\Delta(G) \leq \frac{n}{2}$ $\quad$ $\diagup \leq \frac{1}{2}$

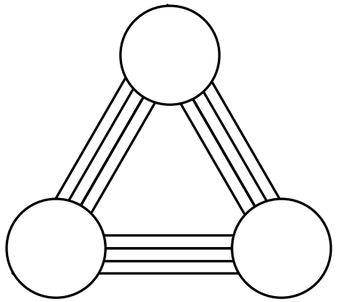
$\therefore + \wedge + \angle + \triangle = 1$

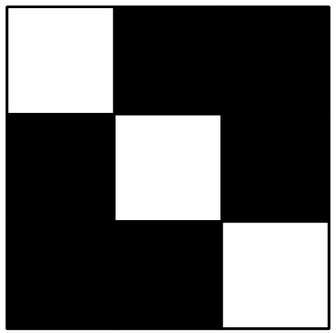
$\diagup = \bigtriangledown + \frac{1}{2}\angle + \frac{1}{2}\diagup$

$\therefore \cdot \diagup = \frac{1}{2}\angle + \frac{1}{2}\diagup$

$[\![\vee]\!] = \frac{1}{3}\wedge$

Flags





Input:

$\triangle \leq 0$

Question:

$/ \leq ?$

$\triangle = \frac{6}{27}$ $\quad$ $/ = \frac{2}{3}$

$\cdot\cdot + / = 1$

$/ = \triangle + \frac{2}{3}\wedge + \frac{1}{3}\cdot\cdot$

$/ \cdot \cdot\cdot = \frac{1}{6}\,\vdots + \frac{1}{3}\,\square + \frac{1}{6}\,\square + \frac{1}{2}\,\square + \frac{1}{2}\,\square + \frac{1}{3}\,\square + \frac{1}{6}\,\square$

$\triangle$-free graphs have at most $\frac{n^2}{4}$ edges.

$\triangle = 0 \to / \leq \frac{1}{2}$

$\Delta(G) \leq \frac{n}{2}$ $\quad$ $/ \leq \frac{1}{2}$

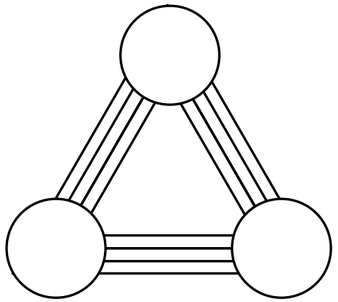
$\cdot\cdot + \wedge + \angle + \triangle = 1$

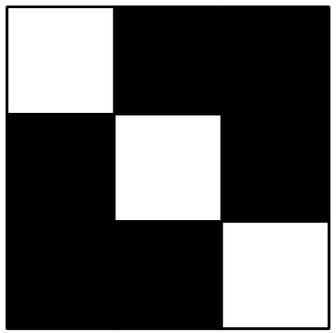
$/ = \triangledown + \frac{1}{2}\,\square + \frac{1}{2}\,\square$

$\cdot\cdot \cdot / = \frac{1}{2}\,\square + \frac{1}{2}\,\square$

$[\![\triangledown]\!] = \frac{1}{3}\wedge$

Flags





Input:

$\triangle \leq 0$

Output:

$\edge \leq \frac{1}{2} + \varepsilon$

Question:

$\edge \leq ?$

$\triangle = \frac{6}{27}$ $\quad$ $\edge = \frac{2}{3}$

$\vee + \edge = 1$

$\edge = \triangle + \frac{2}{3}\wedge + \frac{1}{3}\vee$

$\edge \cdot \vee = \frac{1}{6}\,\mathbb{I}\!\mathbb{I} + \frac{1}{3}\sqcup + \frac{1}{6}\square + \frac{1}{2}\diagup\!\diagup + \frac{1}{2}\diagdown\!\diagdown + \frac{1}{3}\boxtimes + \frac{1}{6}\boxtimes$

$\triangle$-free graphs have at most $\frac{n^2}{4}$ edges.

$\triangle = 0 \to \edge \leq \frac{1}{2}$

$\Delta(G) \leq \frac{n}{2}$ $\quad$ $\edge \leq \frac{1}{2}$

$\vee + \wedge + \angle + \triangle = 1$

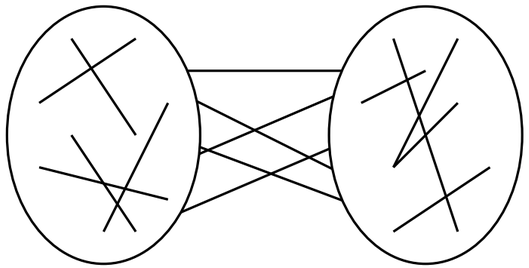
$\edge = \nabla + \frac{1}{2}\,\mathrm{Y} + \frac{1}{2}\,\mathrm{V}$

$\vee \cdot \edge = \frac{1}{2}\,\mathrm{Y} + \frac{1}{2}\,\mathrm{V}$

$[\![\mathrm{V}]\!] = \frac{1}{3}\wedge$

Problem

Balanced bipartition



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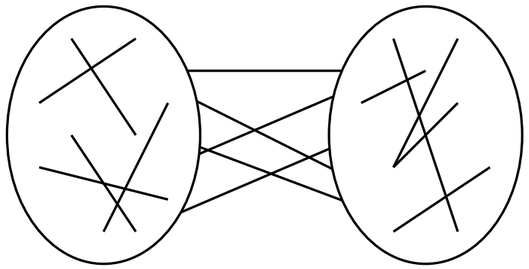

⊠-free, $\delta(G) \geq \frac{n}{2}$, distance at most $\frac{n^2}{9}$

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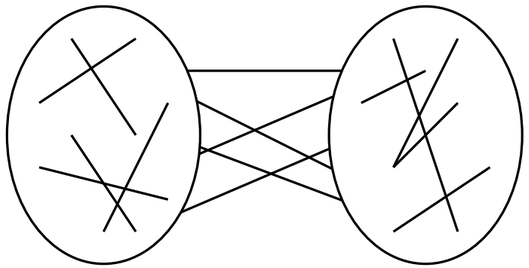

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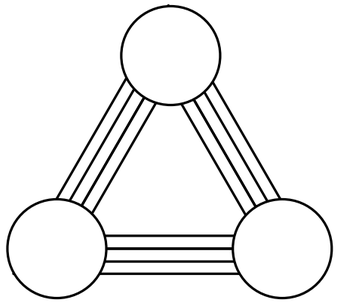

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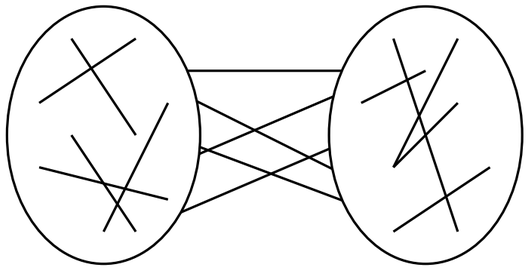

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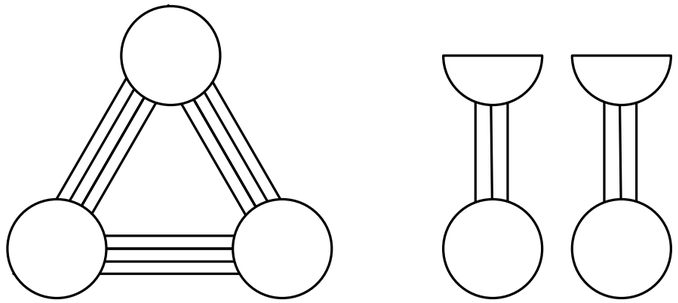

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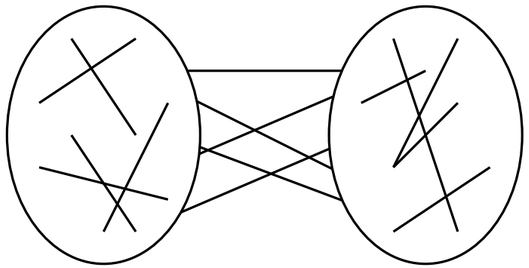

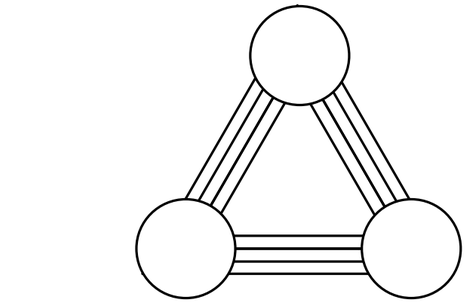

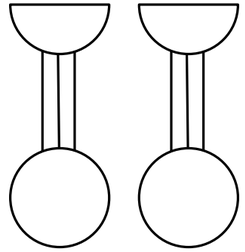

Question:

$$[\![(\tfrac{1}{3} - \triangle)^2]\!] \leq ?$$

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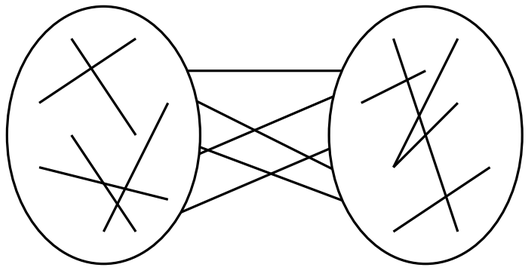

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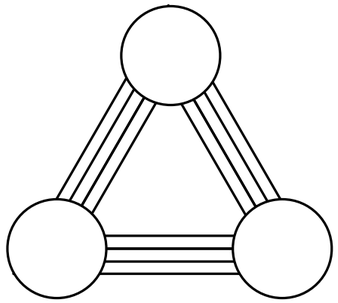

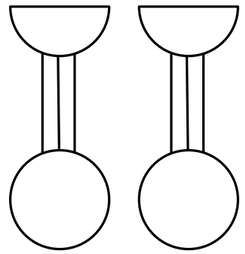

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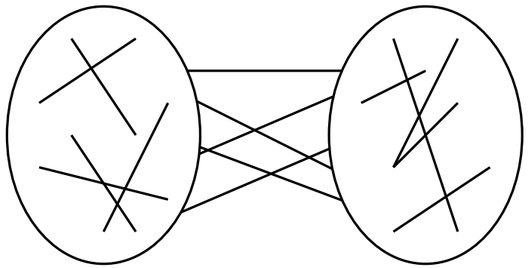
Every balanced bipartition has at least $\frac{n^2}{9}$ edges.

