

A note on computer-assisted proofs in flag algebras

Ref. Ignacy Buczek

Graphons

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$$f(x) = -\frac{x^4}{4} + 7x \quad f_{max} = \frac{21\sqrt[3]{7}}{4}$$

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$$G : \{0, \dots, n-1\}^2 \mapsto \{0, 1\}$$

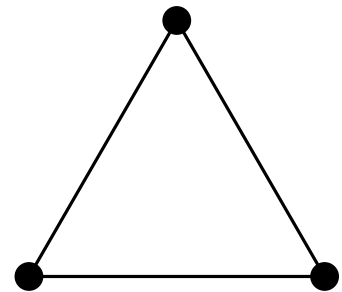
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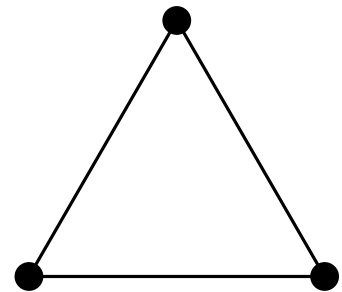
0	1	1
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$$W : [0, 1]^2 \mapsto [0, 1]$$

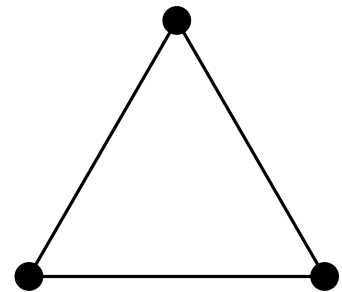
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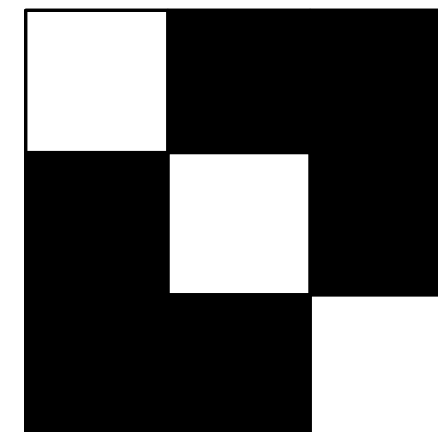
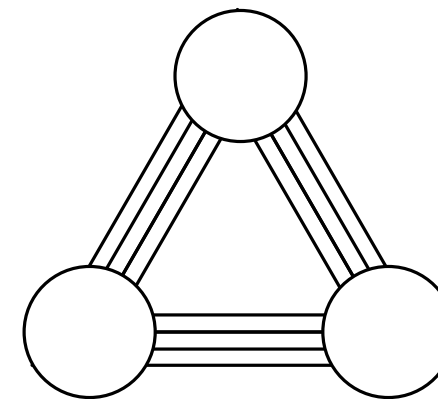
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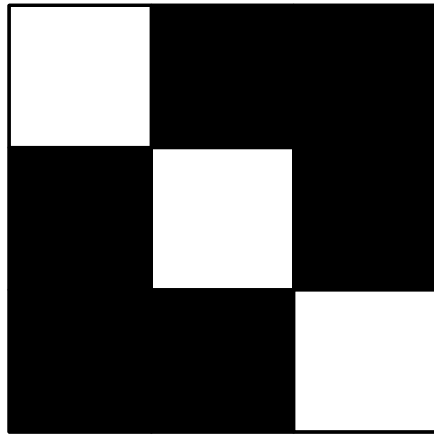
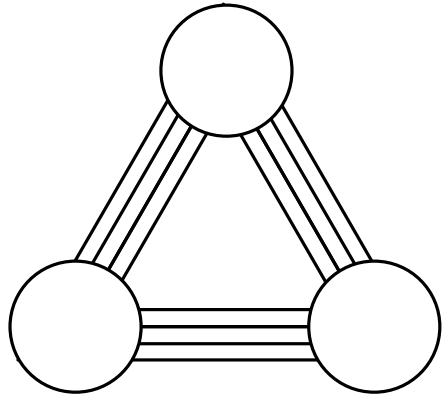
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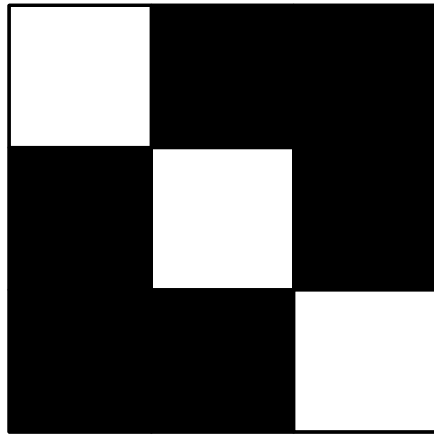
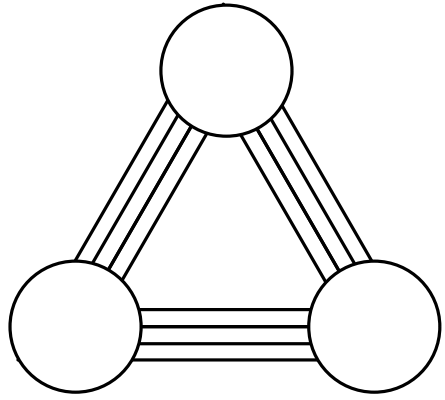


Flags



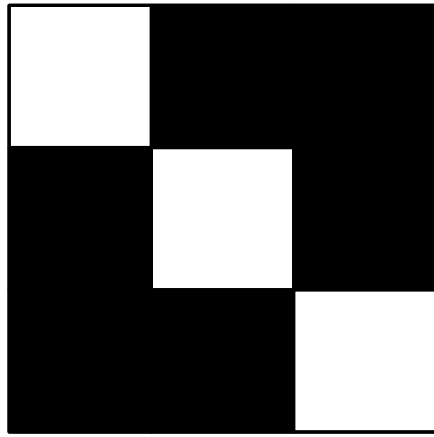
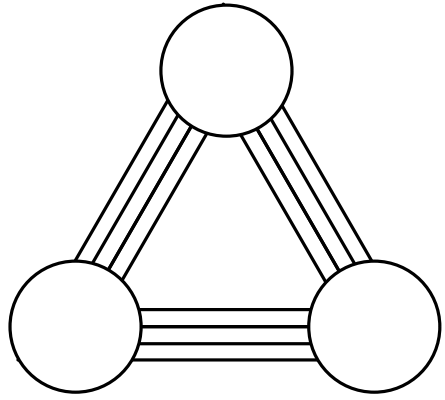
A large dashed rectangular box, divided into four quadrants by a horizontal and a vertical dashed line, intended for drawing or writing.

Flags



$$\Delta = \frac{6}{27}$$

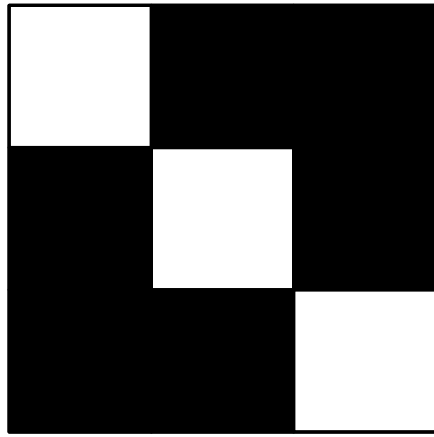
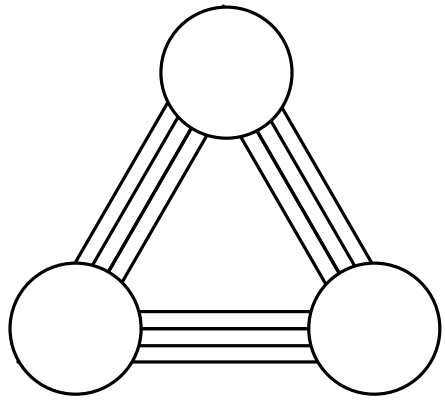
Flags



$$\triangle = \frac{6}{27}$$

$$\text{---} =$$

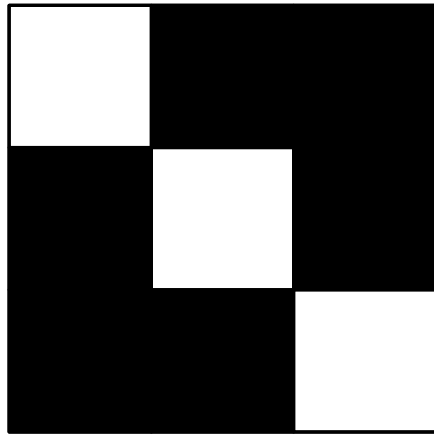
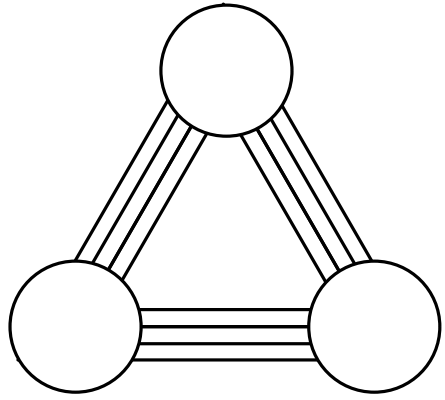
Flags



$$\triangle = \frac{6}{27}$$

$$\text{rod} = \frac{2}{3}$$

Flags

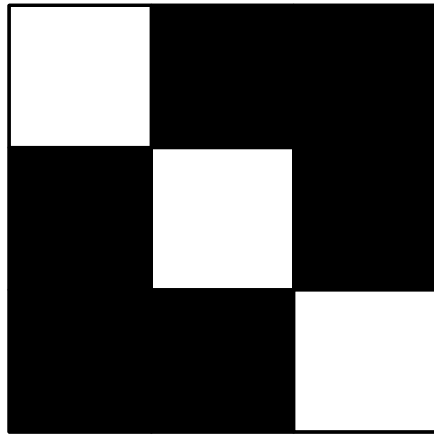
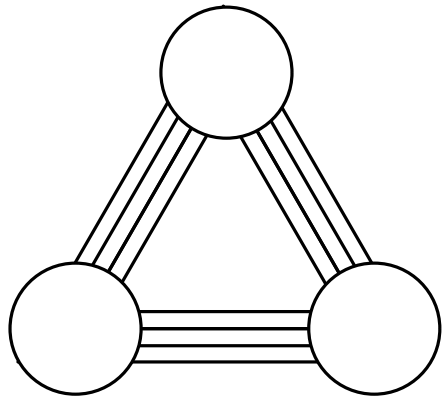


$$\triangle = \frac{6}{27}$$

$$/ = \frac{2}{3}$$

$$\triangle + / = 1$$

Flags

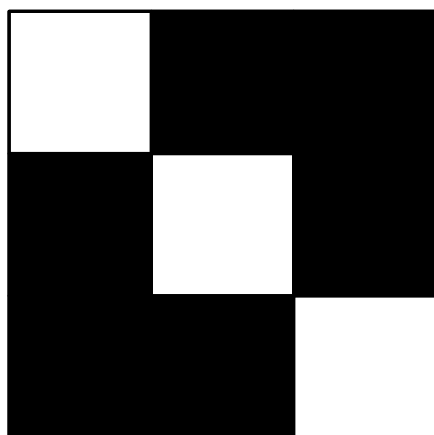
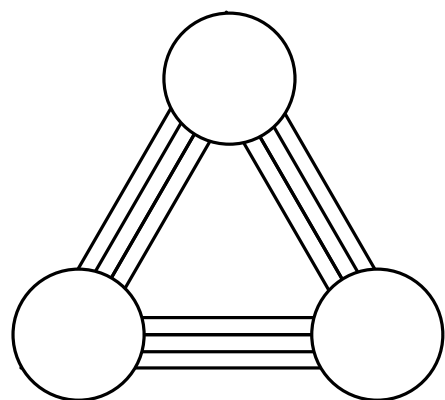


$$\Delta = \frac{6}{27} \quad \text{and} \quad \text{edge} = \frac{2}{3}$$

$$\text{edge} + \text{edge} = 1$$

$$\text{edge} = \Delta + \frac{2}{3} \text{edge} + \frac{1}{3} \text{edge}$$

Flags



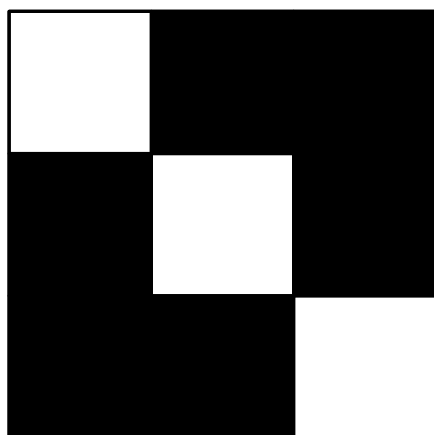
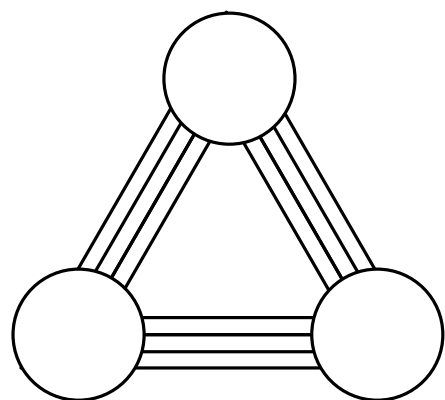
$$\triangle = \frac{6}{27} \quad \text{rod} = \frac{2}{3}$$

$$\text{rod} + \text{rod} = 1$$

$$\text{rod} = \triangle + \frac{2}{3} \text{rod} + \frac{1}{3} \text{rod}$$

$$\text{rod} \cdot \text{rod} =$$

Flags



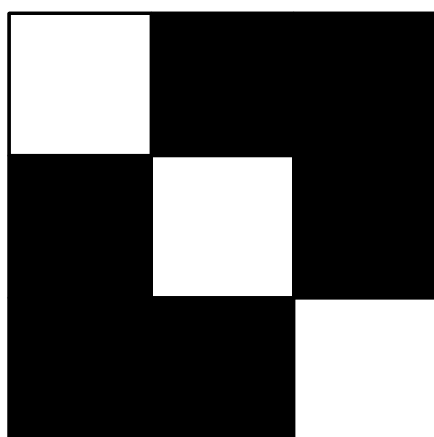
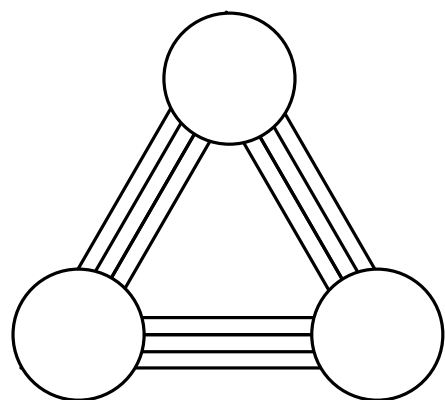
$$\Delta = \frac{6}{27} \quad \text{edge} = \frac{2}{3}$$

$$\text{dot} + \text{edge} = 1$$

$$\text{edge} = \Delta + \frac{2}{3} \text{edge} + \frac{1}{3} \text{dot}$$

$$\text{edge} \cdot \text{dot} = \frac{1}{6} \text{dot} \cdot \text{dot} + \frac{1}{3} \text{dot} \cdot \text{edge} + \frac{1}{6} \text{edge} \cdot \text{edge} + \frac{1}{2} \text{edge} \cdot \text{dot} + \frac{1}{2} \text{edge} \cdot \text{edge} + \frac{1}{3} \text{edge} \cdot \text{edge} + \frac{1}{6} \text{edge} \cdot \text{edge}$$

Flags



$$\Delta = \frac{6}{27} \quad \text{flag} = \frac{2}{3}$$

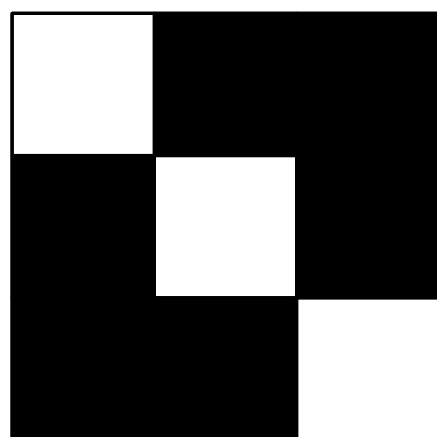
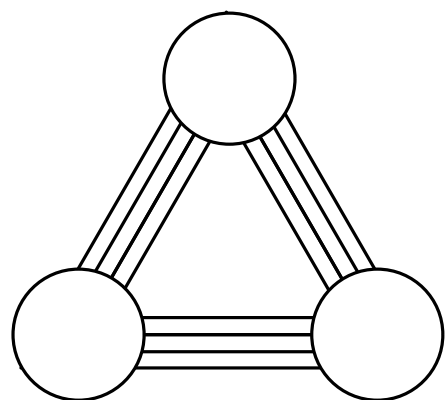
$$\text{flag} + \text{flag} = 1$$

$$\text{flag} = \Delta + \frac{2}{3} \text{flag} + \frac{1}{3} \text{flag}$$

$$\text{flag} \cdot \text{flag} = \frac{1}{6} \text{flag} + \frac{1}{3} \text{flag} + \frac{1}{6} \text{flag} + \frac{1}{2} \text{flag} + \frac{1}{2} \text{flag} + \frac{1}{3} \text{flag} + \frac{1}{6} \text{flag}$$

Δ -free graphs have at most $\frac{n^2}{4}$ edges.

Flags



$$\Delta = \frac{6}{27} \quad \rho = \frac{2}{3}$$

$$\rho + \rho = 1$$

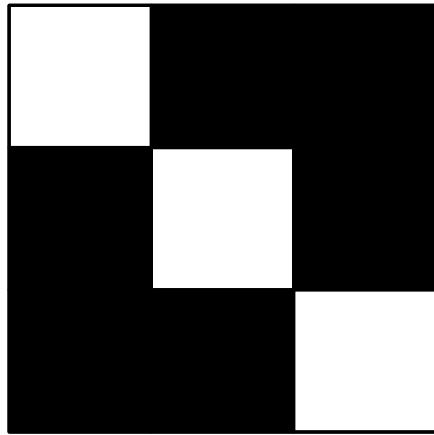
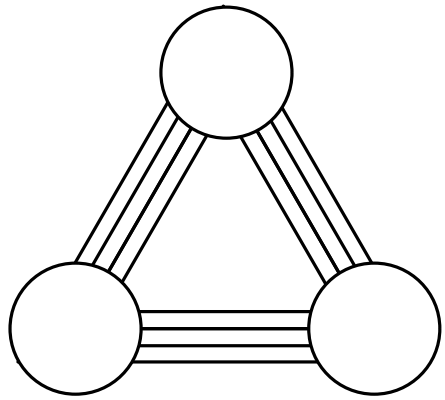
$$\rho = \Delta + \frac{2}{3} \Lambda + \frac{1}{3} \Gamma$$

$$\rho \cdot \rho = \frac{1}{6} \Gamma\Gamma + \frac{1}{3} \Gamma\Lambda + \frac{1}{6} \Lambda\Lambda + \frac{1}{2} \Gamma\Delta + \frac{1}{2} \Lambda\Delta + \frac{1}{3} \Delta\Delta + \frac{1}{6} \Delta\Delta$$

Δ -free graphs have at most $\frac{n^2}{4}$ edges.

$$\Delta = 0 \rightarrow \rho \leq \frac{1}{2}$$

Flags



$$\Delta = \frac{6}{27} \quad f = \frac{2}{3}$$

$$f + f = 1$$

$$f = \Delta + \frac{2}{3} \Delta + \frac{1}{3} f$$

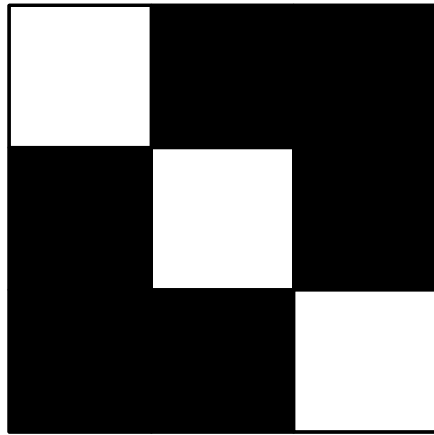
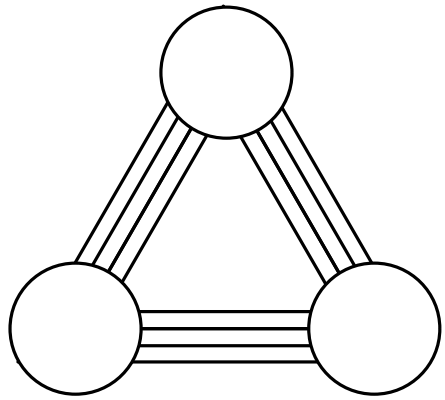
$$f \cdot f = \frac{1}{6} f + \frac{1}{3} f + \frac{1}{6} f + \frac{1}{2} f + \frac{1}{2} f + \frac{1}{3} f + \frac{1}{6} f$$

Δ -free graphs have at most $\frac{n^2}{4}$ edges.

$$\Delta = 0 \rightarrow f \leq \frac{1}{2}$$

$$\Delta(G) \leq \frac{n}{2} \quad f \leq \frac{1}{2}$$

Flags



$$\Delta = \frac{6}{27} \quad f = \frac{2}{3}$$

$$f + f = 1$$

$$f = \Delta + \frac{2}{3} \Lambda + \frac{1}{3} \bullet\bullet$$

$$f \cdot \bullet\bullet = \frac{1}{6} \bullet\bullet\bullet + \frac{1}{3} \bullet\bullet\bullet + \frac{1}{6} \bullet\bullet\bullet + \frac{1}{2} \bullet\bullet\bullet + \frac{1}{2} \bullet\bullet\bullet + \frac{1}{3} \bullet\bullet\bullet + \frac{1}{6} \bullet\bullet\bullet$$

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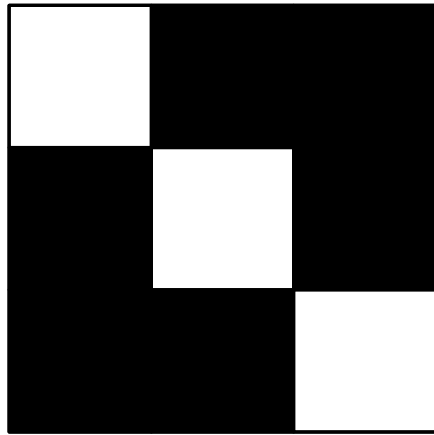
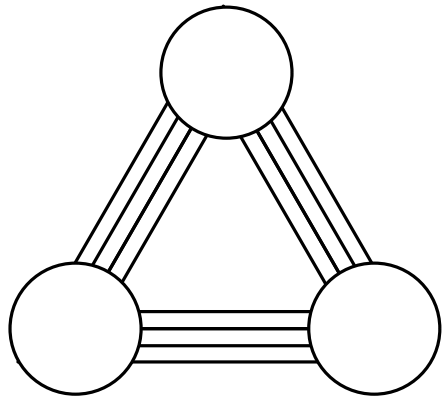
$$\bullet\bullet + \bullet\bullet + \bullet\bullet + \Delta = 1$$

$$f = \nabla + \frac{1}{2} \bullet\bullet + \frac{1}{2} \bullet\bullet$$

$$\bullet\bullet \cdot f = \frac{1}{2} \bullet\bullet + \frac{1}{2} \bullet\bullet$$

$$[\nabla] = \frac{1}{3} \Lambda$$

Flags



Input:

$$\Delta \leq 0$$

$$\Delta = \frac{6}{27} \quad \rho = \frac{2}{3}$$

$$\rho + \rho = 1$$

$$\rho = \Delta + \frac{2}{3} \Lambda + \frac{1}{3} \bullet\bullet$$

$$\rho \cdot \rho = \frac{1}{6} \bullet\bullet\bullet + \frac{1}{3} \bullet\bullet\bullet + \frac{1}{6} \bullet\bullet\bullet + \frac{1}{2} \bullet\bullet\bullet + \frac{1}{2} \bullet\bullet\bullet + \frac{1}{3} \bullet\bullet\bullet + \frac{1}{6} \bullet\bullet\bullet$$

Δ -free graphs have at most $\frac{n^2}{4}$ edges.

$$\Delta = 0 \rightarrow \rho \leq \frac{1}{2}$$

$$\Delta(G) \leq \frac{n}{2} \quad \rho \leq \frac{1}{2}$$

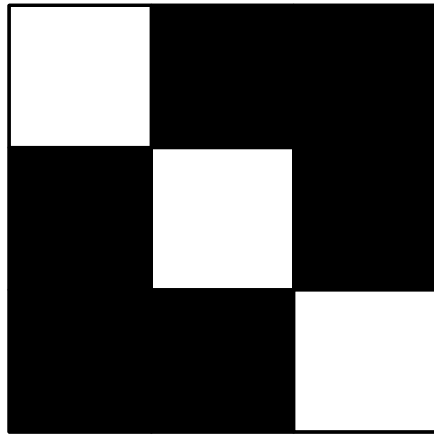
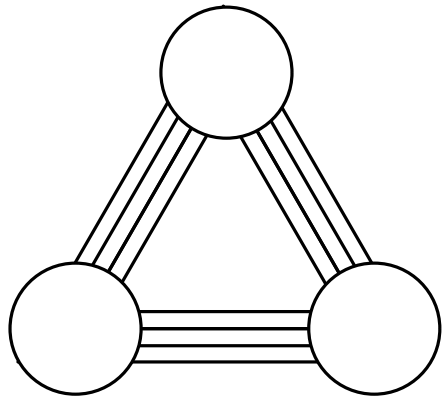
$$\bullet\bullet + \bullet\bullet + \bullet\bullet + \Delta = 1$$

$$\rho = \nabla + \frac{1}{2} \nabla + \frac{1}{2} \bullet\bullet$$

$$\bullet \cdot \rho = \frac{1}{2} \nabla + \frac{1}{2} \bullet\bullet$$

$$\llbracket \nabla \rrbracket = \frac{1}{3} \Lambda$$

Flags



Input:

$$\Delta \leq 0$$

Question:

$$f \leq ?$$

$$\Delta = \frac{6}{27} \quad f = \frac{2}{3}$$

$$f + f = 1$$

$$f = \Delta + \frac{2}{3} \Lambda + \frac{1}{3} \text{---}$$

$$f \cdot f = \frac{1}{6} \text{---} + \frac{1}{3} \text{---} + \frac{1}{6} \text{---} + \frac{1}{2} \text{---} + \frac{1}{2} \text{---} + \frac{1}{3} \text{---} + \frac{1}{6} \text{---}$$

Δ -free graphs have at most $\frac{n^2}{4}$ edges.

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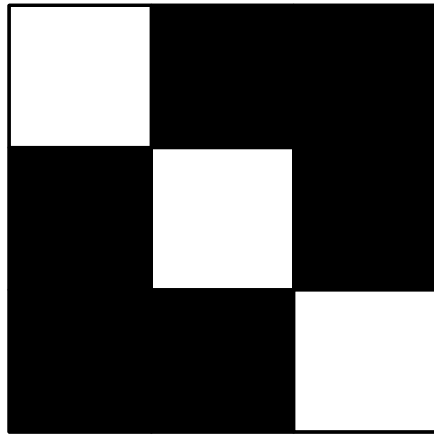
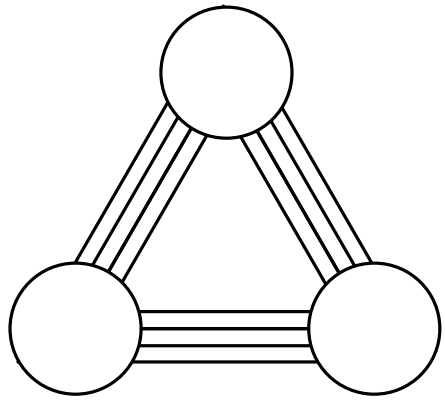
$$\text{---} + \text{---} + \text{---} + \Delta = 1$$

$$f = \nabla + \frac{1}{2} \text{---} + \frac{1}{2} \text{---}$$

$$\text{---} \cdot f = \frac{1}{2} \text{---} + \frac{1}{2} \text{---}$$

$$\llbracket \nabla \rrbracket = \frac{1}{3} \Lambda$$

Flags



Input:

$$\Delta \leq 0$$

Question:

$$f \leq ?$$

Output:

$$f \leq \frac{1}{2} + \varepsilon$$

$$\Delta = \frac{6}{27} \quad f = \frac{2}{3}$$

$$f + f = 1$$

$$f = \Delta + \frac{2}{3} \Lambda + \frac{1}{3} \text{---}$$

$$f \cdot f = \frac{1}{6} \text{---} + \frac{1}{3} \text{---} + \frac{1}{6} \text{---} + \frac{1}{2} \text{---} + \frac{1}{2} \text{---} + \frac{1}{3} \text{---} + \frac{1}{6} \text{---}$$

Δ -free graphs have at most $\frac{n^2}{4}$ edges.

$$\Delta = 0 \rightarrow f \leq \frac{1}{2}$$

$$\Delta(G) \leq \frac{n}{2} \quad f \leq \frac{1}{2}$$

$$\text{---} + \text{---} + \text{---} + \Delta = 1$$

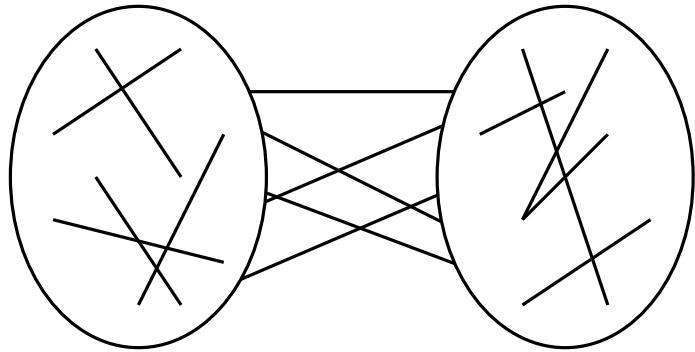
$$f = \nabla + \frac{1}{2} \text{---} + \frac{1}{2} \text{---}$$

$$\text{---} \cdot f = \frac{1}{2} \text{---} + \frac{1}{2} \text{---}$$

$$\llbracket \nabla \rrbracket = \frac{1}{3} \Lambda$$

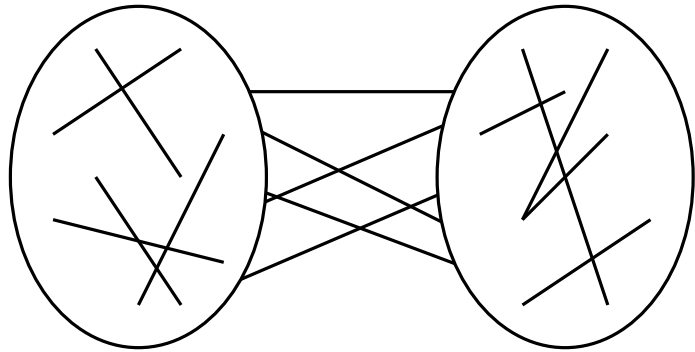
Problem

Balanced bipartition



Problem

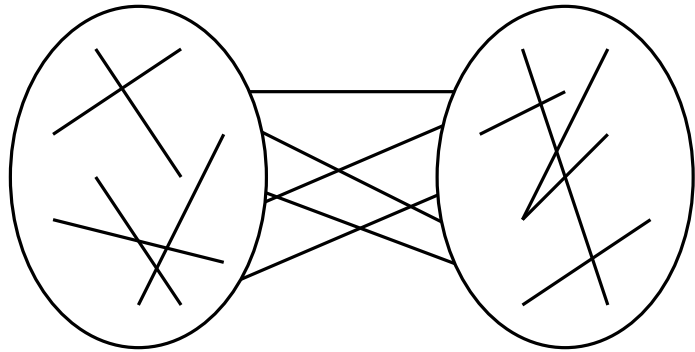
Balanced bipartition



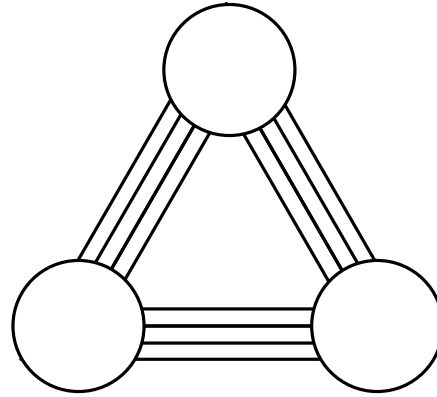
\square -free, $\delta(G) \geq \frac{n}{2}$, distance at most $\frac{n^2}{9}$

Problem

Balanced bipartition

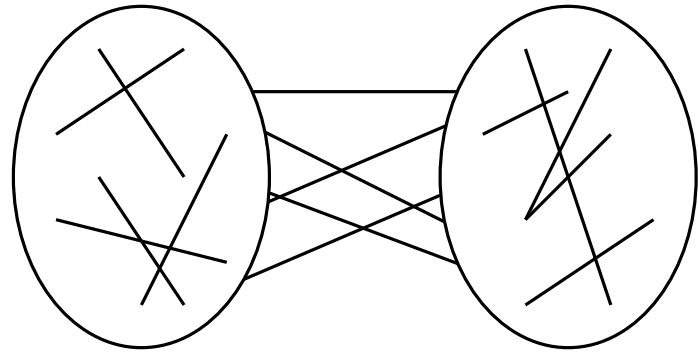


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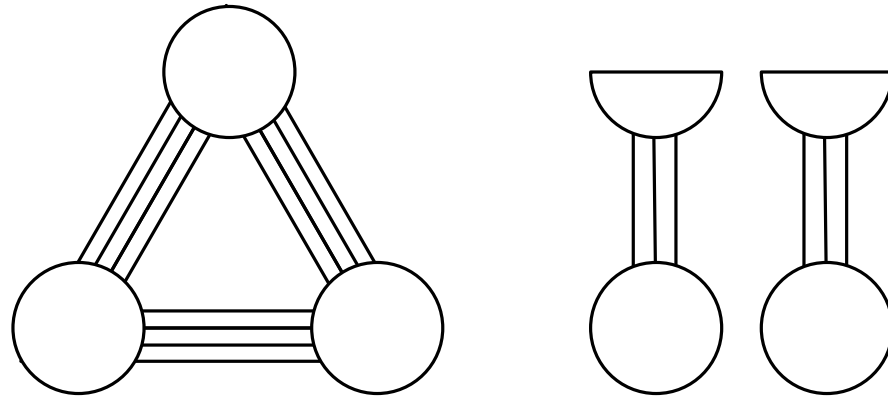