Problem

Balanced bipartition



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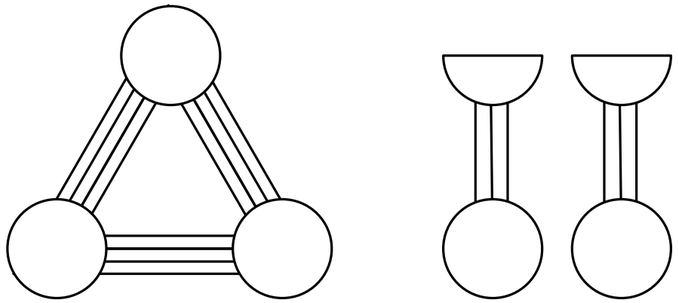

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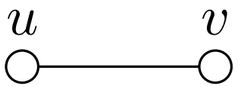

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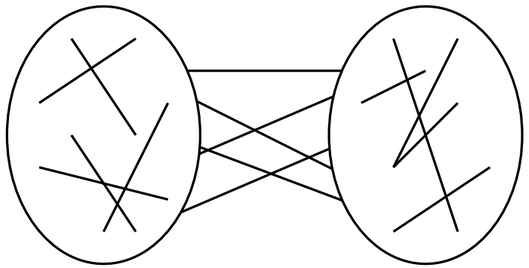

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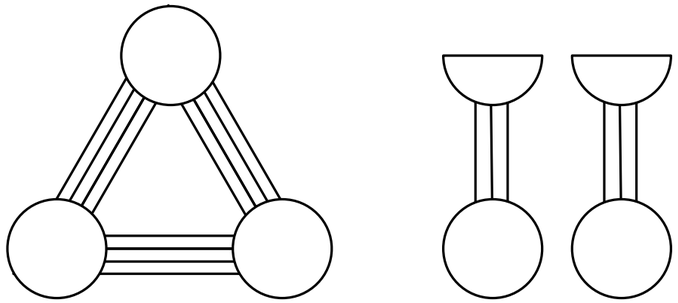

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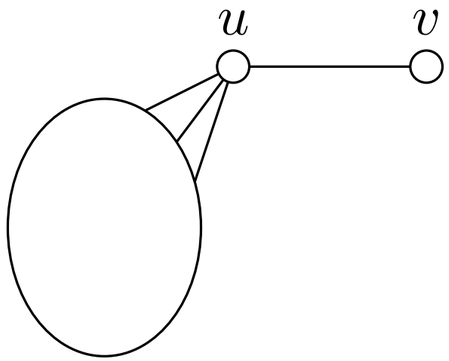

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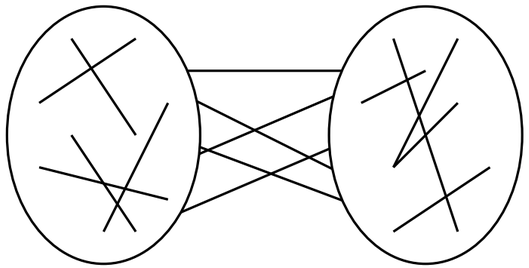

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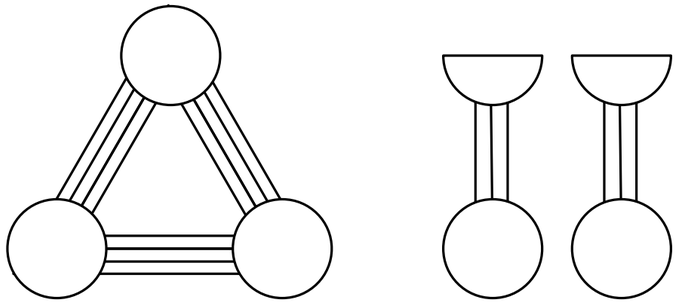

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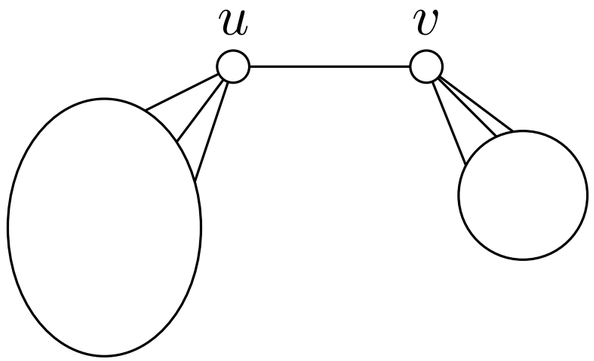

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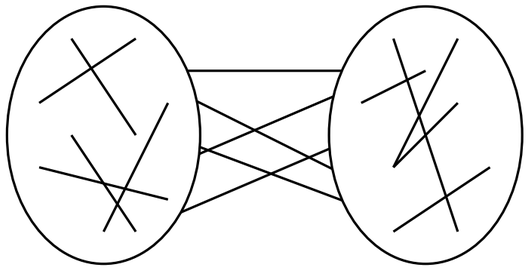

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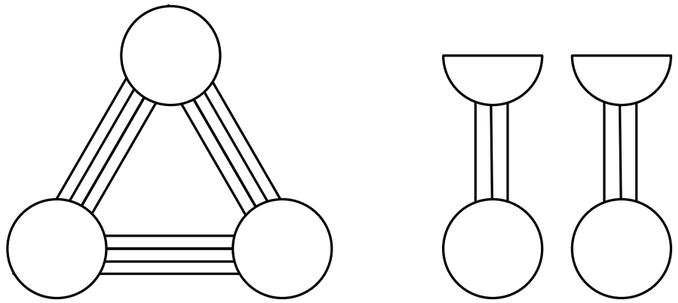

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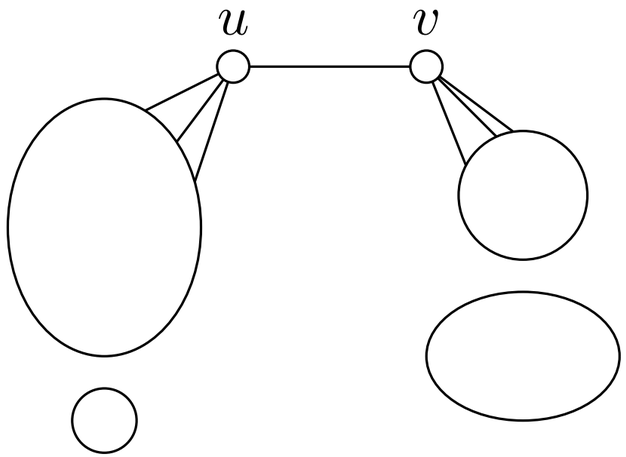

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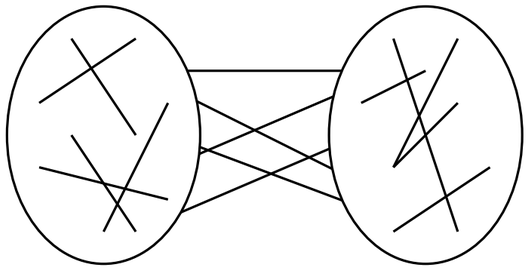

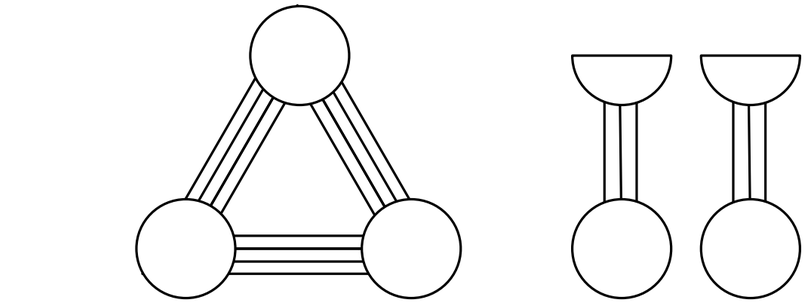

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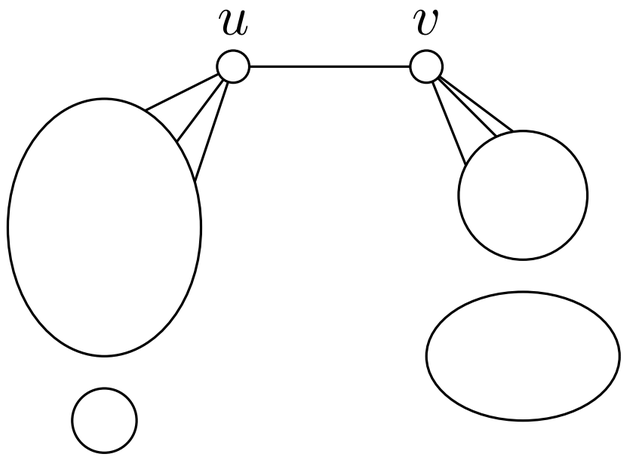
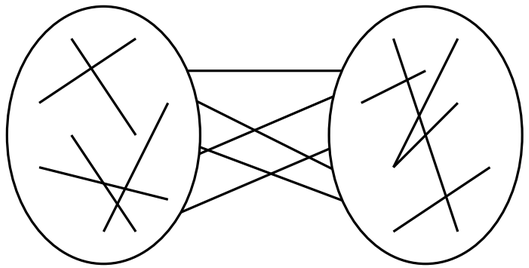
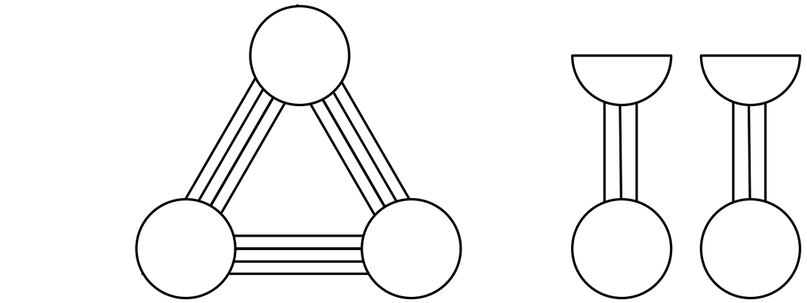


$$\square + \square + (\square + \square) \cdot \frac{\frac{1}{2} - \angle}{\triangle + \cdot} + (\square + \square) \cdot \frac{\frac{1}{2} - \angle}{\triangle + \cdot} + (\square + \square) \cdot \left[\left(\frac{\frac{1}{2} - \angle}{\triangle + \cdot} \right)^2 + \left(\frac{\frac{1}{2} - \angle}{\triangle + \cdot} \right)^2 \right] \geq \frac{2}{9}$$