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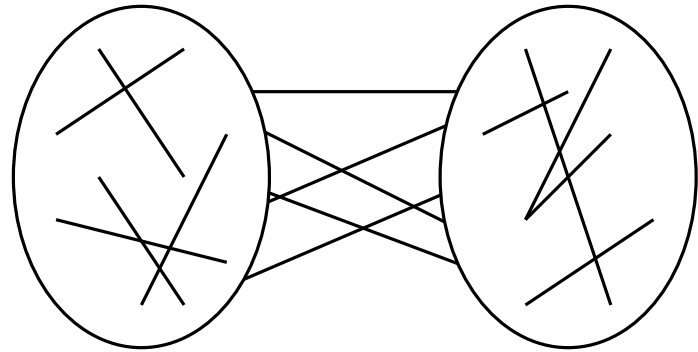


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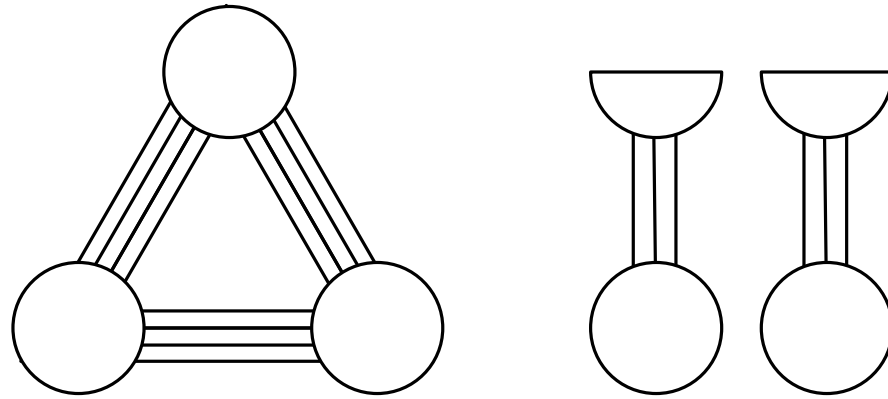


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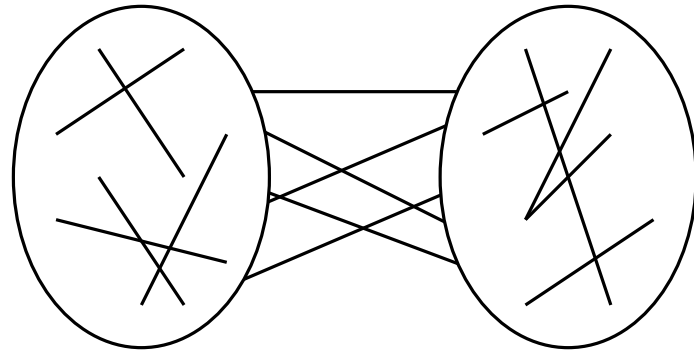


Question:

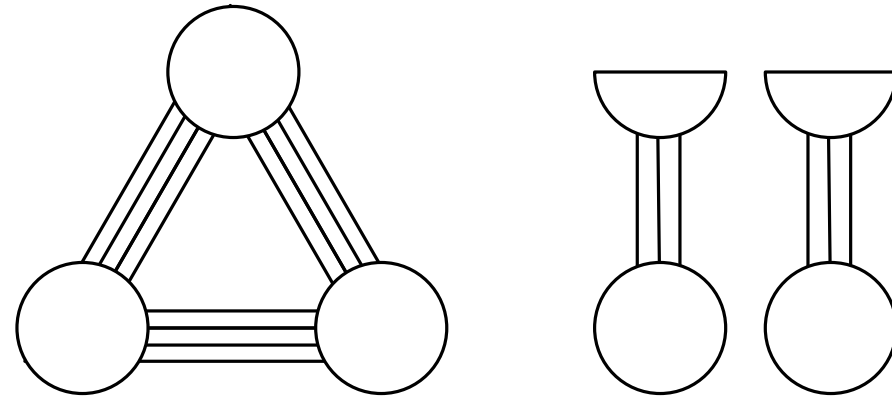
$$\left[\left(\frac{1}{3} - \triangle \right)^2 \right] \leq ?$$

Problem

Balanced bipartition



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Question:

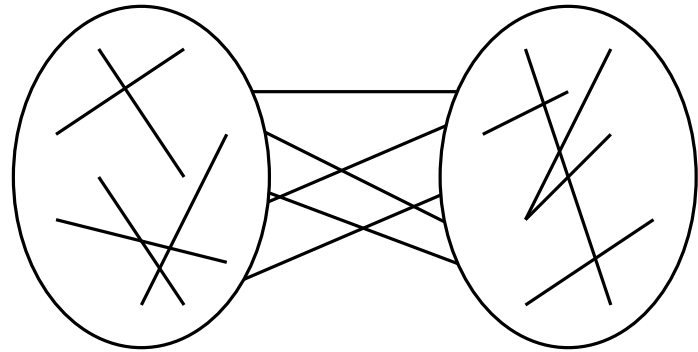
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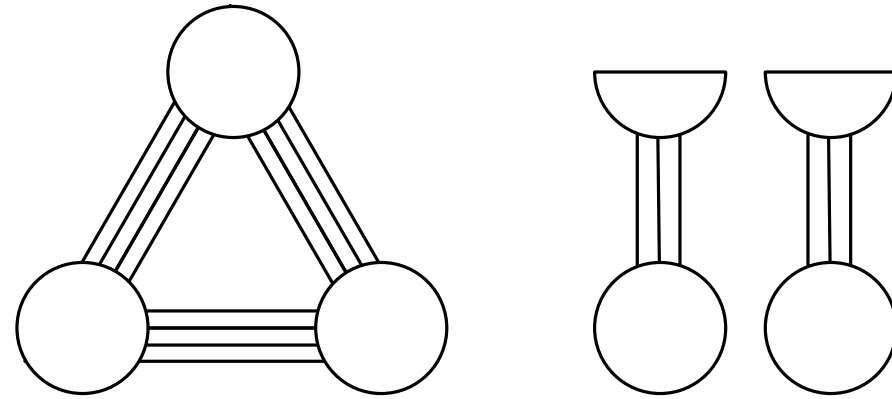
Every balanced bipartition has at least $\frac{n^2}{9}$ edges.

Problem

Balanced bipartition



\square -free, $\delta(G) \geq \frac{n}{2}$, distance at most $\frac{n^2}{9}$

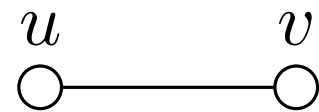


Question:

$$\left[\left(\frac{1}{3} - \triangle \right)^2 \right] \leq ?$$

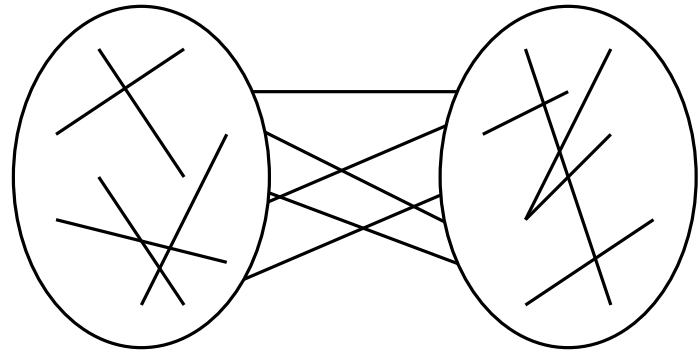
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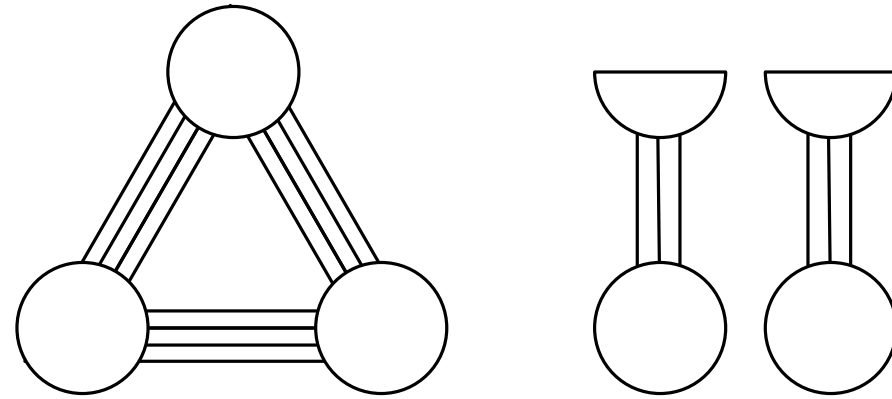


Problem

Balanced bipartition



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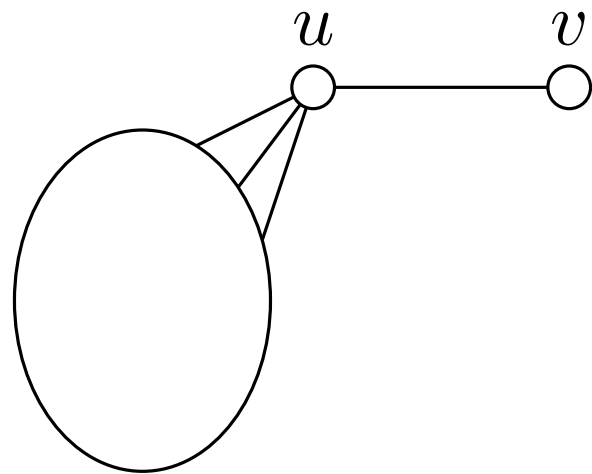


Question:

$$\left[\left(\frac{1}{3} - \epsilon \right)^2 \right] \leq ?$$

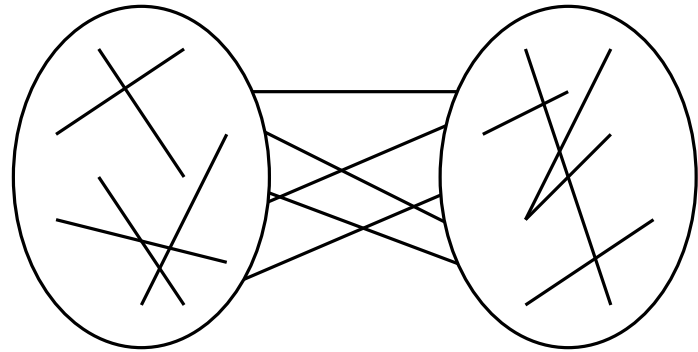
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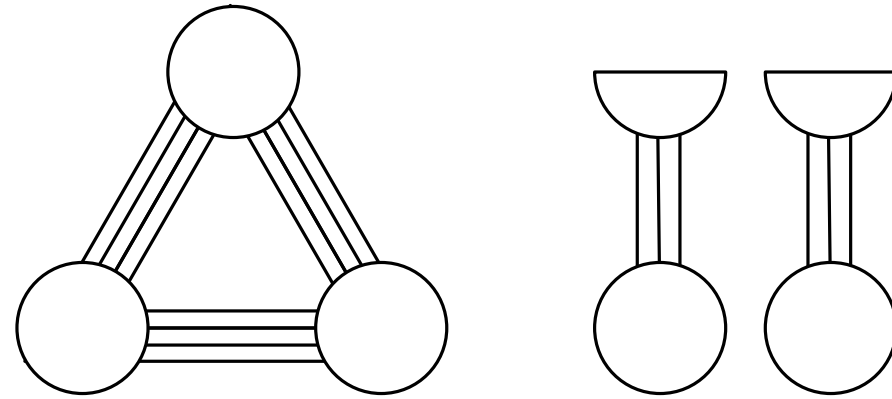


Problem

Balanced bipartition



\square -free, $\delta(G) \geq \frac{n}{2}$, distance at most $\frac{n^2}{9}$

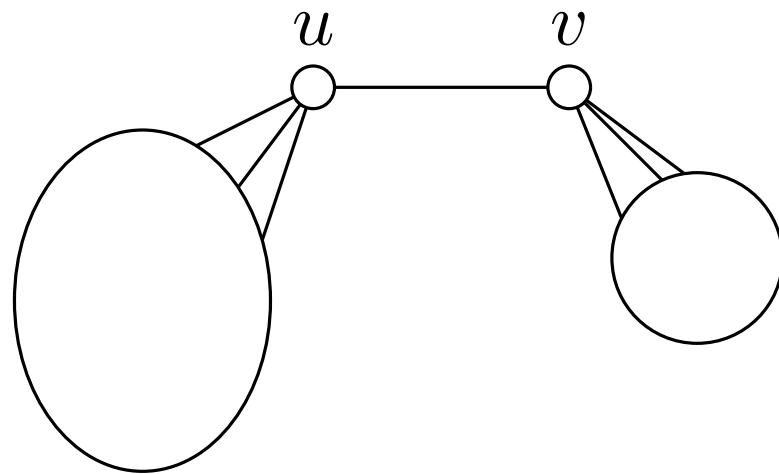


Question:

$$\left[\left(\frac{1}{3} - \alpha \right)^2 \right] \leq ?$$

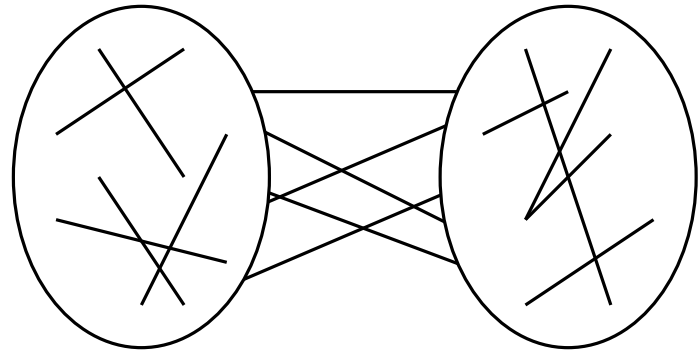
Input:

Every balanced bipartition has at least $\frac{n^2}{9}$ edges.

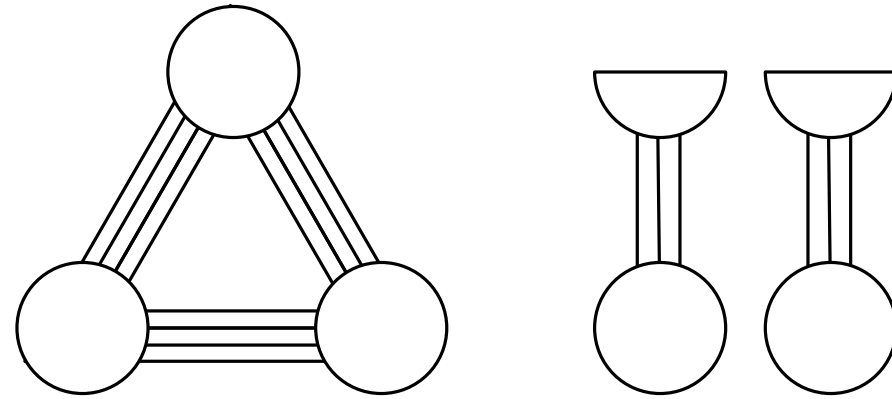


Problem

Balanced bipartition



\square -free, $\delta(G) \geq \frac{n}{2}$, distance at most $\frac{n^2}{9}$

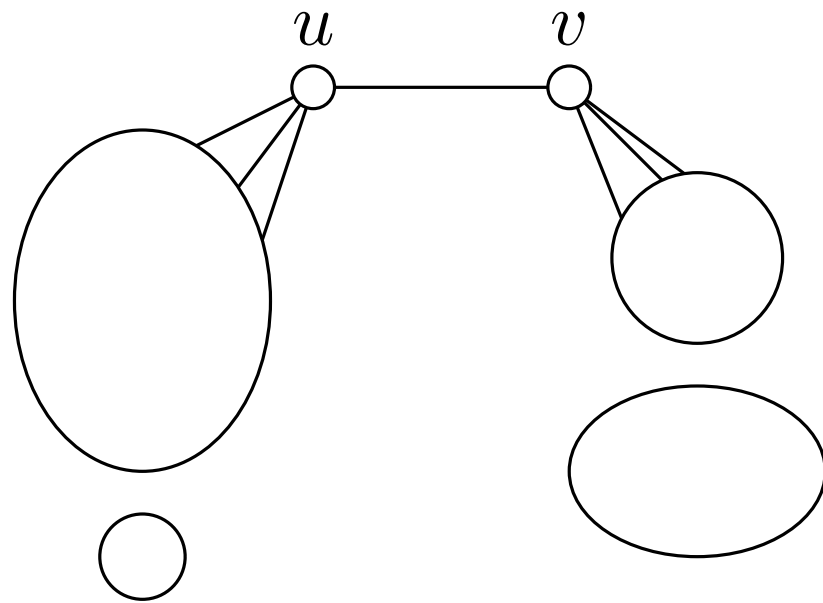


Question:

$$\left[\left(\frac{1}{3} - \epsilon \right)^2 \right] \leq ?$$

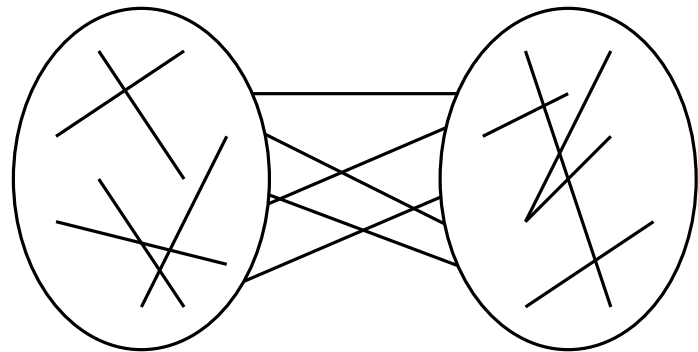
Input:

Every balanced bipartition has at least $\frac{n^2}{9}$ edges.