Problem

Balanced bipartition





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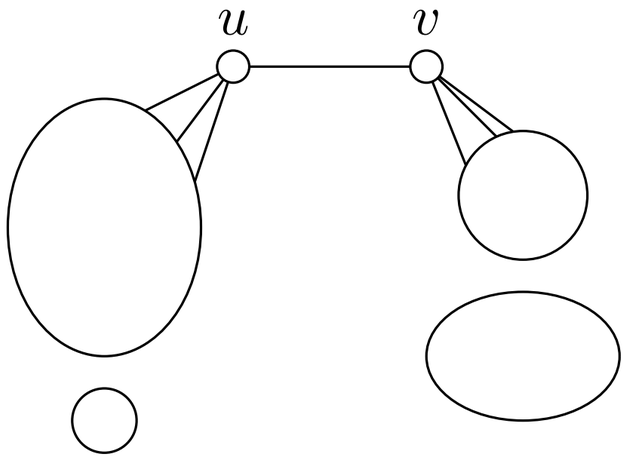

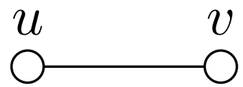

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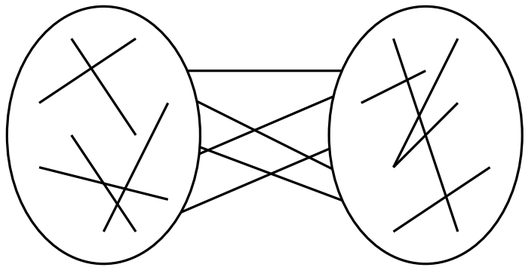

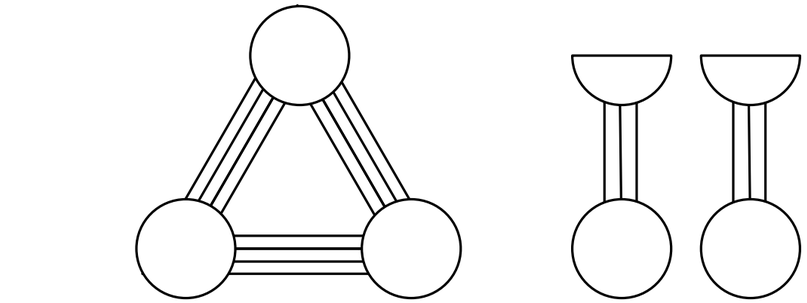

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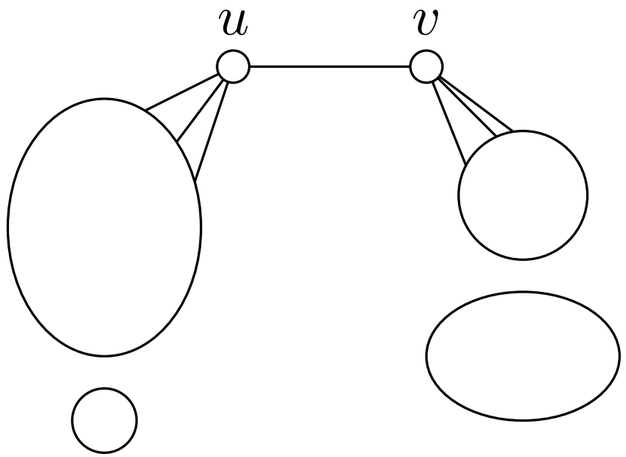

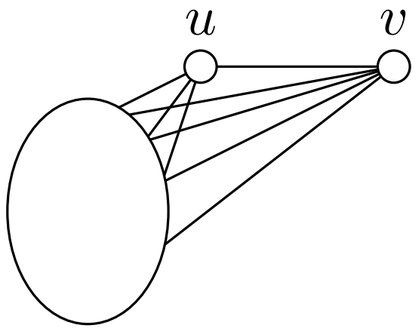

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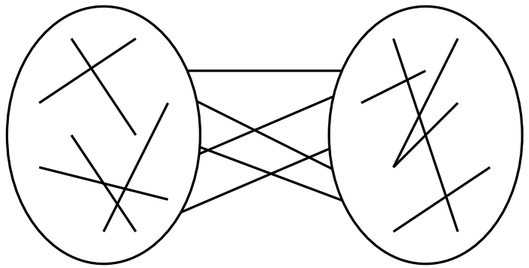
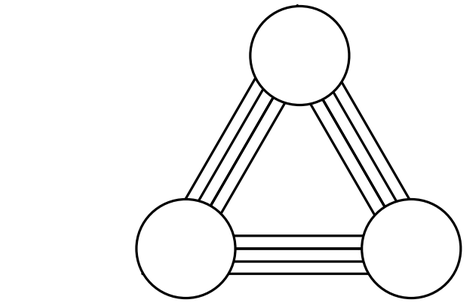
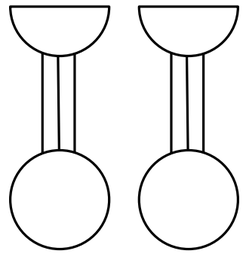
Problem

Balanced bipartition







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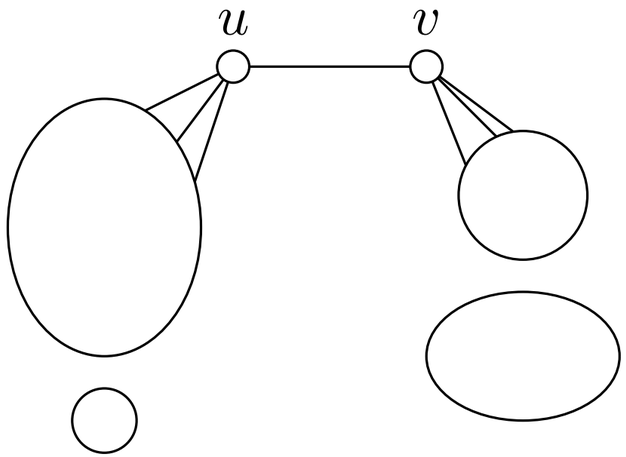

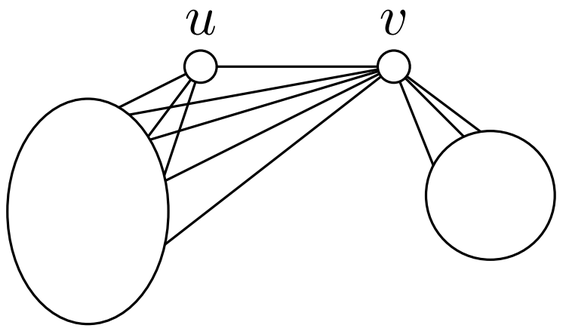

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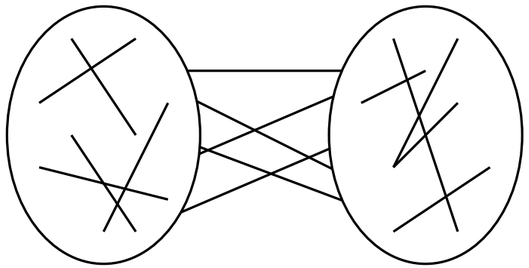

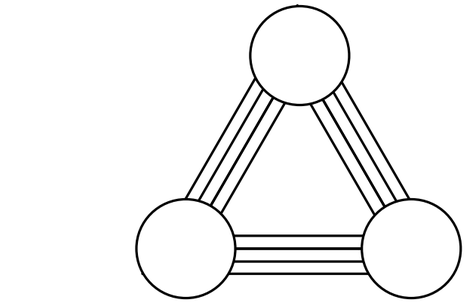

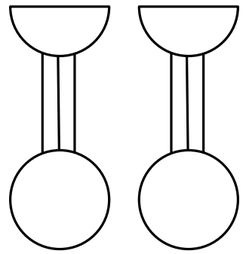

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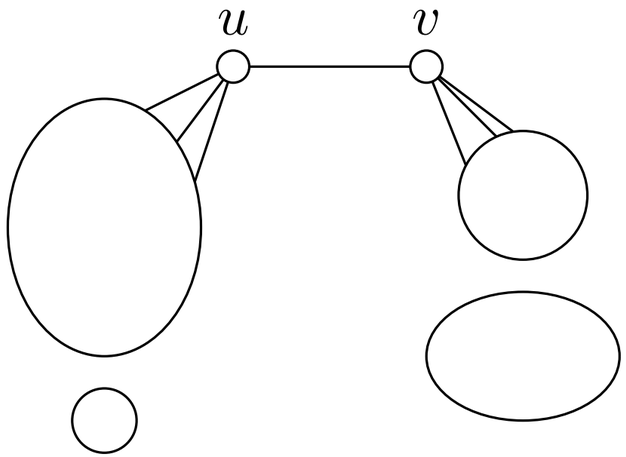

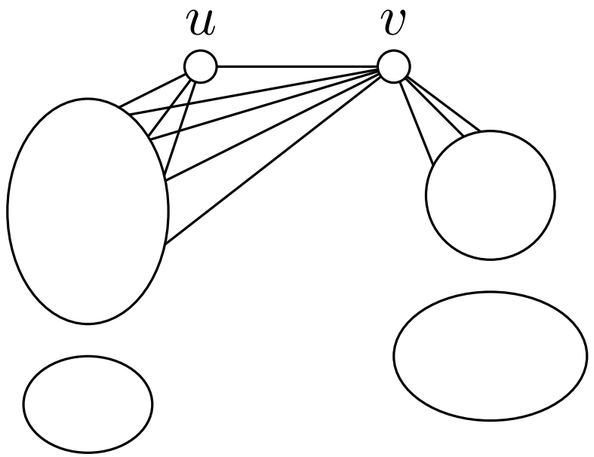

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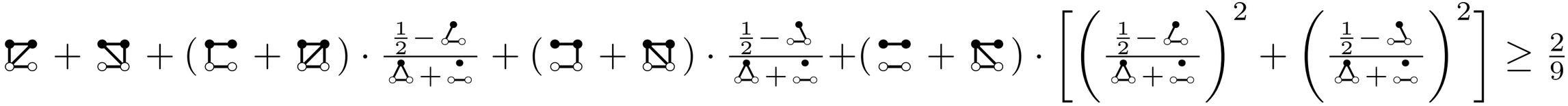
Inequalities