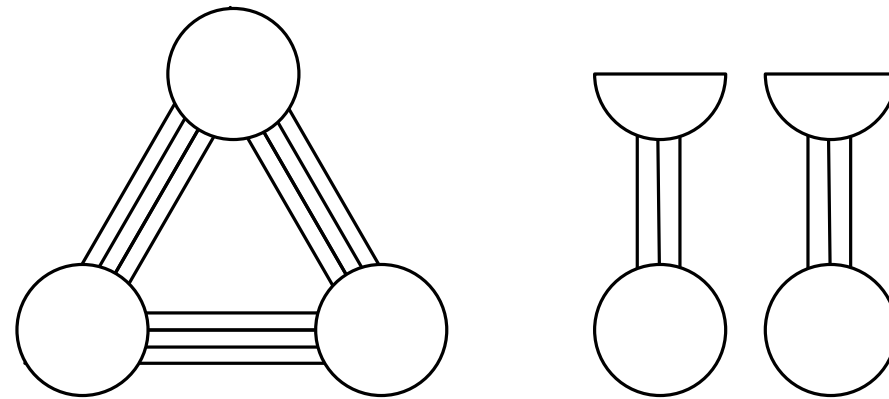


Problem

Balanced bipartition



\square -free, $\delta(G) \geq \frac{n}{2}$, distance at most $\frac{n^2}{9}$

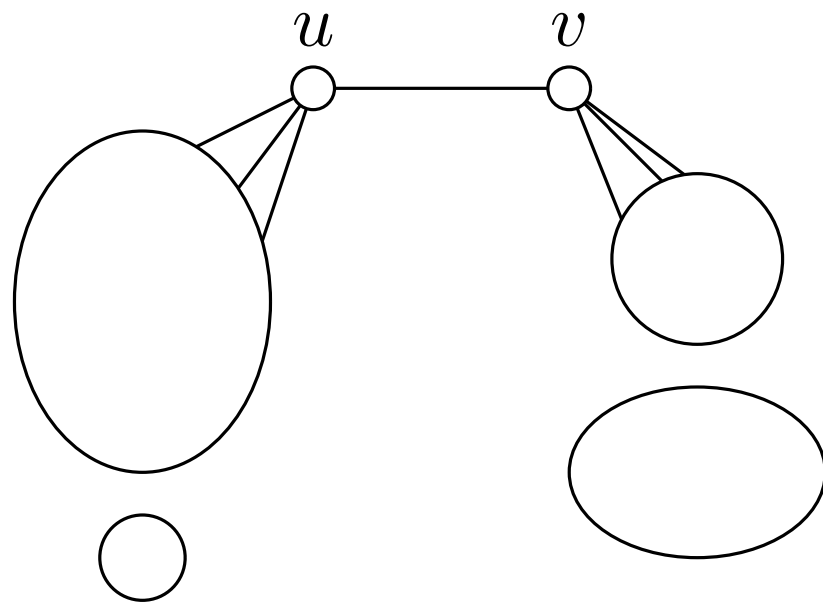


Question:

$$\left[\left(\frac{1}{3} - \triangle \right)^2 \right] \leq ?$$

Input:

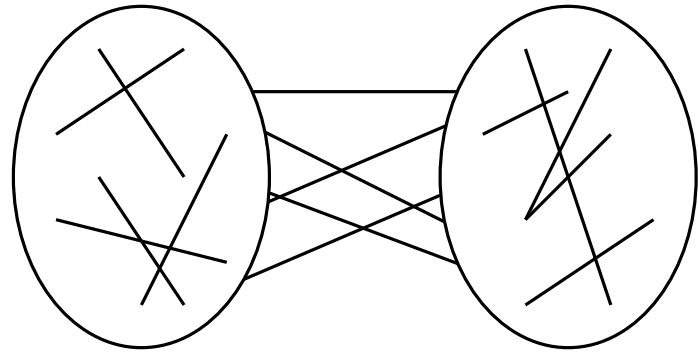
Every balanced bipartition has at least $\frac{n^2}{9}$ edges.



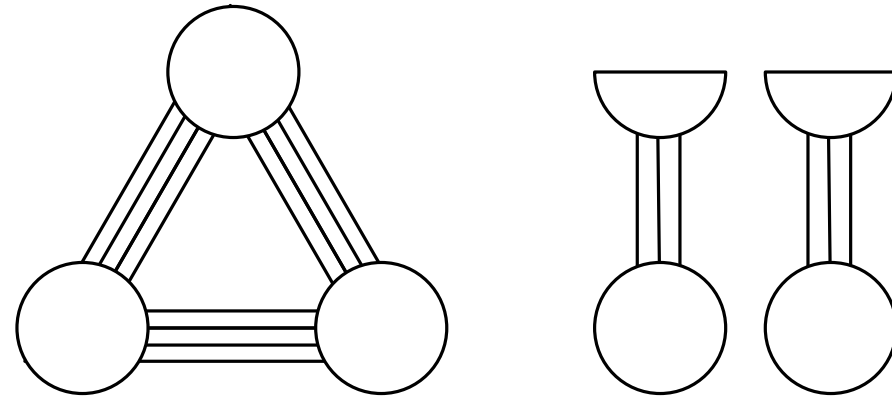
$$\square + \square + (\square + \square) \cdot \frac{\frac{1}{2} - \triangle}{\triangle + \circ} + (\square + \square) \cdot \frac{\frac{1}{2} - \triangle}{\triangle + \circ} + (\square + \square) \cdot \left[\left(\frac{\frac{1}{2} - \triangle}{\triangle + \circ} \right)^2 + \left(\frac{\frac{1}{2} - \triangle}{\triangle + \circ} \right)^2 \right] \geq \frac{2}{9}$$

Problem

Balanced bipartition



\square -free, $\delta(G) \geq \frac{n}{2}$, distance at most $\frac{n^2}{9}$

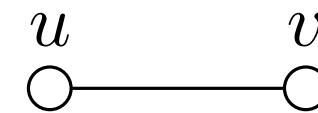
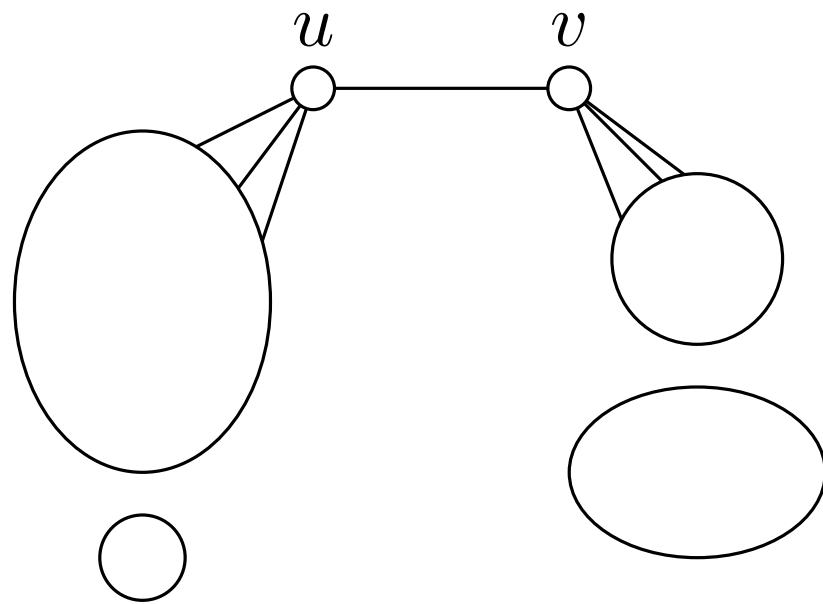


Question:

$$\left\lfloor \left(\frac{1}{3} - \frac{\delta}{n} \right)^2 \right\rfloor \leq ?$$

Input:

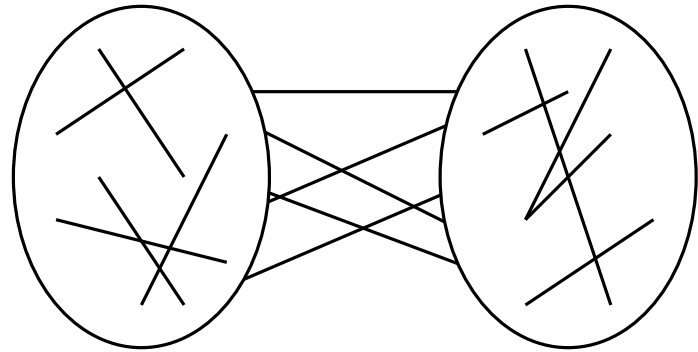
Every balanced bipartition has at least $\frac{n^2}{9}$ edges.



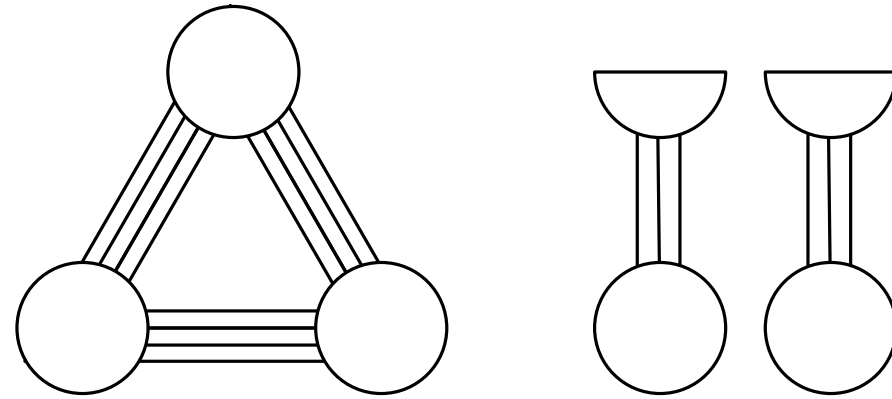
$$\begin{matrix} \square \\ \circ \end{matrix} + \begin{matrix} \square \\ \circ \end{matrix} + \left(\begin{matrix} \square \\ \circ \end{matrix} + \begin{matrix} \square \\ \circ \end{matrix} \right) \cdot \frac{\frac{1}{2} - \frac{\delta}{n}}{\frac{\delta}{n} + \frac{\delta}{n}} + \left(\begin{matrix} \square \\ \circ \end{matrix} + \begin{matrix} \square \\ \circ \end{matrix} \right) \cdot \frac{\frac{1}{2} - \frac{\delta}{n}}{\frac{\delta}{n} + \frac{\delta}{n}} + \left(\begin{matrix} \square \\ \circ \end{matrix} + \begin{matrix} \square \\ \circ \end{matrix} \right) \cdot \left[\left(\frac{\frac{1}{2} - \frac{\delta}{n}}{\frac{\delta}{n} + \frac{\delta}{n}} \right)^2 + \left(\frac{\frac{1}{2} - \frac{\delta}{n}}{\frac{\delta}{n} + \frac{\delta}{n}} \right)^2 \right] \geq \frac{2}{9}$$

Problem

Balanced bipartition



\square -free, $\delta(G) \geq \frac{n}{2}$, distance at most $\frac{n^2}{9}$

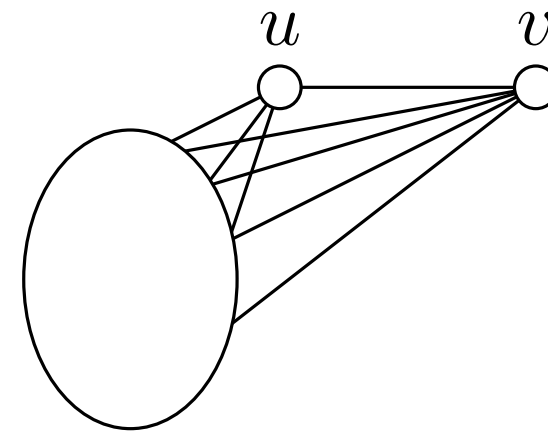
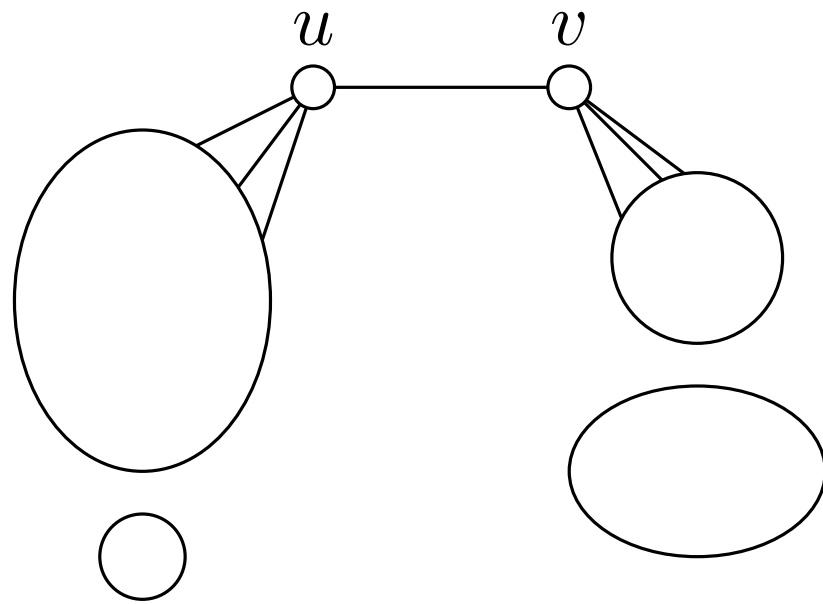


Question:

$$\left\lceil \left(\frac{1}{3} - \frac{\delta}{n} \right)^2 \right\rceil \leq ?$$

Input:

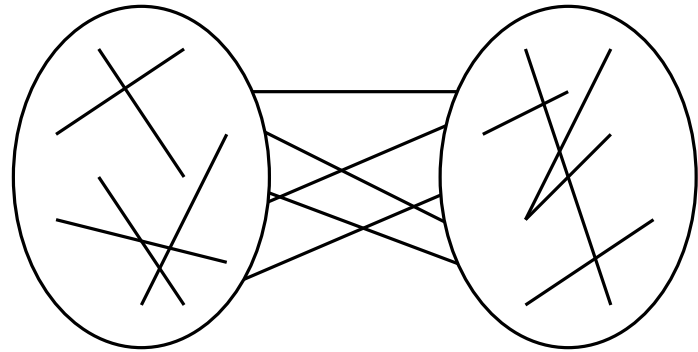
Every balanced bipartition has at least $\frac{n^2}{9}$ edges.



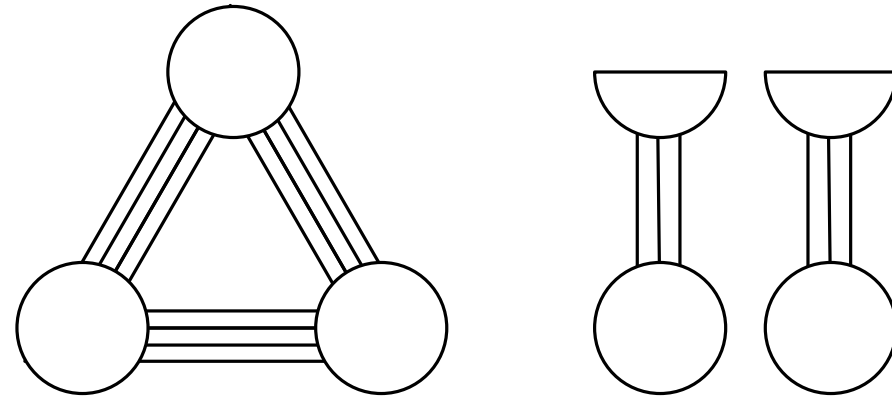
$$\frac{\delta}{n} + \frac{\delta}{n} + \left(\frac{\delta}{n} + \frac{\delta}{n} \right) \cdot \frac{\frac{1}{2} - \frac{\delta}{n}}{\frac{\delta}{n} + \frac{\delta}{n}} + \left(\frac{\delta}{n} + \frac{\delta}{n} \right) \cdot \frac{\frac{1}{2} - \frac{\delta}{n}}{\frac{\delta}{n} + \frac{\delta}{n}} + \left(\frac{\delta}{n} + \frac{\delta}{n} \right) \cdot \left[\left(\frac{\frac{1}{2} - \frac{\delta}{n}}{\frac{\delta}{n} + \frac{\delta}{n}} \right)^2 + \left(\frac{\frac{1}{2} - \frac{\delta}{n}}{\frac{\delta}{n} + \frac{\delta}{n}} \right)^2 \right] \geq \frac{2}{9}$$

Problem

Balanced bipartition



\square -free, $\delta(G) \geq \frac{n}{2}$, distance at most $\frac{n^2}{9}$

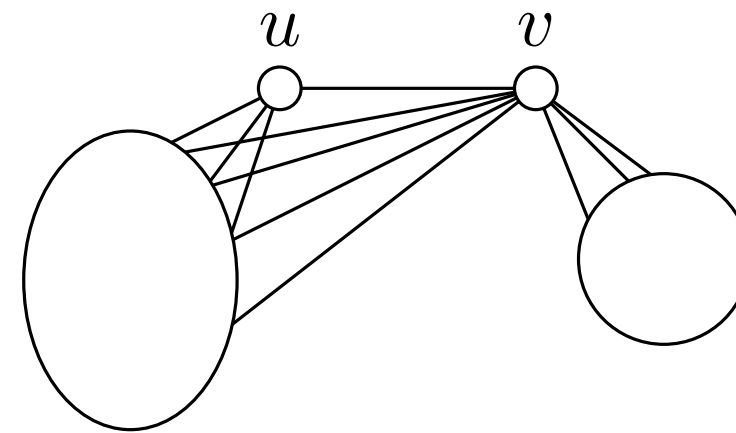
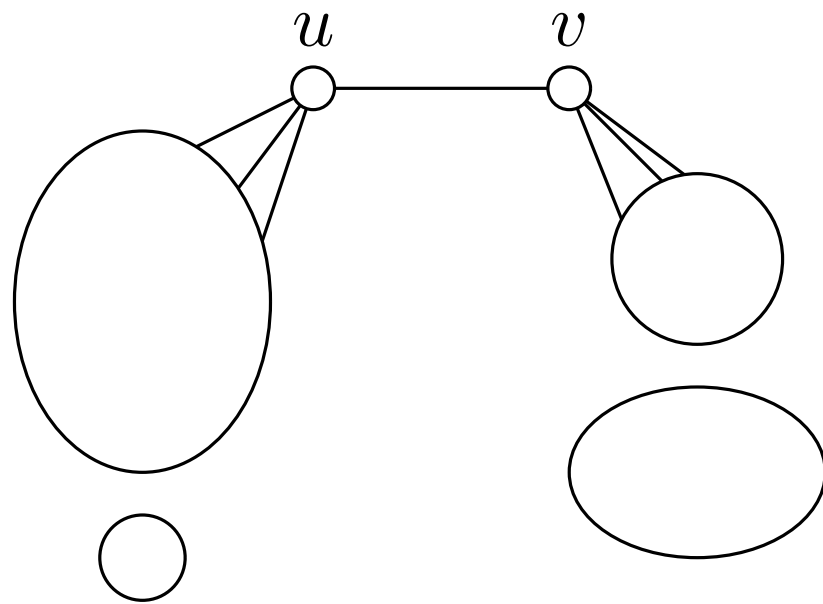


Question:

$$\left\lceil \left(\frac{1}{3} - \frac{\delta}{n} \right)^2 \right\rceil \leq ?$$

Input:

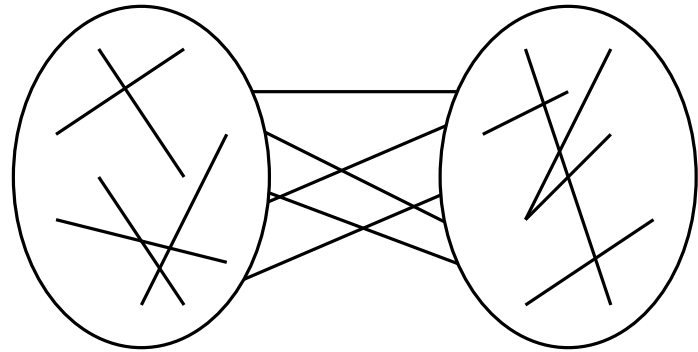
Every balanced bipartition has at least $\frac{n^2}{9}$ edges.



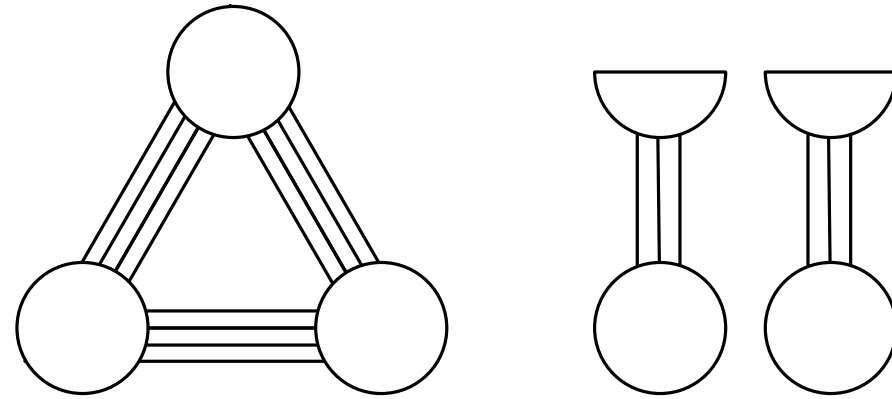
$$\frac{\delta}{n} + \frac{\delta}{n} + \left(\frac{\delta}{n} + \frac{\delta}{n} \right) \cdot \frac{\frac{1}{2} - \frac{\delta}{n}}{\frac{\delta}{n} + \frac{\delta}{n}} + \left(\frac{\delta}{n} + \frac{\delta}{n} \right) \cdot \frac{\frac{1}{2} - \frac{\delta}{n}}{\frac{\delta}{n} + \frac{\delta}{n}} + \left(\frac{\delta}{n} + \frac{\delta}{n} \right) \cdot \left[\left(\frac{\frac{1}{2} - \frac{\delta}{n}}{\frac{\delta}{n} + \frac{\delta}{n}} \right)^2 + \left(\frac{\frac{1}{2} - \frac{\delta}{n}}{\frac{\delta}{n} + \frac{\delta}{n}} \right)^2 \right] \geq \frac{2}{9}$$

Problem

Balanced bipartition



\square -free, $\delta(G) \geq \frac{n}{2}$, distance at most $\frac{n^2}{9}$

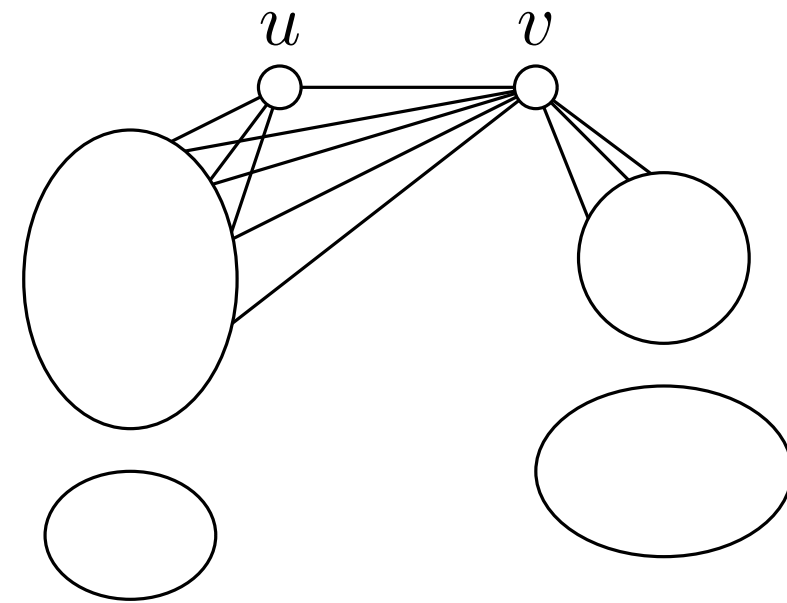
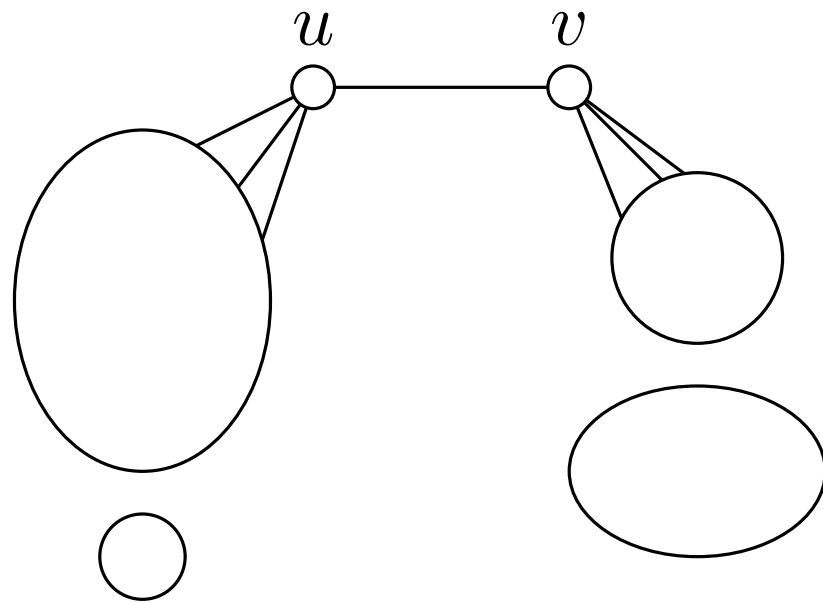


Question:

$$\left\lceil \left(\frac{1}{3} - \epsilon \right)^2 \right\rceil \leq ?$$

Input:

Every balanced bipartition has at least $\frac{n^2}{9}$ edges.



$$\square + \square + (\square + \square) \cdot \frac{\frac{1}{2} - \epsilon}{\epsilon + \epsilon} + (\square + \square) \cdot \frac{\frac{1}{2} - \epsilon}{\epsilon + \epsilon} + (\square + \square) \cdot \left[\left(\frac{\frac{1}{2} - \epsilon}{\epsilon + \epsilon} \right)^2 + \left(\frac{\frac{1}{2} - \epsilon}{\epsilon + \epsilon} \right)^2 \right] \geq \frac{2}{9}$$

Inequalities

$$\begin{aligned}
 & \begin{array}{c} \bullet \\ \diagup \\ \circ \end{array} + \begin{array}{c} \bullet \\ \diagdown \\ \circ \end{array} + \left(\begin{array}{c} \bullet \\ \text{---} \\ \circ \end{array} + \begin{array}{c} \bullet \\ \text{---} \\ \circ \end{array} \right) \cdot \frac{\frac{1}{2} - \begin{array}{c} \bullet \\ \diagup \\ \circ \end{array}}{\begin{array}{c} \triangle \\ \text{---} \\ \circ \end{array} + \begin{array}{c} \bullet \\ \text{---} \\ \circ \end{array}} + \left(\begin{array}{c} \bullet \\ \text{---} \\ \circ \end{array} + \begin{array}{c} \bullet \\ \text{---} \\ \circ \end{array} \right) \cdot \frac{\frac{1}{2} - \begin{array}{c} \bullet \\ \diagdown \\ \circ \end{array}}{\begin{array}{c} \triangle \\ \text{---} \\ \circ \end{array} + \begin{array}{c} \bullet \\ \text{---} \\ \circ \end{array}} + \left(\begin{array}{c} \bullet \\ \text{---} \\ \circ \end{array} + \begin{array}{c} \bullet \\ \text{---} \\ \circ \end{array} \right) \cdot \left[\left(\frac{\frac{1}{2} - \begin{array}{c} \bullet \\ \diagup \\ \circ \end{array}}{\begin{array}{c} \triangle \\ \text{---} \\ \circ \end{array} + \begin{array}{c} \bullet \\ \text{---} \\ \circ \end{array}} \right)^2 + \left(\frac{\frac{1}{2} - \begin{array}{c} \bullet \\ \diagdown \\ \circ \end{array}}{\begin{array}{c} \triangle \\ \text{---} \\ \circ \end{array} + \begin{array}{c} \bullet \\ \text{---} \\ \circ \end{array}} \right)^2 \right] \geq \frac{2}{9}
 \end{aligned}$$

Inequalities

$$\begin{aligned}
 & \left(\begin{array}{c} \bullet \\ \diagup \\ \circ \end{array} + \begin{array}{c} \bullet \\ \diagdown \\ \circ \end{array} + \left(\begin{array}{c} \bullet \bullet \\ \circ \circ \end{array} + \begin{array}{c} \bullet \bullet \\ \diagup \diagdown \\ \circ \circ \end{array} \right) \cdot \frac{\frac{1}{2} - \begin{array}{c} \bullet \\ \diagup \\ \circ \end{array}}{\begin{array}{c} \triangle \\ + \\ \circ \circ \end{array}} + \left(\begin{array}{c} \bullet \bullet \\ \circ \circ \end{array} + \begin{array}{c} \bullet \bullet \\ \diagup \diagdown \\ \circ \circ \end{array} \right) \cdot \frac{\frac{1}{2} - \begin{array}{c} \bullet \\ \diagdown \\ \circ \end{array}}{\begin{array}{c} \triangle \\ + \\ \circ \circ \end{array}} + \left(\begin{array}{c} \bullet \bullet \\ \circ \circ \end{array} + \begin{array}{c} \bullet \bullet \\ \diagup \diagdown \\ \circ \circ \end{array} \right) \cdot \left[\left(\frac{\frac{1}{2} - \begin{array}{c} \bullet \\ \diagup \\ \circ \end{array}}{\begin{array}{c} \triangle \\ + \\ \circ \circ \end{array}} \right)^2 + \left(\frac{\frac{1}{2} - \begin{array}{c} \bullet \\ \diagdown \\ \circ \end{array}}{\begin{array}{c} \triangle \\ + \\ \circ \circ \end{array}} \right)^2 \right] \geq \frac{2}{9}
 \end{aligned}$$

$$\begin{aligned}
 & \begin{array}{c} \bullet \\ \diagup \\ \circ \end{array} + \left(\begin{array}{c} \bullet \bullet \\ \circ \circ \end{array} + \begin{array}{c} \bullet \bullet \\ \circ \circ \end{array} \right) \cdot \frac{\frac{1}{2} - \begin{array}{c} \bullet \\ \diagup \\ \circ \end{array}}{\begin{array}{c} \circ \circ \\ + \\ \circ \circ \end{array}} + \left(\begin{array}{c} \bullet \bullet \\ \diagup \diagdown \\ \circ \circ \end{array} + \begin{array}{c} \bullet \bullet \\ \diagdown \diagup \\ \circ \circ \end{array} \right) \cdot \frac{\frac{1}{2} - \begin{array}{c} \triangle \\ \circ \circ \end{array}}{\begin{array}{c} \circ \circ \\ + \\ \circ \circ \end{array}} + \left(\begin{array}{c} \bullet \bullet \\ \circ \circ \end{array} + \begin{array}{c} \bullet \bullet \\ \circ \circ \end{array} + \begin{array}{c} \bullet \bullet \\ \diagup \diagdown \\ \circ \circ \end{array} \right) \cdot \left[\left(\frac{\frac{1}{2} - \begin{array}{c} \bullet \\ \diagup \\ \circ \end{array}}{\begin{array}{c} \circ \circ \\ + \\ \circ \circ \end{array}} \right)^2 + \left(\frac{\frac{1}{2} - \begin{array}{c} \triangle \\ \circ \circ \end{array}}{\begin{array}{c} \circ \circ \\ + \\ \circ \circ \end{array}} \right)^2 \right] \geq \frac{2}{9}
 \end{aligned}$$

Inequalities

$$\begin{aligned}
 & \left(\begin{array}{c} \bullet \\ \diagup \\ \circ \end{array} + \begin{array}{c} \bullet \\ \diagdown \\ \circ \end{array} + \left(\begin{array}{c} \bullet \\ \text{---} \\ \circ \end{array} + \begin{array}{c} \bullet \\ \text{---} \\ \circ \end{array} \right) \cdot \frac{\frac{1}{2} - \begin{array}{c} \bullet \\ \diagup \\ \circ \end{array}}{\begin{array}{c} \triangle \\ + \\ \circ \end{array}} + \left(\begin{array}{c} \bullet \\ \text{---} \\ \circ \end{array} + \begin{array}{c} \bullet \\ \text{---} \\ \circ \end{array} \right) \cdot \frac{\frac{1}{2} - \begin{array}{c} \bullet \\ \diagdown \\ \circ \end{array}}{\begin{array}{c} \triangle \\ + \\ \circ \end{array}} + \left(\begin{array}{c} \bullet \\ \text{---} \\ \circ \end{array} + \begin{array}{c} \bullet \\ \text{---} \\ \circ \end{array} \right) \cdot \left[\left(\frac{\frac{1}{2} - \begin{array}{c} \bullet \\ \diagup \\ \circ \end{array}}{\begin{array}{c} \triangle \\ + \\ \circ \end{array}} \right)^2 + \left(\frac{\frac{1}{2} - \begin{array}{c} \bullet \\ \diagdown \\ \circ \end{array}}{\begin{array}{c} \triangle \\ + \\ \circ \end{array}} \right)^2 \right] \geq \frac{2}{9}
 \end{aligned}$$

$$\begin{aligned}
 & \begin{array}{c} \bullet \\ \diagup \\ \circ \end{array} + \left(\begin{array}{c} \bullet \\ \text{---} \\ \circ \end{array} + \begin{array}{c} \bullet \\ \text{---} \\ \circ \end{array} \right) \cdot \frac{\frac{1}{2} - \begin{array}{c} \bullet \\ \diagup \\ \circ \end{array}}{\begin{array}{c} \triangle \\ + \\ \circ \end{array}} + \left(\begin{array}{c} \bullet \\ \diagup \\ \circ \end{array} + \begin{array}{c} \bullet \\ \diagdown \\ \circ \end{array} \right) \cdot \frac{\frac{1}{2} - \triangle}{\begin{array}{c} \triangle \\ + \\ \circ \end{array}} + \left(\begin{array}{c} \bullet \\ \text{---} \\ \circ \end{array} + \begin{array}{c} \bullet \\ \text{---} \\ \circ \end{array} + \begin{array}{c} \bullet \\ \text{---} \\ \circ \end{array} \right) \cdot \left[\left(\frac{\frac{1}{2} - \begin{array}{c} \bullet \\ \diagup \\ \circ \end{array}}{\begin{array}{c} \triangle \\ + \\ \circ \end{array}} \right)^2 + \left(\frac{\frac{1}{2} - \triangle}{\begin{array}{c} \triangle \\ + \\ \circ \end{array}} \right)^2 \right] \geq \frac{2}{9}
 \end{aligned}$$

$$\left[\left(\frac{1}{3} - \begin{array}{c} \bullet \\ \triangle \\ \circ \end{array} \right)^2 \right]$$