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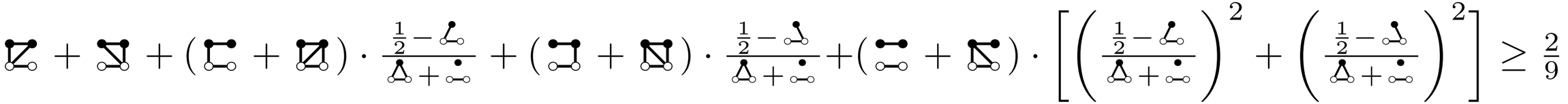

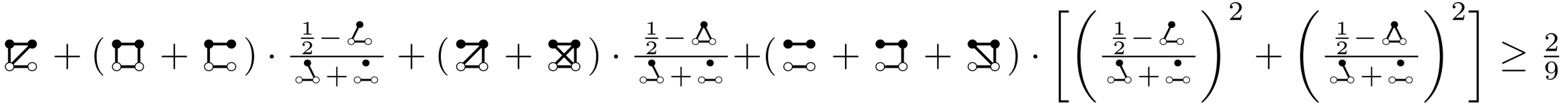

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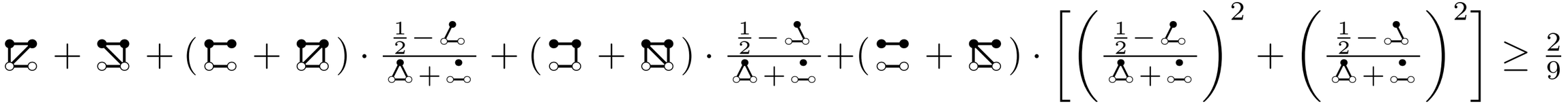
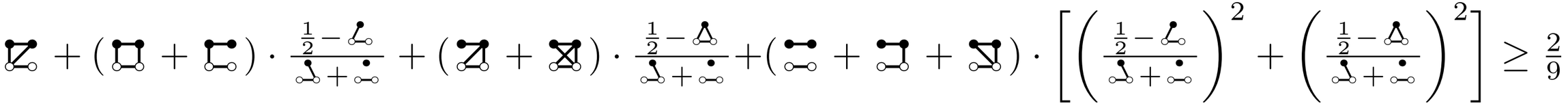
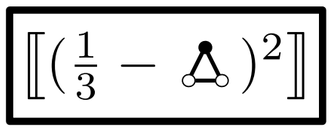
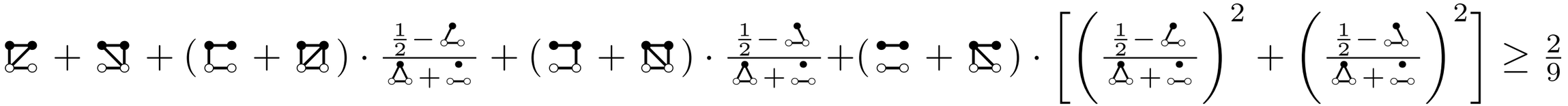
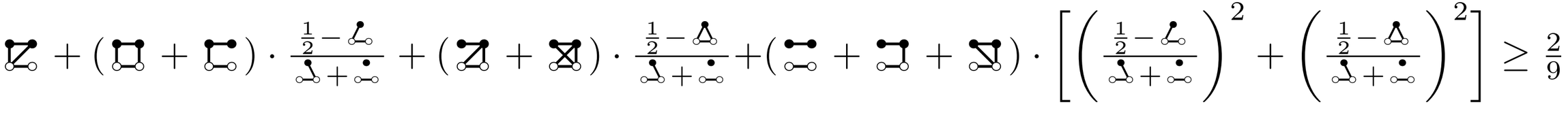


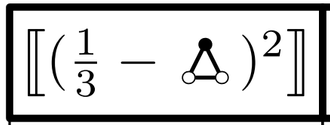




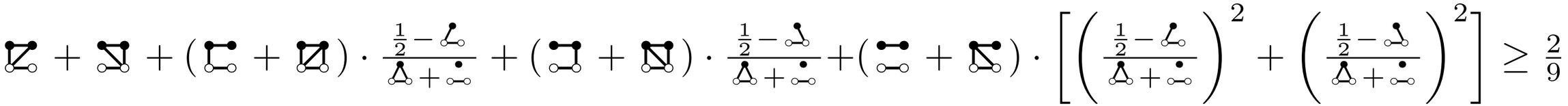
Inequalities

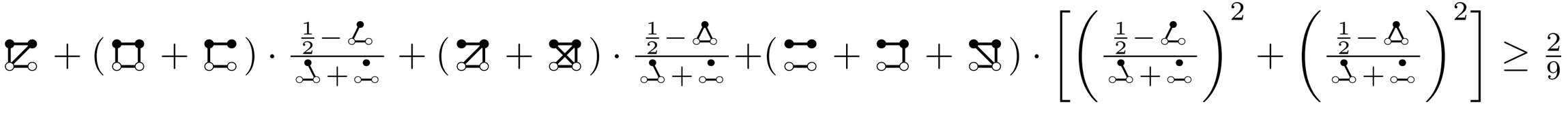




		0.005
		0.0363
		0.0314
		0.0805
		0.2203
		0.0098
		0.0031
		0.0968
		0.1278
		0.1703
		0.2232

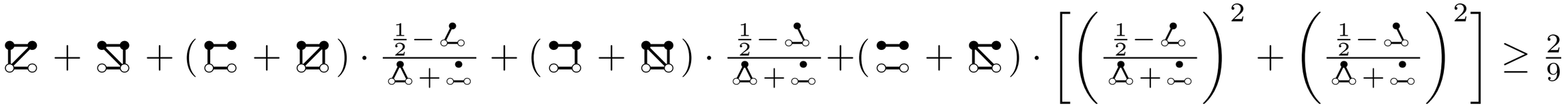
Inequalities

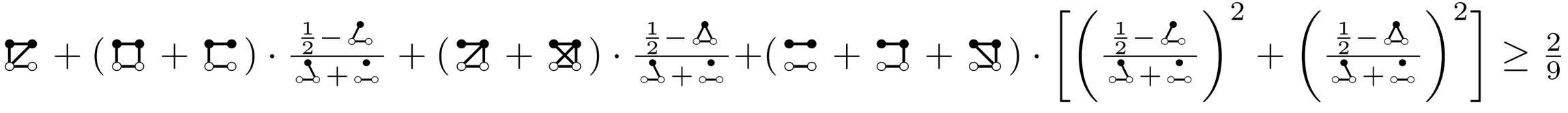


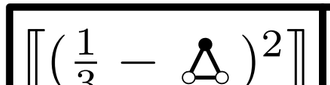


$[\![(\frac{1}{3}-$ $)^2]\!]$	0.0000	0.005
	0.0370	0.0363
	0.0000	0.0314
	0.0000	0.0805
	0.2963	0.2203
	0.0000	0.0098
	0.0000	0.0031
	0.0000	0.0968
	0.0000	0.1278
	0.2222	0.1703
	0.4444	0.2232

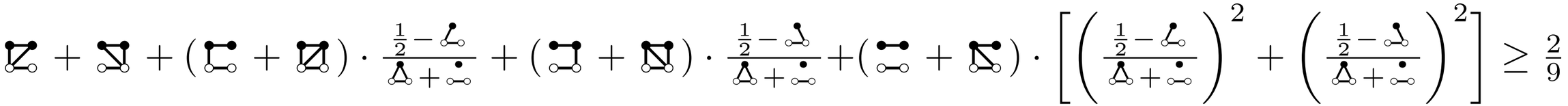
Inequalities

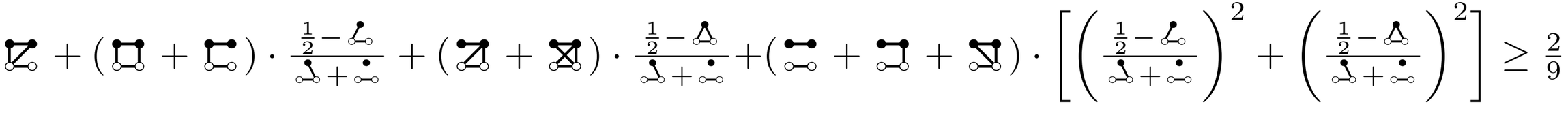




	0.0000	0.005
	0.0370	0.0363
	0.0000	0.0314
	0.0000	0.0805
	0.2963	0.2203
	0.0000	0.0098
	0.0000	0.0031
	0.0000	0.0968
	0.0000	0.1278
	0.2222	0.1703
	0.4444	0.2232

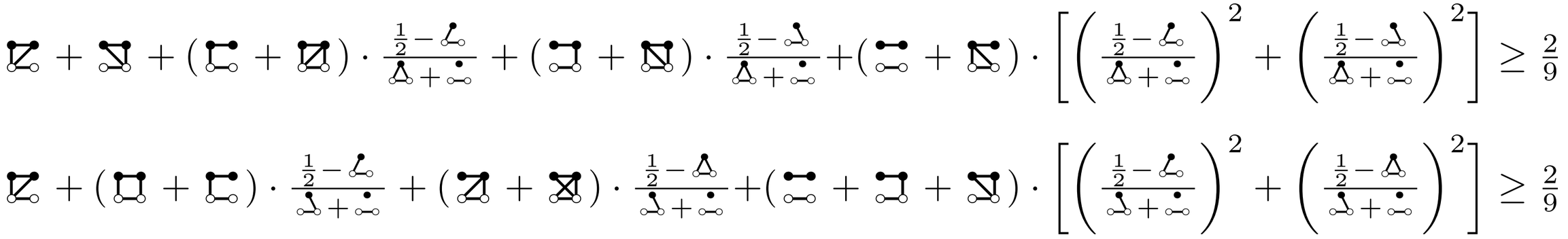
Inequalities





$[\![(\frac{1}{3}-$ $)^2]\!]$	0.0000	0.005	0.002
	0.0370	0.0363	0.0379
	0.0000	0.0314	0.0249
	0.0000	0.0805	0.0499
	0.2963	0.2203	0.2498
	0.0000	0.0098	0.0000
	0.0000	0.0031	0.0000
	0.0000	0.0968	0.0499
	0.0000	0.1278	0.0999
	0.2222	0.1703	0.1874
	0.4444	0.2232	0.2998