$$0.005$$

Inequalities

$$\begin{matrix} \bullet & \bullet \\ \diagdown & \diagup \\ \circ & \circ \end{matrix} + \begin{matrix} \bullet & \bullet \\ \diagup & \diagdown \\ \circ & \circ \end{matrix} + \left(\begin{matrix} \bullet & \bullet \\ \circ & \circ \end{matrix} + \begin{matrix} \bullet & \bullet \\ \diagdown & \diagup \\ \circ & \circ \end{matrix} \right) \cdot \frac{\frac{1}{2} - \begin{matrix} \bullet \\ \diagdown \\ \circ \end{matrix}}{\begin{matrix} \triangle \\ + \\ \circ \end{matrix}} + \left(\begin{matrix} \bullet & \bullet \\ \circ & \circ \end{matrix} + \begin{matrix} \bullet & \bullet \\ \diagup & \diagdown \\ \circ & \circ \end{matrix} \right) \cdot \frac{\frac{1}{2} - \begin{matrix} \bullet \\ \diagup \\ \circ \end{matrix}}{\begin{matrix} \triangle \\ + \\ \circ \end{matrix}} + \left(\begin{matrix} \bullet & \bullet \\ \circ & \circ \end{matrix} + \begin{matrix} \bullet & \bullet \\ \diagdown & \diagup \\ \circ & \circ \end{matrix} \right) \cdot \left[\left(\frac{\frac{1}{2} - \begin{matrix} \bullet \\ \diagdown \\ \circ \end{matrix}}{\begin{matrix} \triangle \\ + \\ \circ \end{matrix}} \right)^2 + \left(\frac{\frac{1}{2} - \begin{matrix} \bullet \\ \diagup \\ \circ \end{matrix}}{\begin{matrix} \triangle \\ + \\ \circ \end{matrix}} \right)^2 \right] \geq \frac{2}{9}$$

$$\begin{matrix} \bullet & \bullet \\ \diagdown & \diagup \\ \circ & \circ \end{matrix} + \left(\begin{matrix} \bullet & \bullet \\ \circ & \circ \end{matrix} + \begin{matrix} \bullet & \bullet \\ \circ & \circ \end{matrix} \right) \cdot \frac{\frac{1}{2} - \begin{matrix} \bullet \\ \diagdown \\ \circ \end{matrix}}{\begin{matrix} \triangle \\ + \\ \circ \end{matrix}} + \left(\begin{matrix} \bullet & \bullet \\ \diagdown & \diagup \\ \circ & \circ \end{matrix} + \begin{matrix} \bullet & \bullet \\ \diagup & \diagdown \\ \circ & \circ \end{matrix} \right) \cdot \frac{\frac{1}{2} - \begin{matrix} \triangle \\ \circ \end{matrix}}{\begin{matrix} \triangle \\ + \\ \circ \end{matrix}} + \left(\begin{matrix} \bullet & \bullet \\ \circ & \circ \end{matrix} + \begin{matrix} \bullet & \bullet \\ \circ & \circ \end{matrix} + \begin{matrix} \bullet & \bullet \\ \diagdown & \diagup \\ \circ & \circ \end{matrix} \right) \cdot \left[\left(\frac{\frac{1}{2} - \begin{matrix} \bullet \\ \diagdown \\ \circ \end{matrix}}{\begin{matrix} \triangle \\ + \\ \circ \end{matrix}} \right)^2 + \left(\frac{\frac{1}{2} - \begin{matrix} \triangle \\ \circ \end{matrix}}{\begin{matrix} \triangle \\ + \\ \circ \end{matrix}} \right)^2 \right] \geq \frac{2}{9}$$

$\left[\left(\frac{1}{3} - \begin{matrix} \triangle \\ \circ \end{matrix} \right)^2 \right]$	0.0000	0.005
$\begin{matrix} \bullet & \bullet \\ \bullet & \bullet \end{matrix}$	0.0370	0.0363
$\begin{matrix} \bullet & \bullet \\ \bullet & \bullet \end{matrix}$	0.0000	0.0314
$\begin{matrix} \bullet & \bullet \\ \bullet & \bullet \end{matrix}$	0.0000	0.0805
$\begin{matrix} \bullet & \bullet \\ \diagdown & \diagup \\ \bullet & \bullet \end{matrix}$	0.2963	0.2203
$\begin{matrix} \bullet & \bullet \\ \diagup & \diagdown \\ \bullet & \bullet \end{matrix}$	0.0000	0.0098
$\begin{matrix} \bullet & \bullet \\ \bullet & \bullet \end{matrix}$	0.0000	0.0031
$\begin{matrix} \bullet & \bullet \\ \bullet & \bullet \end{matrix}$	0.0000	0.0968
$\begin{matrix} \bullet & \bullet \\ \diagdown & \diagup \\ \bullet & \bullet \end{matrix}$	0.0000	0.1278
$\begin{matrix} \bullet & \bullet \\ \circ & \circ \end{matrix}$	0.2222	0.1703
$\begin{matrix} \bullet & \bullet \\ \diagdown & \diagup \\ \circ & \circ \end{matrix}$	0.4444	0.2232

Inequalities

$$\begin{matrix} \diagup \\ \circ \end{matrix} + \begin{matrix} \circ \\ \diagdown \end{matrix} + \left(\begin{matrix} \circ \\ \circ \end{matrix} + \begin{matrix} \circ \\ \diagup \end{matrix} \right) \cdot \frac{\frac{1}{2} - \begin{matrix} \circ \\ \diagdown \end{matrix}}{\begin{matrix} \triangle \\ + \\ \circ \end{matrix}} + \left(\begin{matrix} \circ \\ \circ \end{matrix} + \begin{matrix} \circ \\ \diagdown \end{matrix} \right) \cdot \frac{\frac{1}{2} - \begin{matrix} \circ \\ \diagdown \end{matrix}}{\begin{matrix} \triangle \\ + \\ \circ \end{matrix}} + \left(\begin{matrix} \circ \\ \circ \end{matrix} + \begin{matrix} \circ \\ \diagup \end{matrix} \right) \cdot \left[\left(\frac{\frac{1}{2} - \begin{matrix} \circ \\ \diagdown \end{matrix}}{\begin{matrix} \triangle \\ + \\ \circ \end{matrix}} \right)^2 + \left(\frac{\frac{1}{2} - \begin{matrix} \circ \\ \diagdown \end{matrix}}{\begin{matrix} \triangle \\ + \\ \circ \end{matrix}} \right)^2 \right] \geq \frac{2}{9}$$

$$\begin{matrix} \circ \\ \diagup \end{matrix} + \left(\begin{matrix} \circ \\ \circ \end{matrix} + \begin{matrix} \circ \\ \circ \end{matrix} \right) \cdot \frac{\frac{1}{2} - \begin{matrix} \circ \\ \diagdown \end{matrix}}{\begin{matrix} \circ \\ + \\ \circ \end{matrix}} + \left(\begin{matrix} \circ \\ \diagup \end{matrix} + \begin{matrix} \circ \\ \diagdown \end{matrix} \right) \cdot \frac{\frac{1}{2} - \begin{matrix} \triangle \\ \circ \end{matrix}}{\begin{matrix} \circ \\ + \\ \circ \end{matrix}} + \left(\begin{matrix} \circ \\ \circ \end{matrix} + \begin{matrix} \circ \\ \circ \end{matrix} + \begin{matrix} \circ \\ \diagup \end{matrix} \right) \cdot \left[\left(\frac{\frac{1}{2} - \begin{matrix} \circ \\ \diagdown \end{matrix}}{\begin{matrix} \circ \\ + \\ \circ \end{matrix}} \right)^2 + \left(\frac{\frac{1}{2} - \begin{matrix} \triangle \\ \circ \end{matrix}}{\begin{matrix} \circ \\ + \\ \circ \end{matrix}} \right)^2 \right] \geq \frac{2}{9}$$

$\left[\left(\frac{1}{3} - \begin{matrix} \triangle \\ \circ \end{matrix} \right)^2 \right]$	0.0000	0.005	0.002
$\begin{matrix} \circ \\ \circ \end{matrix}$	0.0370	0.0363	0.0379
$\begin{matrix} \circ \\ \diagdown \end{matrix}$	0.0000	0.0314	0.0249
$\begin{matrix} \circ \\ \diagup \end{matrix}$	0.0000	0.0805	0.0499
$\begin{matrix} \circ \\ \diagdown \end{matrix} \begin{matrix} \diagup \\ \circ \end{matrix}$	0.2963	0.2203	0.2498
$\begin{matrix} \circ \\ \diagup \end{matrix} \begin{matrix} \diagdown \\ \circ \end{matrix}$	0.0000	0.0098	0.0000
$\begin{matrix} \circ \\ \circ \end{matrix} \begin{matrix} \circ \\ \circ \end{matrix}$	0.0000	0.0031	0.0000
$\begin{matrix} \circ \\ \circ \end{matrix} \begin{matrix} \circ \\ \circ \end{matrix}$	0.0000	0.0968	0.0499
$\begin{matrix} \circ \\ \diagup \end{matrix} \begin{matrix} \diagdown \\ \circ \end{matrix}$	0.0000	0.1278	0.0999
$\begin{matrix} \circ \\ \circ \end{matrix} \begin{matrix} \circ \\ \circ \end{matrix}$	0.2222	0.1703	0.1874
$\begin{matrix} \circ \\ \diagup \end{matrix} \begin{matrix} \diagdown \\ \circ \end{matrix}$	0.4444	0.2232	0.2998

Inequalities

$$\begin{matrix} \text{diag} & + & \text{diag} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \left[\left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 + \left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 \right] \end{matrix} \geq \frac{2}{9}$$

$$\begin{matrix} \text{diag} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag} & + & \text{diag}) \cdot \left[\left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 + \left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 \right] \end{matrix} \geq \frac{2}{9}$$

$\left[\left(\frac{1}{3} - \text{diag} \right)^2 \right]$	0.0000	0.005	0.002
$\begin{matrix} \bullet & \bullet \\ \vdots & \vdots \end{matrix}$	0.0370	0.0363	0.0379
$\begin{matrix} \bullet & \bullet \\ \vdots & \vdots \end{matrix}$	0.0000	0.0314	0.0249
$\begin{matrix} \bullet & \bullet \\ \vdots & \vdots \end{matrix}$	0.0000	0.0805	0.0499
$\begin{matrix} \bullet & \bullet \\ \vdots & \vdots \end{matrix}$	0.2963	0.2203	0.2498
$\begin{matrix} \bullet & \bullet \\ \vdots & \vdots \end{matrix}$	0.0000	0.0098	0.0000
$\begin{matrix} \bullet & \bullet \\ \vdots & \vdots \end{matrix}$	0.0000	0.0031	0.0000
$\begin{matrix} \bullet & \bullet \\ \vdots & \vdots \end{matrix}$	0.0000	0.0968	0.0499
$\begin{matrix} \bullet & \bullet \\ \vdots & \vdots \end{matrix}$	0.0000	0.1278	0.0999
$\begin{matrix} \bullet & \bullet \\ \vdots & \vdots \end{matrix}$	0.2222	0.1703	0.1874
$\begin{matrix} \bullet & \bullet \\ \vdots & \vdots \end{matrix}$	0.4444	0.2232	0.2998

$\text{diag} = \text{const}$

Inequalities

$$\begin{matrix} \text{diag} & + & \text{diag} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \left[\left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 + \left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 \right] \end{matrix} \geq \frac{2}{9}$$

$$\begin{matrix} \text{diag} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag} & + & \text{diag}) \cdot \left[\left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 + \left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 \right] \end{matrix} \geq \frac{2}{9}$$

$\left[\left(\frac{1}{3} - \text{diag} \right)^2 \right]$	0.0000	0.005	0.002
$\begin{matrix} \bullet & \bullet \\ \vdots & \vdots \\ \bullet & \bullet \end{matrix}$	0.0370	0.0363	0.0379
$\begin{matrix} \bullet & \bullet \\ \vdots & \vdots \\ \bullet & \bullet \end{matrix}$	0.0000	0.0314	0.0249
$\begin{matrix} \bullet & \bullet \\ \vdots & \vdots \\ \bullet & \bullet \end{matrix}$	0.0000	0.0805	0.0499
$\begin{matrix} \bullet & \bullet \\ \vdots & \vdots \\ \bullet & \bullet \end{matrix}$	0.2963	0.2203	0.2498
$\begin{matrix} \bullet & \bullet \\ \vdots & \vdots \\ \bullet & \bullet \end{matrix}$	0.0000	0.0098	0.0000
$\begin{matrix} \bullet & \bullet \\ \vdots & \vdots \\ \bullet & \bullet \end{matrix}$	0.0000	0.0031	0.0000
$\begin{matrix} \bullet & \bullet \\ \vdots & \vdots \\ \bullet & \bullet \end{matrix}$	0.0000	0.0968	0.0499
$\begin{matrix} \bullet & \bullet \\ \vdots & \vdots \\ \bullet & \bullet \end{matrix}$	0.0000	0.1278	0.0999
$\begin{matrix} \bullet & \bullet \\ \vdots & \vdots \\ \bullet & \bullet \end{matrix}$	0.2222	0.1703	0.1874
$\begin{matrix} \bullet & \bullet \\ \vdots & \vdots \\ \bullet & \bullet \end{matrix}$	0.4444	0.2232	0.2998

$\text{diag} = \text{const}$
 $\text{diag} = \frac{2}{7}$

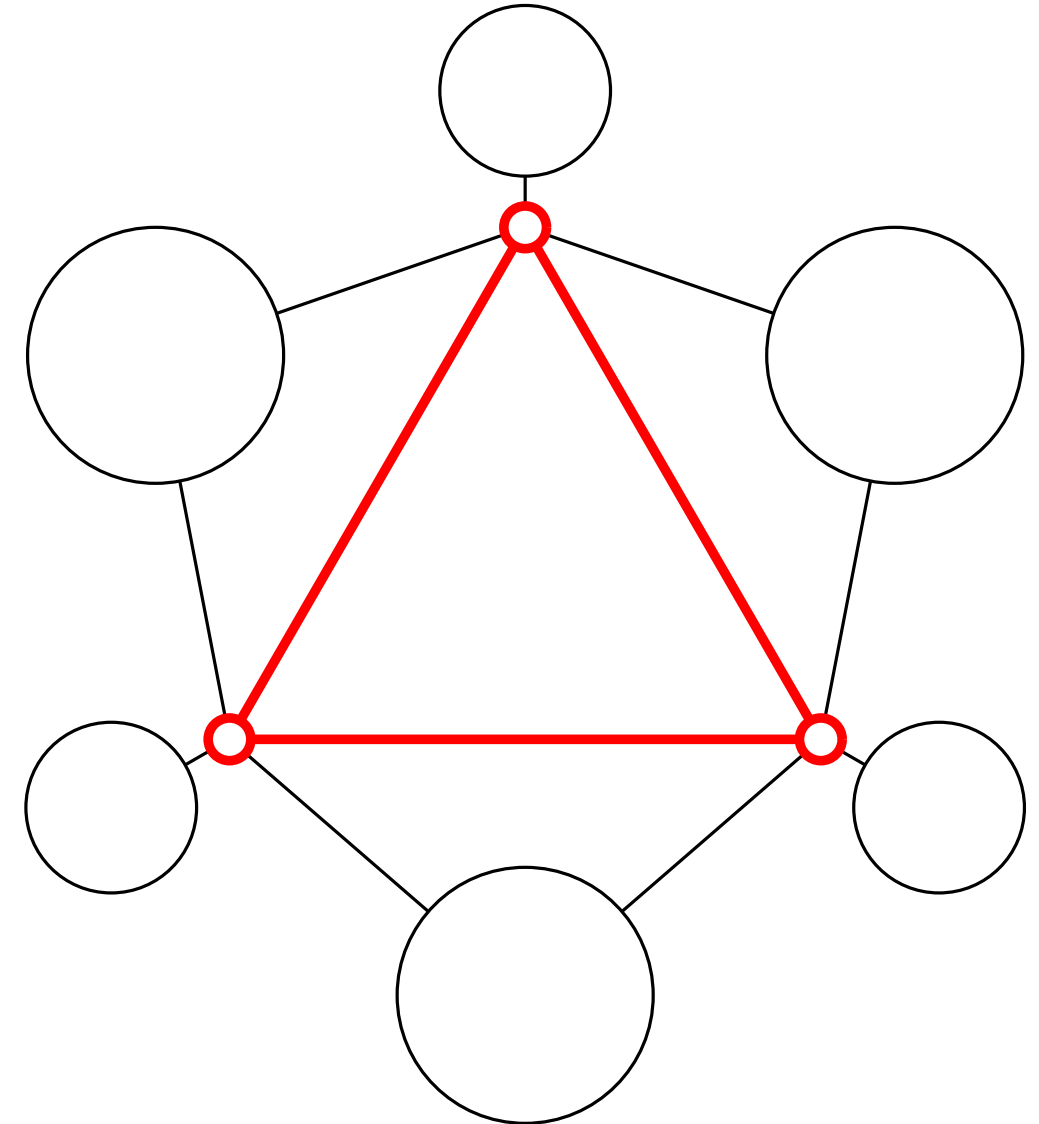
Inequalities

$$\begin{matrix} \text{diag} & + & \text{diag} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \left[\left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 + \left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 \right] \end{matrix} \geq \frac{2}{9}$$

$$\begin{matrix} \text{diag} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag} & + & \text{diag}) \cdot \left[\left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 + \left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 \right] \end{matrix} \geq \frac{2}{9}$$

$\left[\left(\frac{1}{3} - \text{diag} \right)^2 \right]$	0.0000	0.005	0.002
$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	0.0370	0.0363	0.0379
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.0000	0.0314	0.0249
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.0000	0.0805	0.0499
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.2963	0.2203	0.2498
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.0000	0.0098	0.0000
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.0000	0.0031	0.0000
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.0000	0.0968	0.0499
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.0000	0.1278	0.0999
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.2222	0.1703	0.1874
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.4444	0.2232	0.2998

$\text{diag} = \text{const}$
 $\text{diag} = \frac{2}{7}$



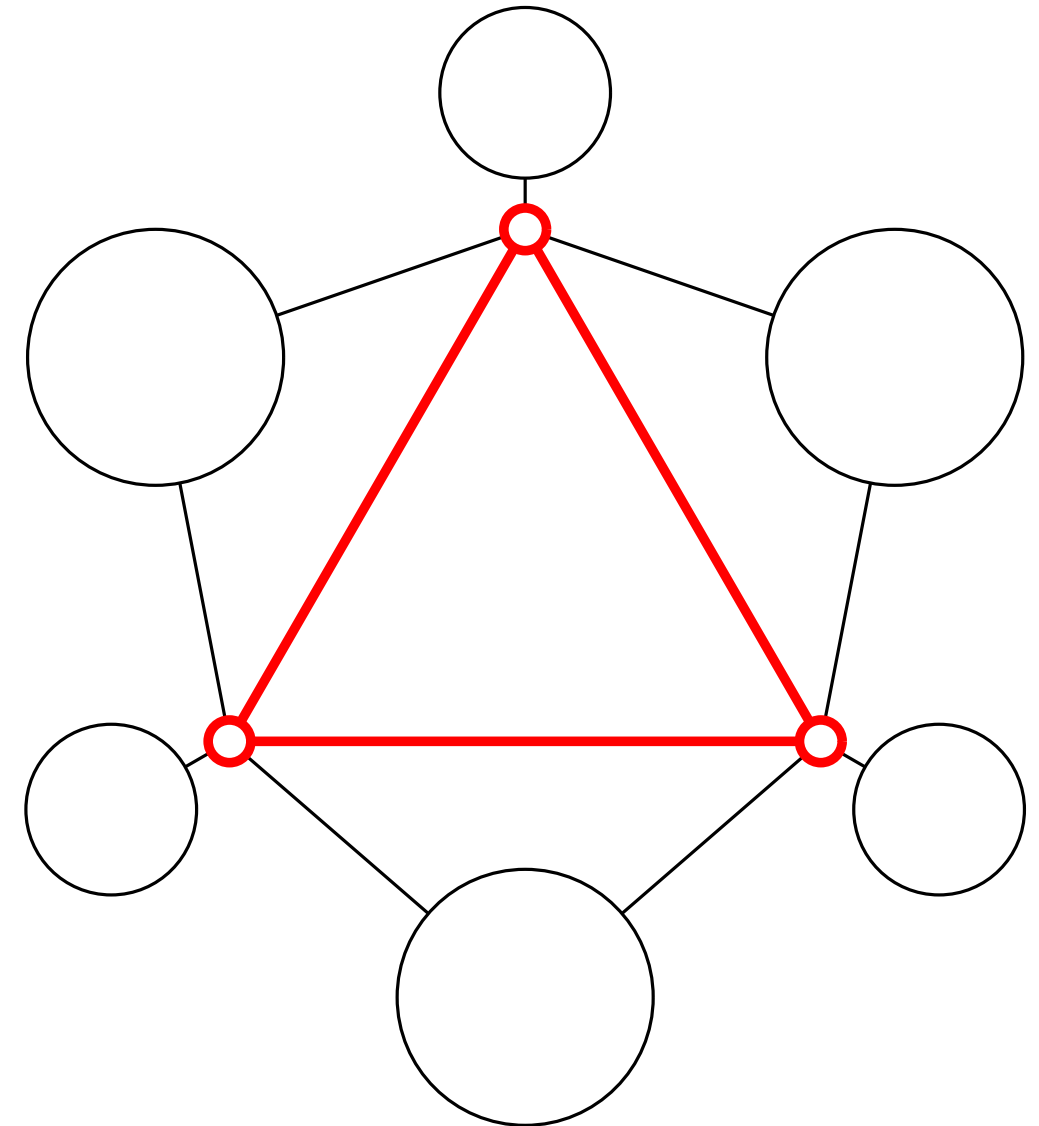
Inequalities

$$\begin{matrix} \text{diag} & + & \text{diag} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \left[\left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 + \left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 \right] \end{matrix} \geq \frac{2}{9}$$

$$\begin{matrix} \text{diag} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag} & + & \text{diag}) \cdot \left[\left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 + \left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 \right] \end{matrix} \geq \frac{2}{9}$$

$\left[\left(\frac{1}{3} - \text{diag} \right)^2 \right]$	0.0000	0.005	0.002
$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	0.0370	0.0363	0.0379
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \end{matrix}$	0.0000	0.0314	0.0249
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \end{matrix}$	0.0000	0.0805	0.0499
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \end{matrix}$	0.2963	0.2203	0.2498
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.0000	0.0098	0.0000
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \end{matrix}$	0.0000	0.0031	0.0000
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ \cdot \end{matrix}$	0.0000	0.0968	0.0499
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \end{matrix}$	0.0000	0.1278	0.0999
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.2222	0.1703	0.1874
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.4444	0.2232	0.2998

- $\text{diag} = \text{const}$
- $\text{diag} = \frac{2}{7}$
- $\text{diag} = 0$
- $\text{diag} = 0$



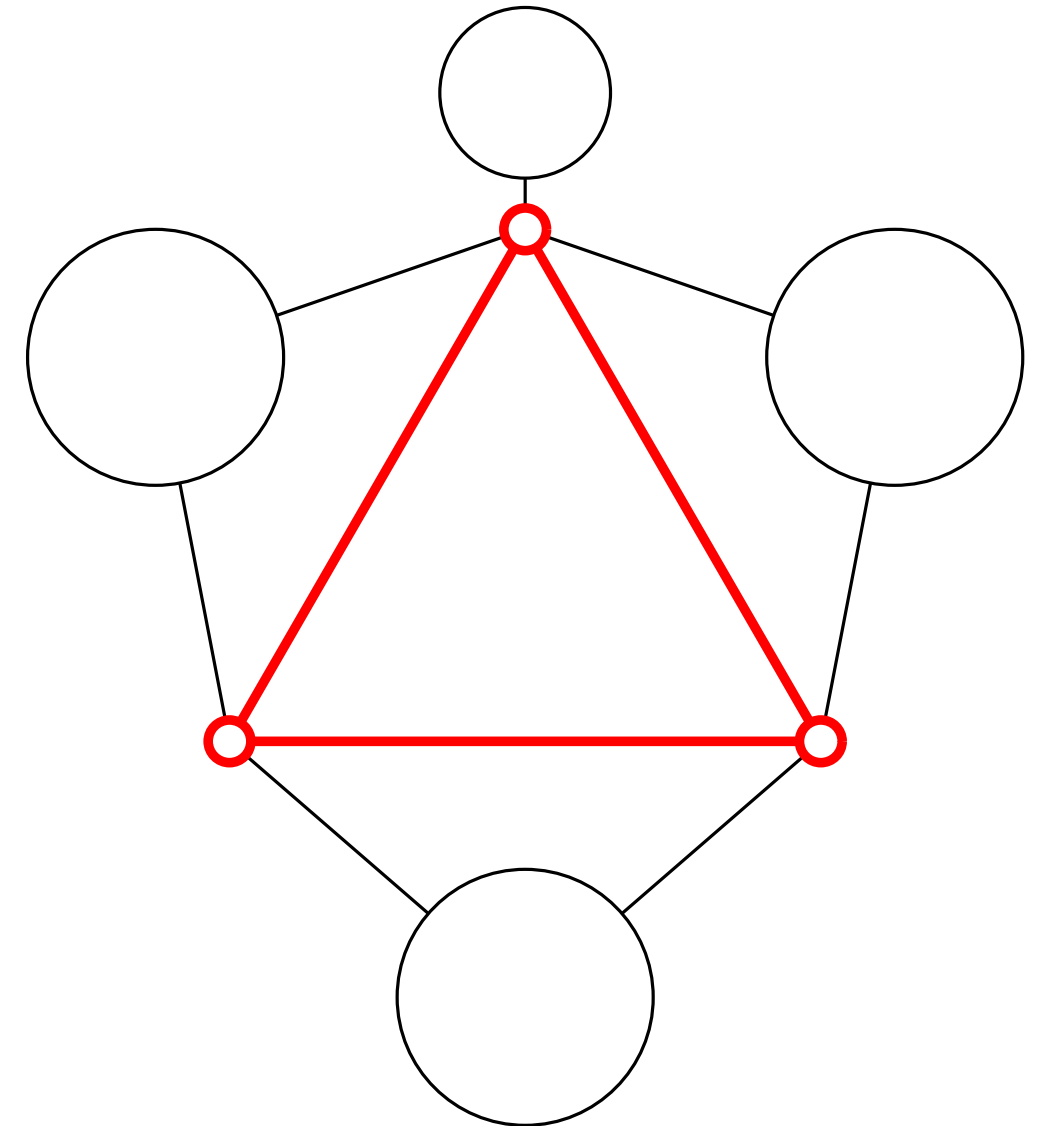
Inequalities

$$\begin{matrix} \text{diag} & + & \text{diag} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \left[\left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 + \left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 \right] \end{matrix} \geq \frac{2}{9}$$

$$\begin{matrix} \text{diag} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag} & + & \text{diag}) \cdot \left[\left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 + \left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 \right] \end{matrix} \geq \frac{2}{9}$$

$\left[\left(\frac{1}{3} - \text{diag} \right)^2 \right]$	0.0000	0.005	0.002
$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	0.0370	0.0363	0.0379
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \end{matrix}$	0.0000	0.0314	0.0249
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \end{matrix}$	0.0000	0.0805	0.0499
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \end{matrix}$	0.2963	0.2203	0.2498
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.0000	0.0098	0.0000
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \end{matrix}$	0.0000	0.0031	0.0000
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ \cdot \end{matrix}$	0.0000	0.0968	0.0499
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \end{matrix}$	0.0000	0.1278	0.0999
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.2222	0.1703	0.1874
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.4444	0.2232	0.2998

- $\text{diag} = \text{const}$
- $\text{diag} = \frac{2}{7}$
- $\text{diag} = 0$
- $\text{diag} = 0$



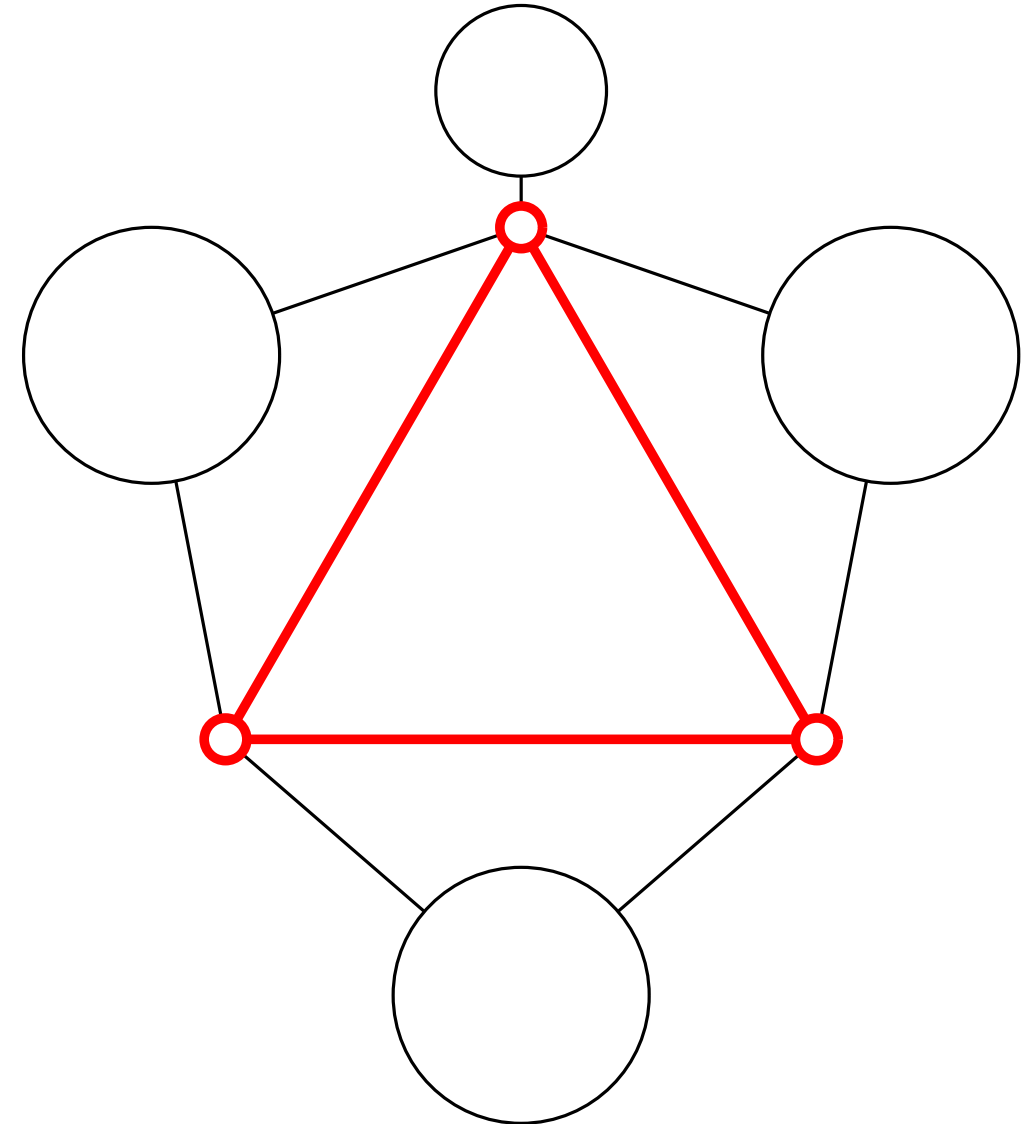
Inequalities

$$\begin{matrix} \text{diag} & + & \text{diag} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \left[\left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 + \left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 \right] \end{matrix} \geq \frac{2}{9}$$

$$\begin{matrix} \text{diag} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag} & + & \text{diag}) \cdot \left[\left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 + \left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 \right] \end{matrix} \geq \frac{2}{9}$$

$\left[\left(\frac{1}{3} - \text{diag} \right)^2 \right]$	0.0000	0.005	0.002
$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	0.0370	0.0363	0.0379
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \end{matrix}$	0.0000	0.0314	0.0249
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \end{matrix}$	0.0000	0.0805	0.0499
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \\ \cdot \end{matrix}$	0.2963	0.2203	0.2498
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \\ \cdot \\ \cdot \end{matrix}$	0.0000	0.0098	0.0000
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.0000	0.0031	0.0000
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.0000	0.0968	0.0499
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.0000	0.1278	0.0999
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.2222	0.1703	0.1874
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.4444	0.2232	0.2998

- $\text{diag} = \text{const}$
- $\text{diag} = \frac{2}{7}$
- $\text{diag} = 0$
- $\text{diag} = 0$
- $\text{diag} = 0$



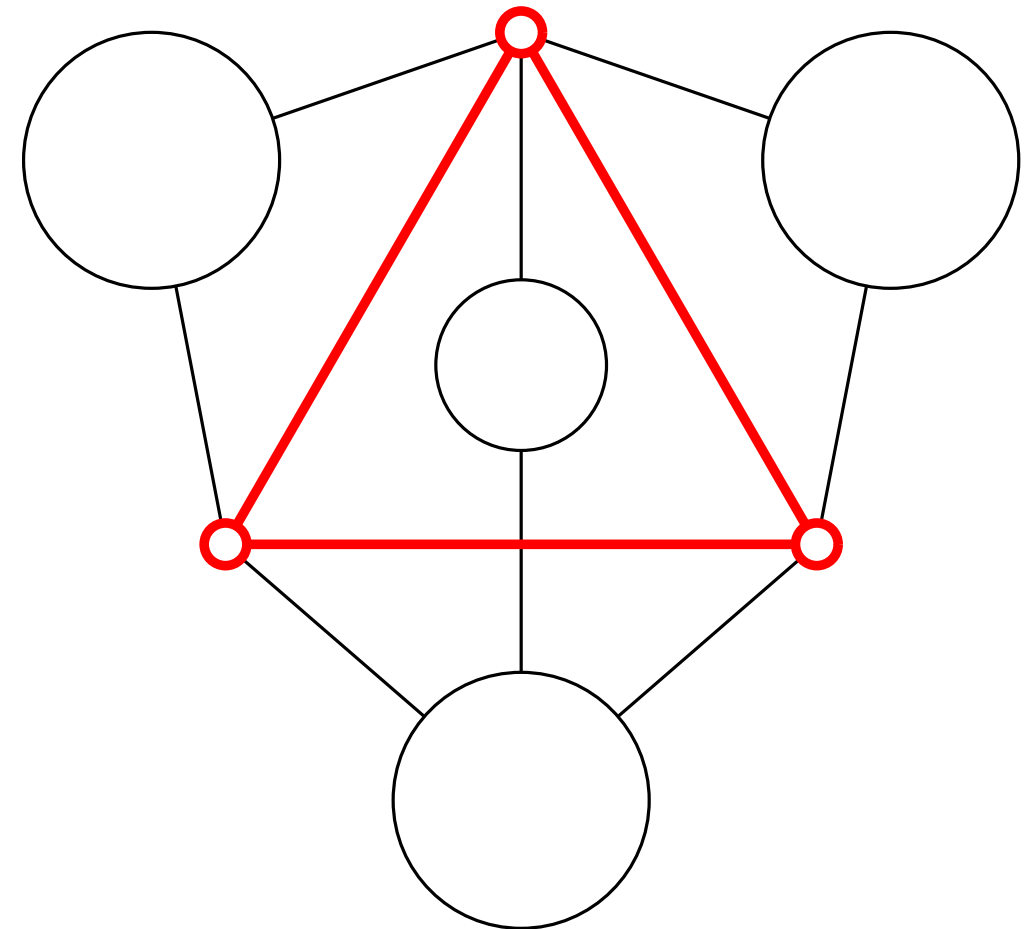
Inequalities

$$\begin{matrix} \text{diag} & + & \text{diag} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \left[\left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 + \left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 \right] \end{matrix} \geq \frac{2}{9}$$

$$\begin{matrix} \text{diag} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag} & + & \text{diag}) \cdot \left[\left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 + \left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 \right] \end{matrix} \geq \frac{2}{9}$$