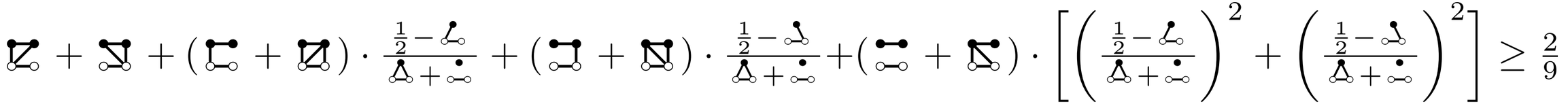
Inequalities

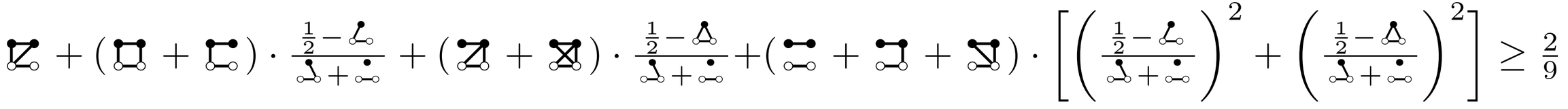


$[\![(\frac{1}{3} - \triangle)^2]\!]$	0.0000	0.005	0.002
::	0.0370	0.0363	0.0379
:|	0.0000	0.0314	0.0249
:⌊	0.0000	0.0805	0.0499
⌊⌊	0.2963	0.2203	0.2498
⌊△	0.0000	0.0098	0.0000
||	0.0000	0.0031	0.0000
⌊□	0.0000	0.0968	0.0499
△△	0.0000	0.1278	0.0999
□□	0.2222	0.1703	0.1874
⊠	0.4444	0.2232	0.2998

$\triangle$ = const

Inequalities



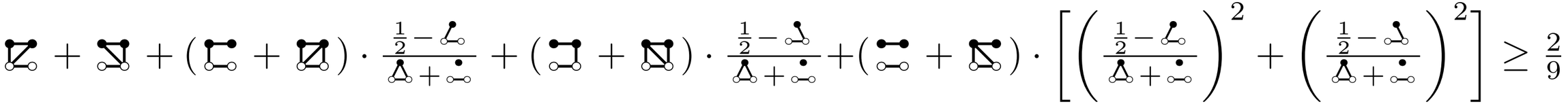


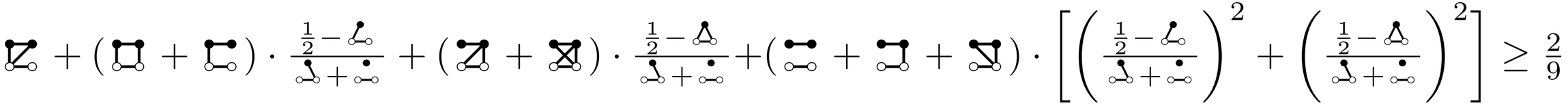
$[\![(\frac{1}{3}-\triangle)^2]\!]$	0.0000	0.005	0.002
::	0.0370	0.0363	0.0379
:|	0.0000	0.0314	0.0249
:⌊	0.0000	0.0805	0.0499
⟍	0.2963	0.2203	0.2498
◿	0.0000	0.0098	0.0000
||	0.0000	0.0031	0.0000
⌊⌊	0.0000	0.0968	0.0499
◿	0.0000	0.1278	0.0999
□	0.2222	0.1703	0.1874
⧅	0.4444	0.2232	0.2998

$\triangle$ = const

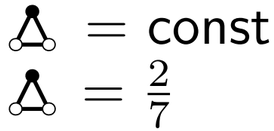
$\triangle = \frac{2}{7}$

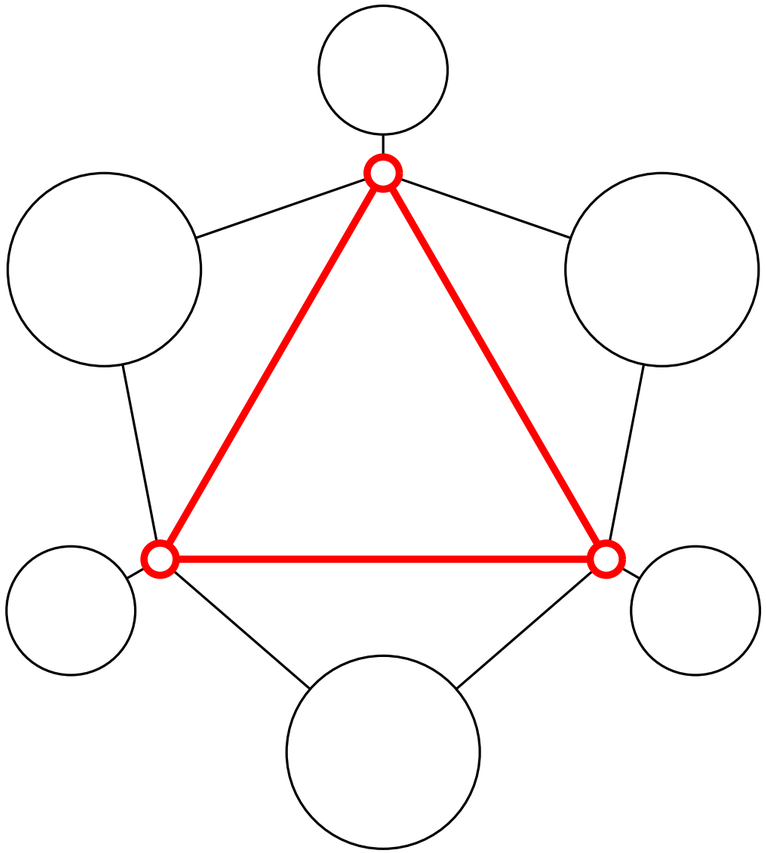
Inequalities



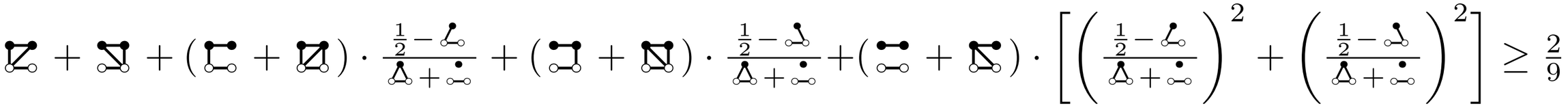


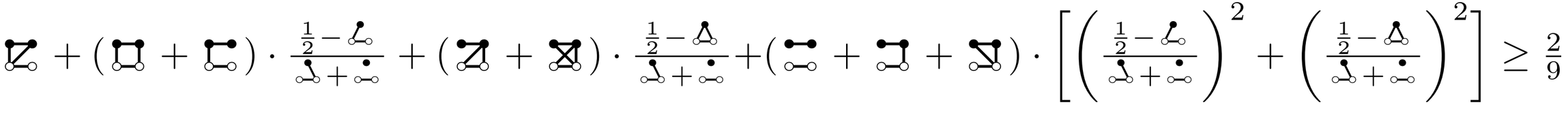
$[\![(\frac{1}{3}-\triangle)^2]\!]$	0.0000	0.005	0.002
::	0.0370	0.0363	0.0379
:|	0.0000	0.0314	0.0249
.|	0.0000	0.0805	0.0499
⟍⟍	0.2963	0.2203	0.2498
⟍△	0.0000	0.0098	0.0000
||	0.0000	0.0031	0.0000
.□	0.0000	0.0968	0.0499
△△	0.0000	0.1278	0.0999
□□	0.2222	0.1703	0.1874
⟍□	0.4444	0.2232	0.2998



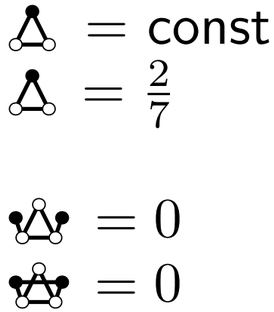


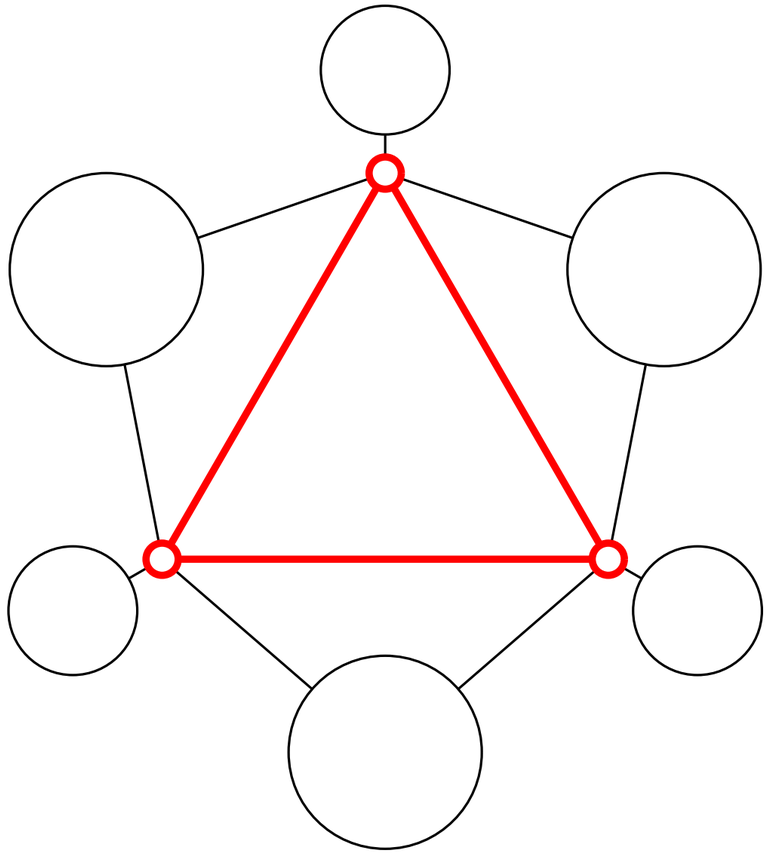
Inequalities



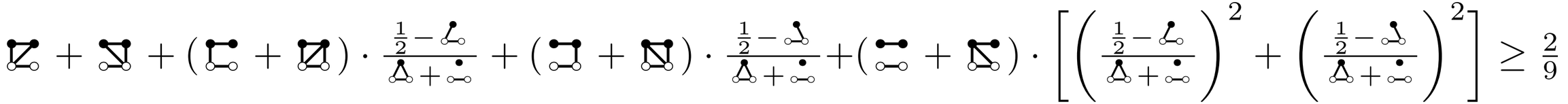


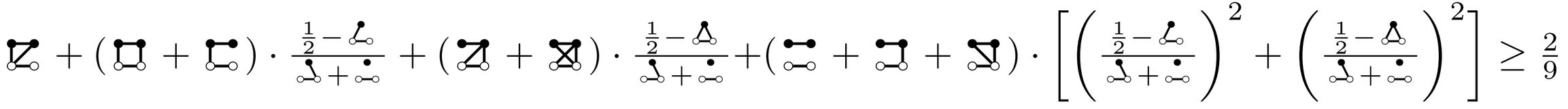
$[\![(\frac{1}{3} - \triangle)^2]\!]$	0.0000	0.005	0.002
::	0.0370	0.0363	0.0379
:|	0.0000	0.0314	0.0249
.|	0.0000	0.0805	0.0499
⟍	0.2963	0.2203	0.2498
◿	0.0000	0.0098	0.0000
||	0.0000	0.0031	0.0000
⊐	0.0000	0.0968	0.0499
◹	0.0000	0.1278	0.0999
□	0.2222	0.1703	0.1874
⧅	0.4444	0.2232	0.2998



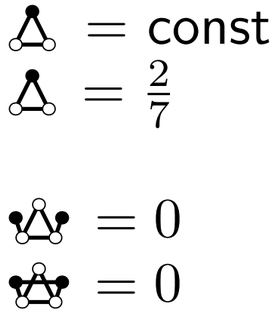


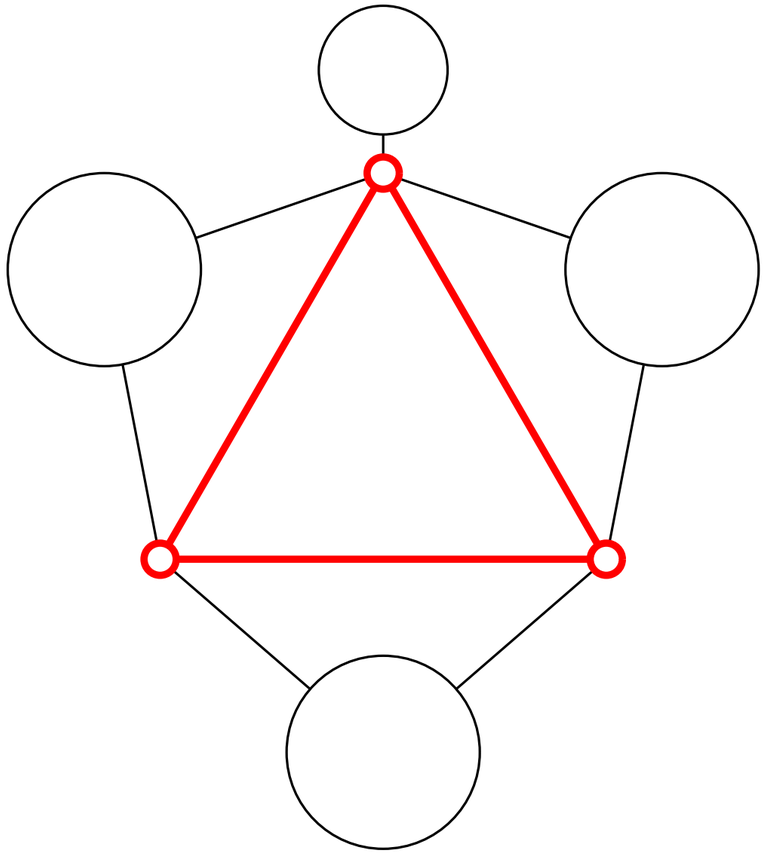
Inequalities



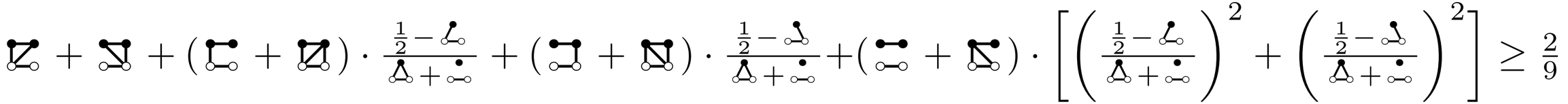


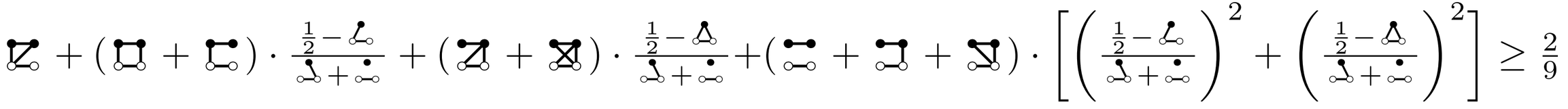
$[\![(\frac{1}{3} - \triangle)^2]\!]$	0.0000	0.005	0.002
::	0.0370	0.0363	0.0379
:|	0.0000	0.0314	0.0249
.|	0.0000	0.0805	0.0499
⟍	0.2963	0.2203	0.2498
◿	0.0000	0.0098	0.0000
||	0.0000	0.0031	0.0000
⊐	0.0000	0.0968	0.0499
◹	0.0000	0.1278	0.0999
□	0.2222	0.1703	0.1874
⧅	0.4444	0.2232	0.2998



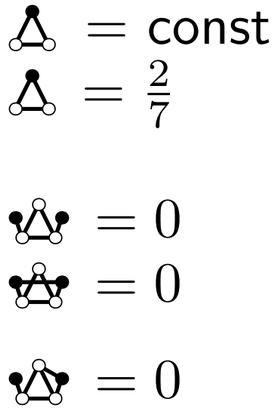


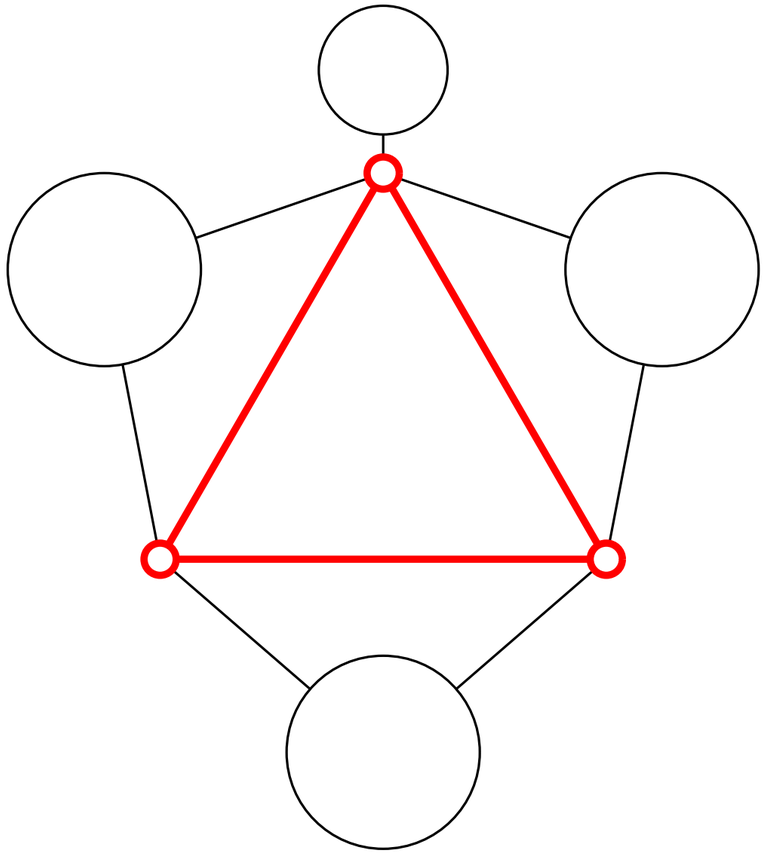
Inequalities



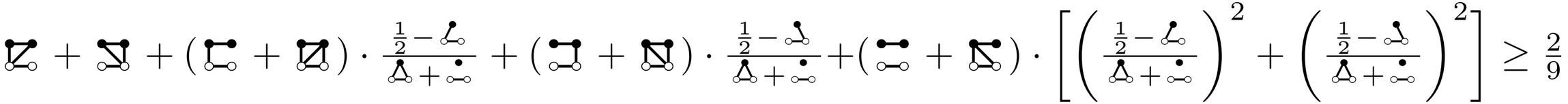


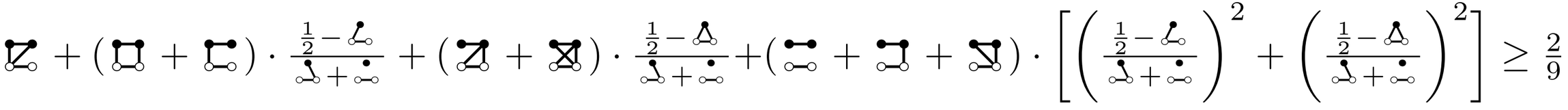
$[\![(\frac{1}{3} - \triangle)^2]\!]$	0.0000	0.005	0.002
::	0.0370	0.0363	0.0379
:|	0.0000	0.0314	0.0249
:L	0.0000	0.0805	0.0499
⋱	0.2963	0.2203	0.2498
◿	0.0000	0.0098	0.0000
||	0.0000	0.0031	0.0000
⊐	0.0000	0.0968	0.0499
◹	0.0000	0.1278	0.0999
□	0.2222	0.1703	0.1874
⧅	0.4444	0.2232	0.2998



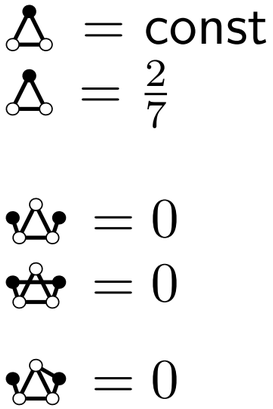


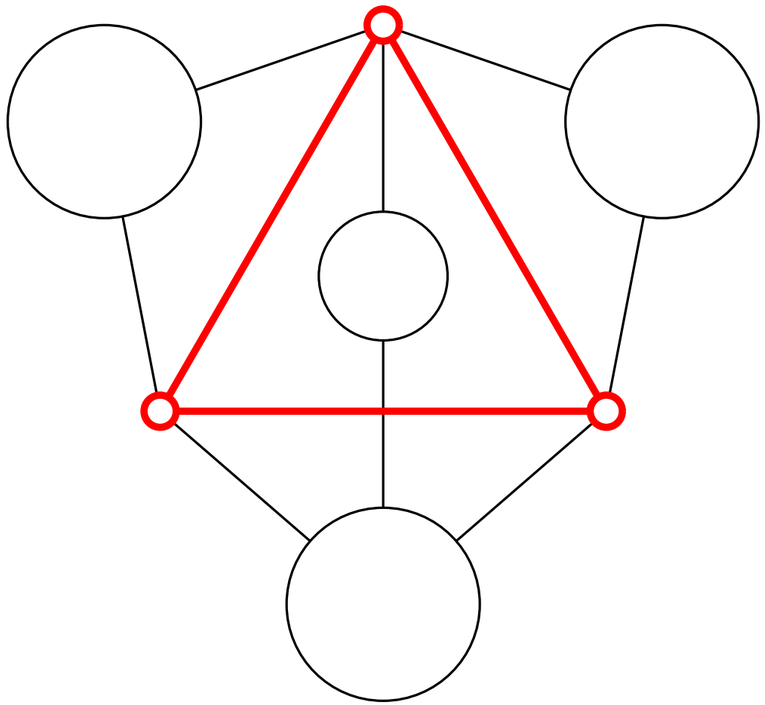
Inequalities



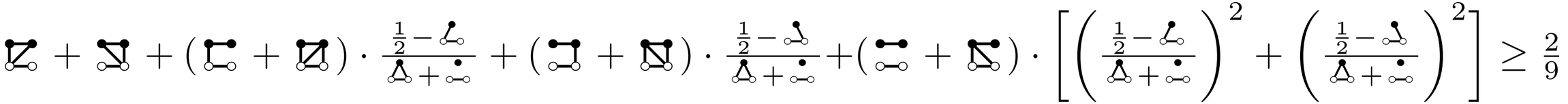


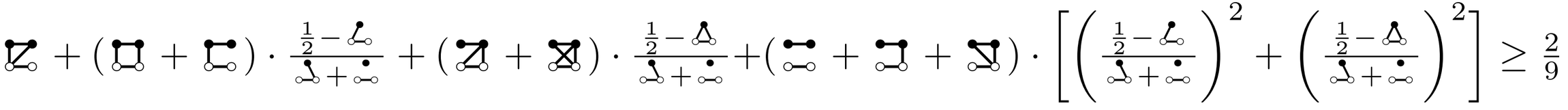
$[\![(\frac{1}{3}-\triangle)^2]\!]$	0.0000	0.005	0.002
::	0.0370	0.0363	0.0379
:|	0.0000	0.0314	0.0249
⌐|	0.0000	0.0805	0.0499
⌐⌐	0.2963	0.2203	0.2498
⌐◺	0.0000	0.0098	0.0000
||	0.0000	0.0031	0.0000
⌐□	0.0000	0.0968	0.0499
⌐◿	0.0000	0.1278	0.0999
□	0.2222	0.1703	0.1874
⌐⌐	0.4444	0.2232	0.2998





Inequalities





$[\![(\frac{1}{3} - \triangle)^2]\!]$	0.0000	0.005	0.002
::	0.0370	0.0363	0.0379
:|	0.0000	0.0314	0.0249
⌐|	0.0000	0.0805	0.0499
⌐⌐	0.2963	0.2203	0.2498
◿	0.0000	0.0098	0.0000
||	0.0000	0.0031	0.0000
⌐□	0.0000	0.0968	0.0499
◿□	0.0000	0.1278	0.0999
□	0.2222	0.1703	0.1874
⌐⌐□	0.4444	0.2232	0.2998

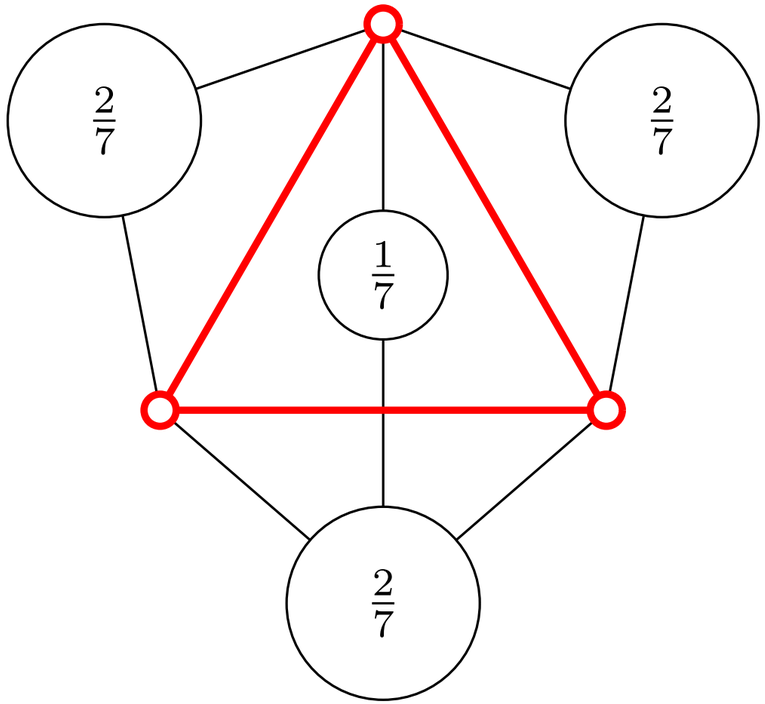
$\triangle$ = const

$\triangle = \frac{2}{7}$

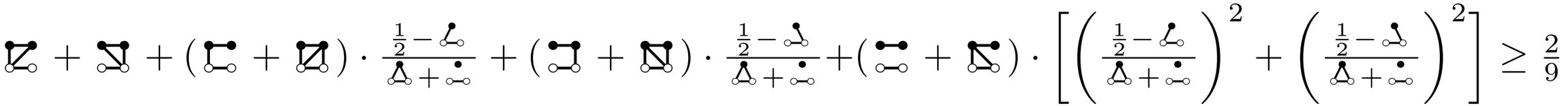
$= 0$

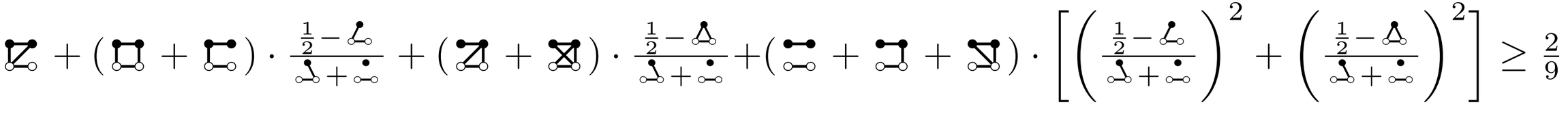
$= 0$

$= 0$

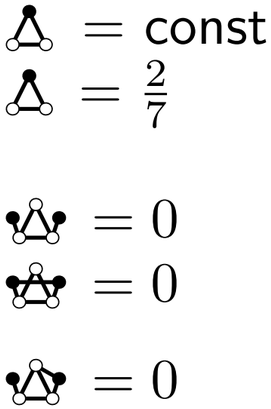


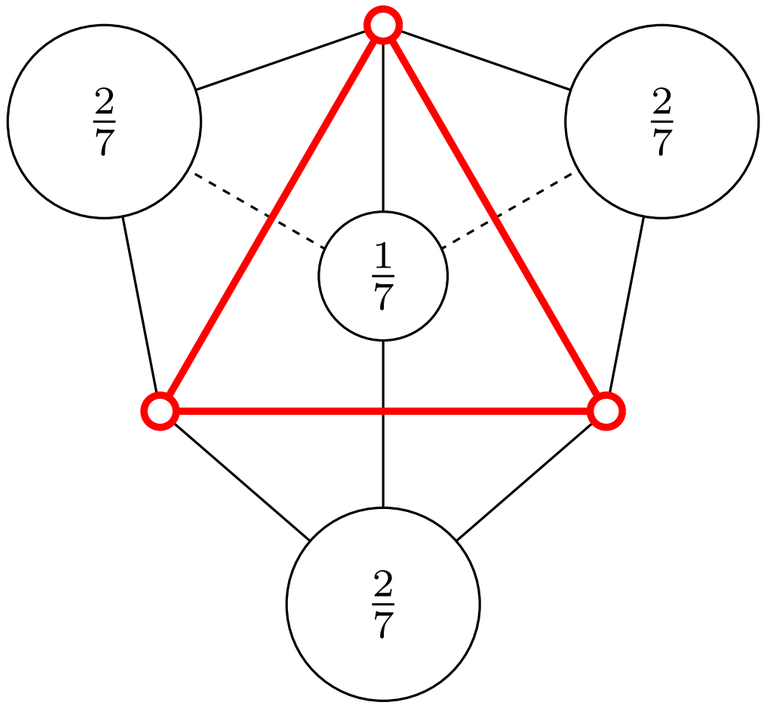
Inequalities



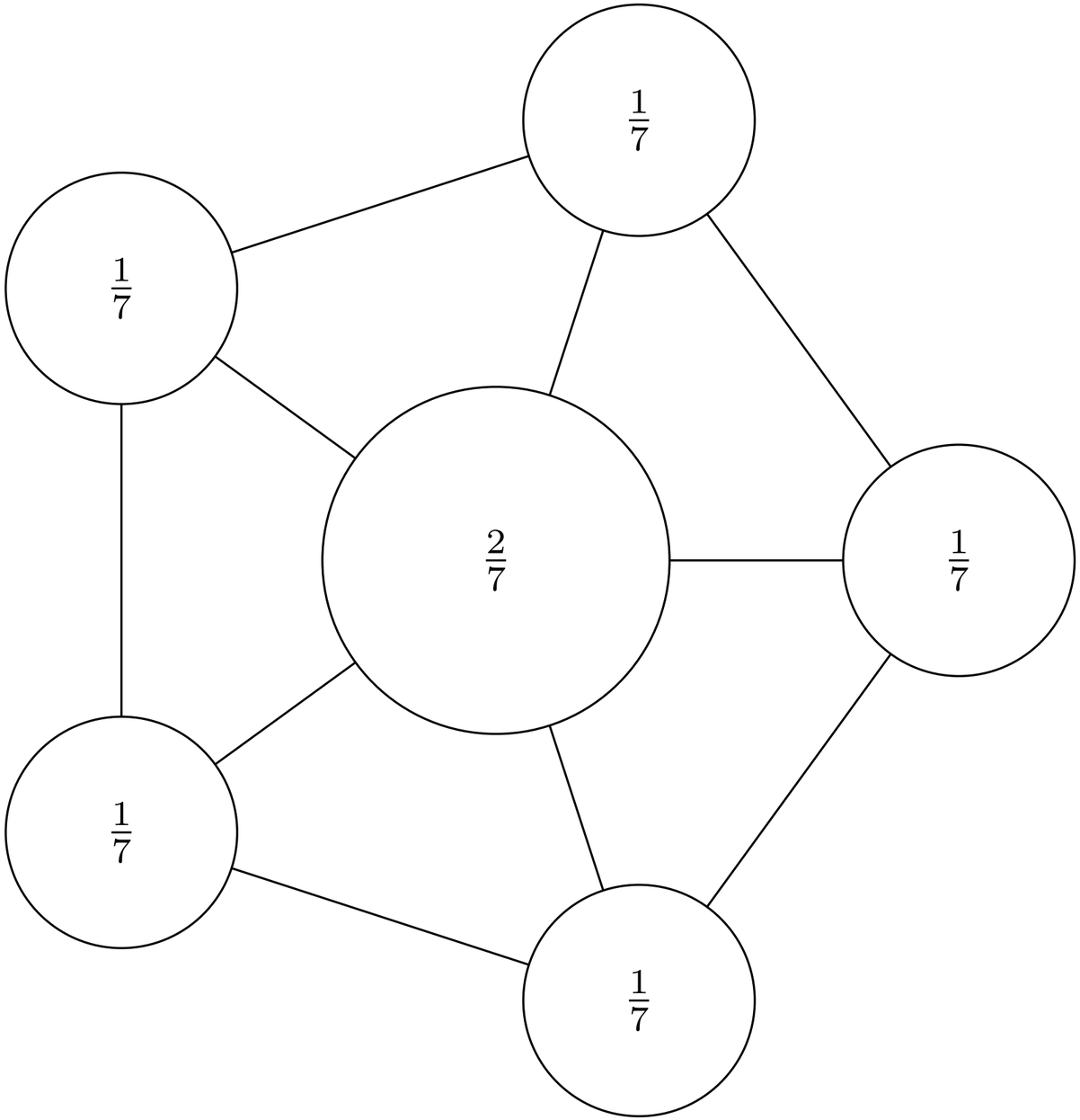


$[\![(\frac{1}{3} - \triangle)^2]\!]$	0.0000	0.005	0.002
	0.0370	0.0363	0.0379
	0.0000	0.0314	0.0249
	0.0000	0.0805	0.0499
	0.2963	0.2203	0.2498
	0.0000	0.0098	0.0000
	0.0000	0.0031	0.0000
	0.0000	0.0968	0.0499
	0.0000	0.1278	0.0999
	0.2222	0.1703	0.1874
	0.4444	0.2232	0.2998

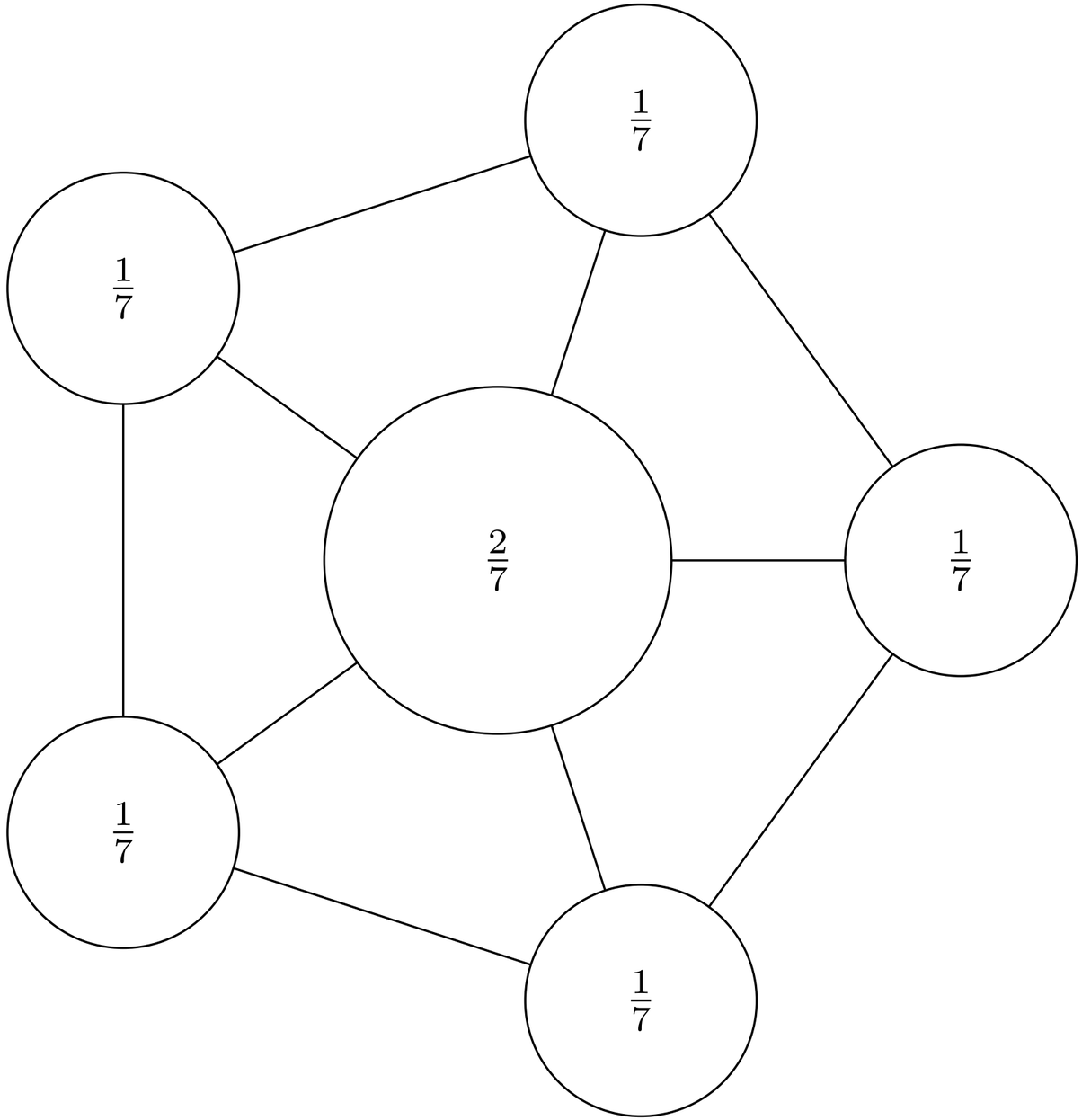




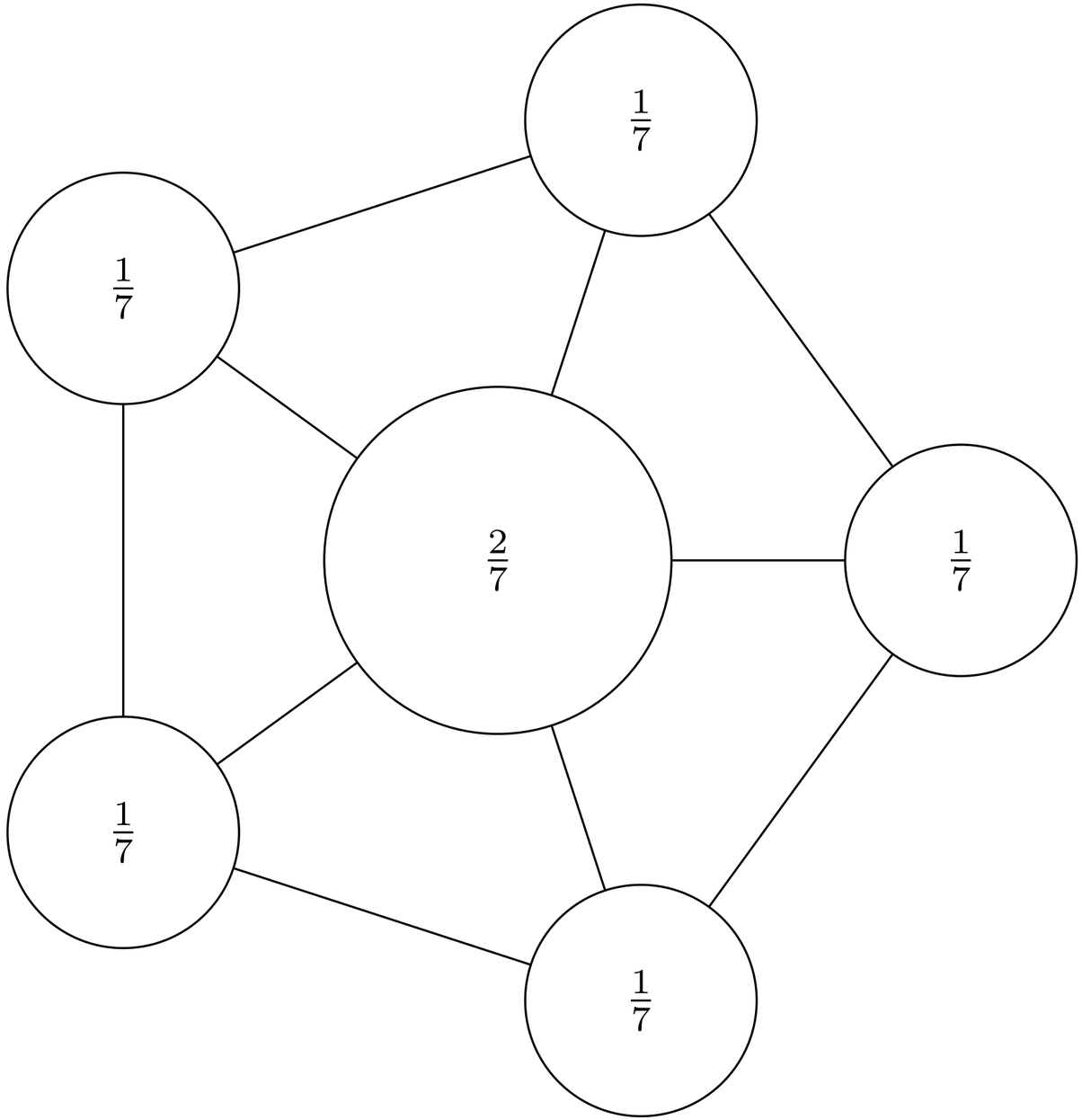
Counterexample



Counterexample



Counterexample



Thank you!