$\left[\left(\frac{1}{3} - \text{diag} \right)^2 \right]$	0.0000	0.005	0.002
	0.0370	0.0363	0.0379
	0.0000	0.0314	0.0249
	0.0000	0.0805	0.0499
	0.2963	0.2203	0.2498
	0.0000	0.0098	0.0000
	0.0000	0.0031	0.0000
	0.0000	0.0968	0.0499
	0.0000	0.1278	0.0999
	0.2222	0.1703	0.1874
	0.4444	0.2232	0.2998

- = const
- = $\frac{2}{7}$
- = 0
- = 0
- = 0



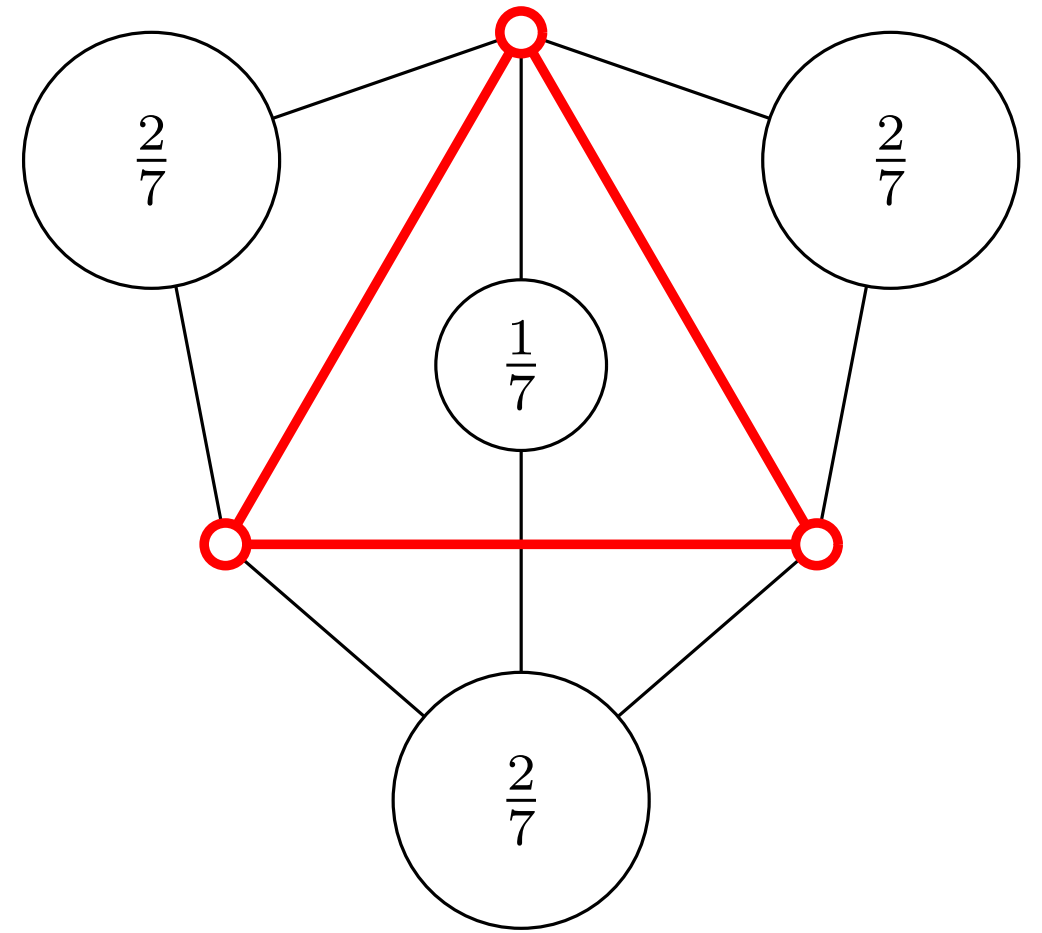
Inequalities

$$\begin{matrix} \text{diag} & + & \text{diag} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \left[\left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 + \left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 \right] \end{matrix} \geq \frac{2}{9}$$

$$\begin{matrix} \text{diag} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag} & + & \text{diag}) \cdot \left[\left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 + \left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 \right] \end{matrix} \geq \frac{2}{9}$$

$\left[\left(\frac{1}{3} - \text{diag} \right)^2 \right]$	0.0000	0.005	0.002
$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	0.0370	0.0363	0.0379
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \end{matrix}$	0.0000	0.0314	0.0249
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \end{matrix}$	0.0000	0.0805	0.0499
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \end{matrix}$	0.2963	0.2203	0.2498
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.0000	0.0098	0.0000
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \end{matrix}$	0.0000	0.0031	0.0000
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ \cdot \end{matrix}$	0.0000	0.0968	0.0499
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \end{matrix}$	0.0000	0.1278	0.0999
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.2222	0.1703	0.1874
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.4444	0.2232	0.2998

- $\text{diag} = \text{const}$
- $\text{diag} = \frac{2}{7}$
- $\text{diag} = 0$
- $\text{diag} = 0$
- $\text{diag} = 0$



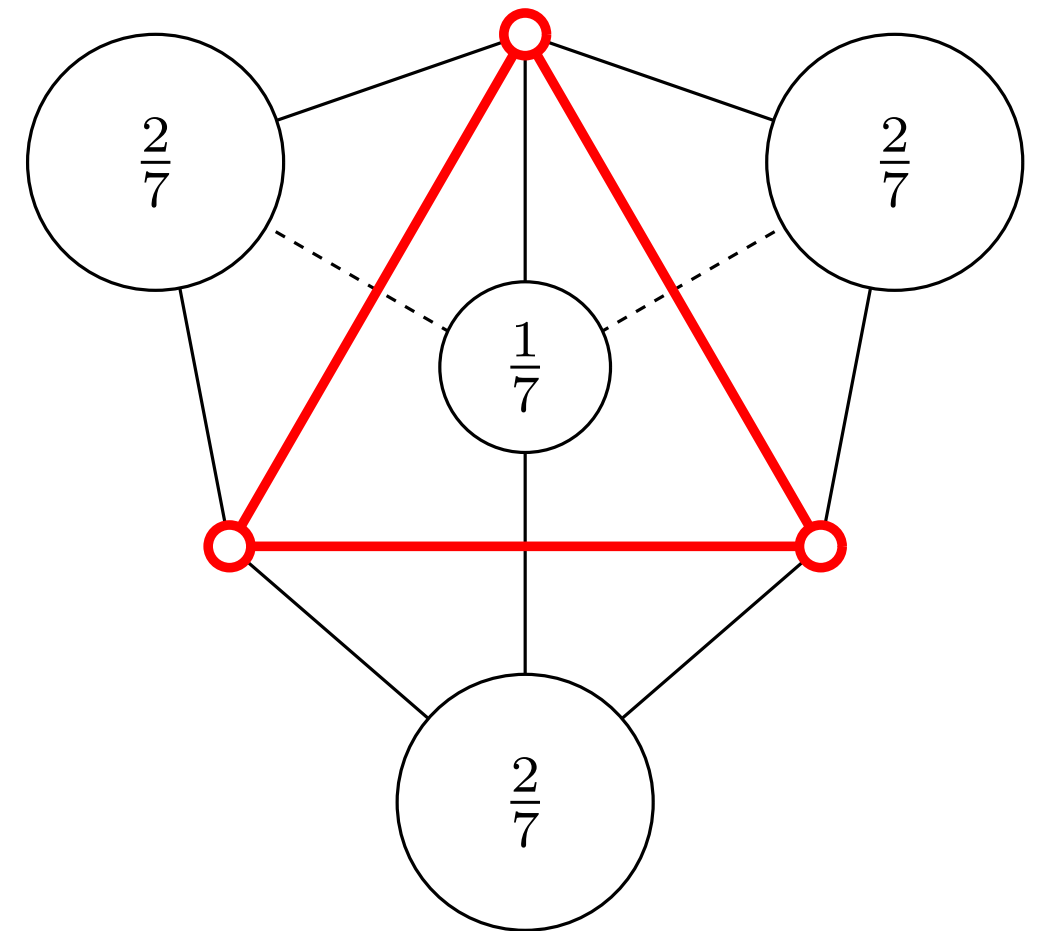
Inequalities

$$\begin{matrix} \text{diag} & + & \text{diag} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \left[\left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 + \left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 \right] \end{matrix} \geq \frac{2}{9}$$

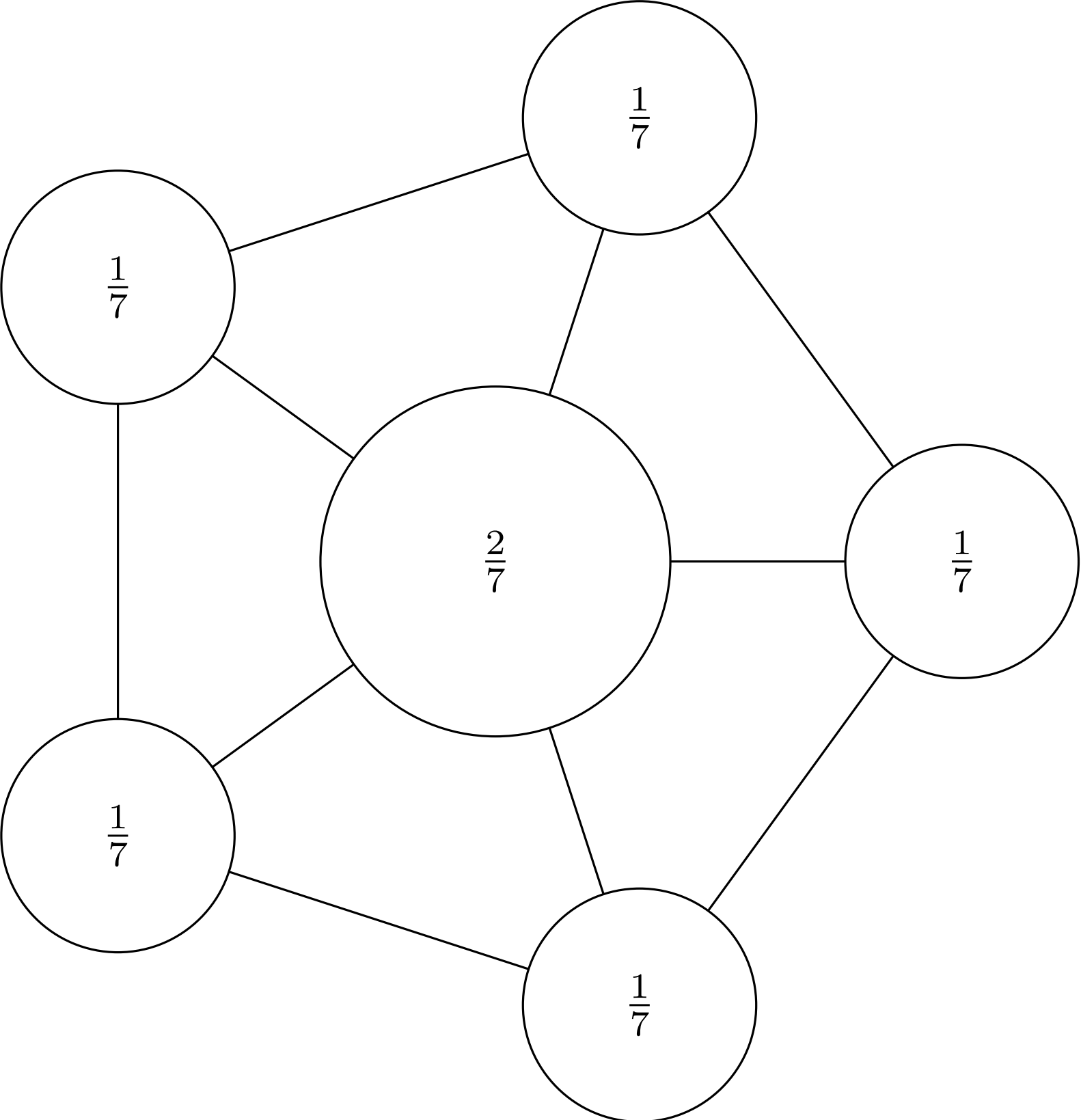
$$\begin{matrix} \text{diag} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag}) \cdot \frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} & + & (\text{diag} & + & \text{diag} & + & \text{diag}) \cdot \left[\left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 + \left(\frac{\frac{1}{2} - \text{diag}}{\text{diag} + \text{diag}} \right)^2 \right] \end{matrix} \geq \frac{2}{9}$$

$\left[\left(\frac{1}{3} - \text{diag} \right)^2 \right]$	0.0000	0.005	0.002
$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	0.0370	0.0363	0.0379
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \end{matrix}$	0.0000	0.0314	0.0249
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \end{matrix}$	0.0000	0.0805	0.0499
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \end{matrix}$	0.2963	0.2203	0.2498
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.0000	0.0098	0.0000
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \end{matrix}$	0.0000	0.0031	0.0000
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ \cdot \end{matrix}$	0.0000	0.0968	0.0499
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \end{matrix}$	0.0000	0.1278	0.0999
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.2222	0.1703	0.1874
$\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}$	0.4444	0.2232	0.2998

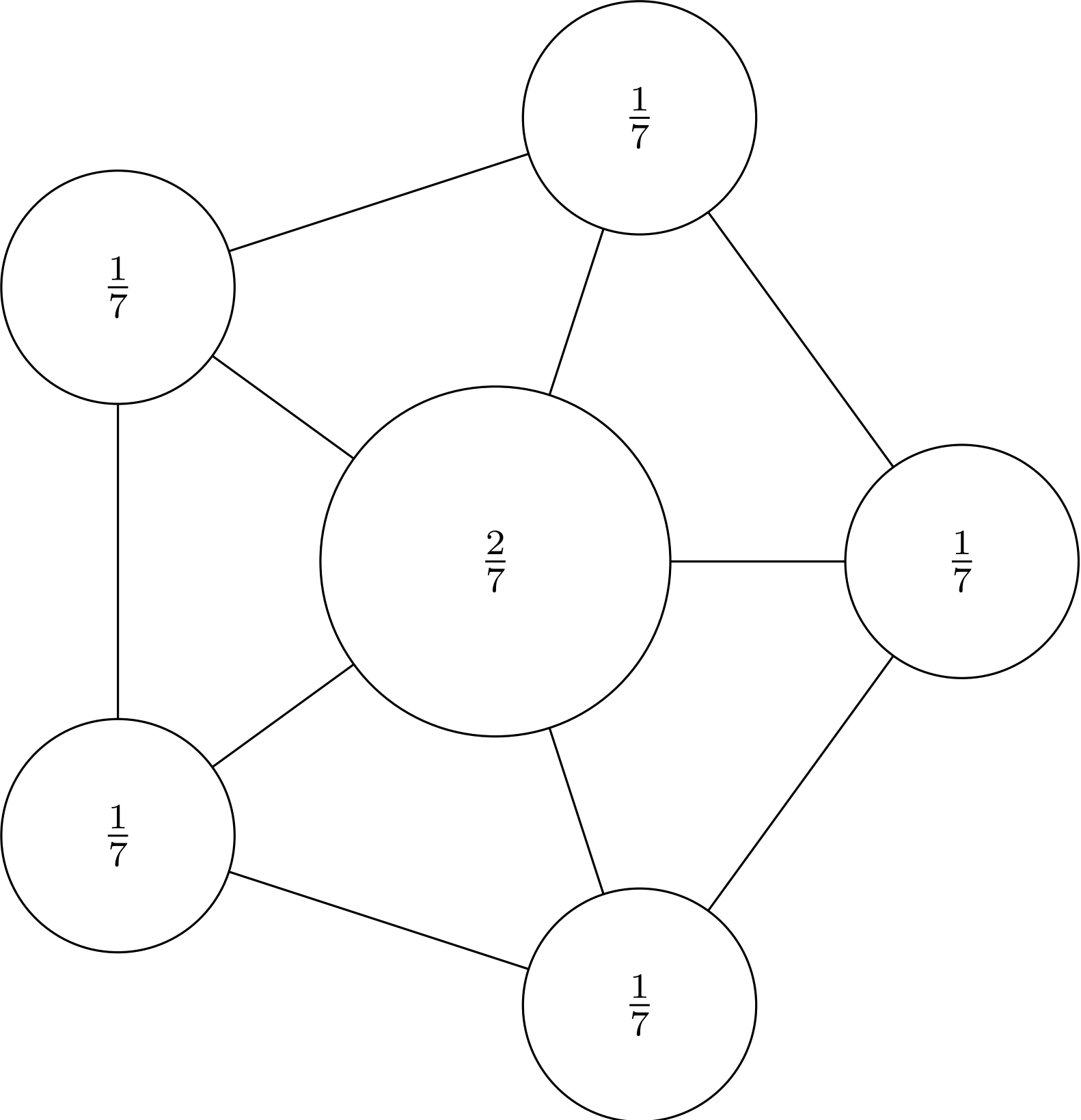
- $\text{diag} = \text{const}$
- $\text{diag} = \frac{2}{7}$
- $\text{diag} = 0$
- $\text{diag} = 0$
- $\text{diag} = 0$



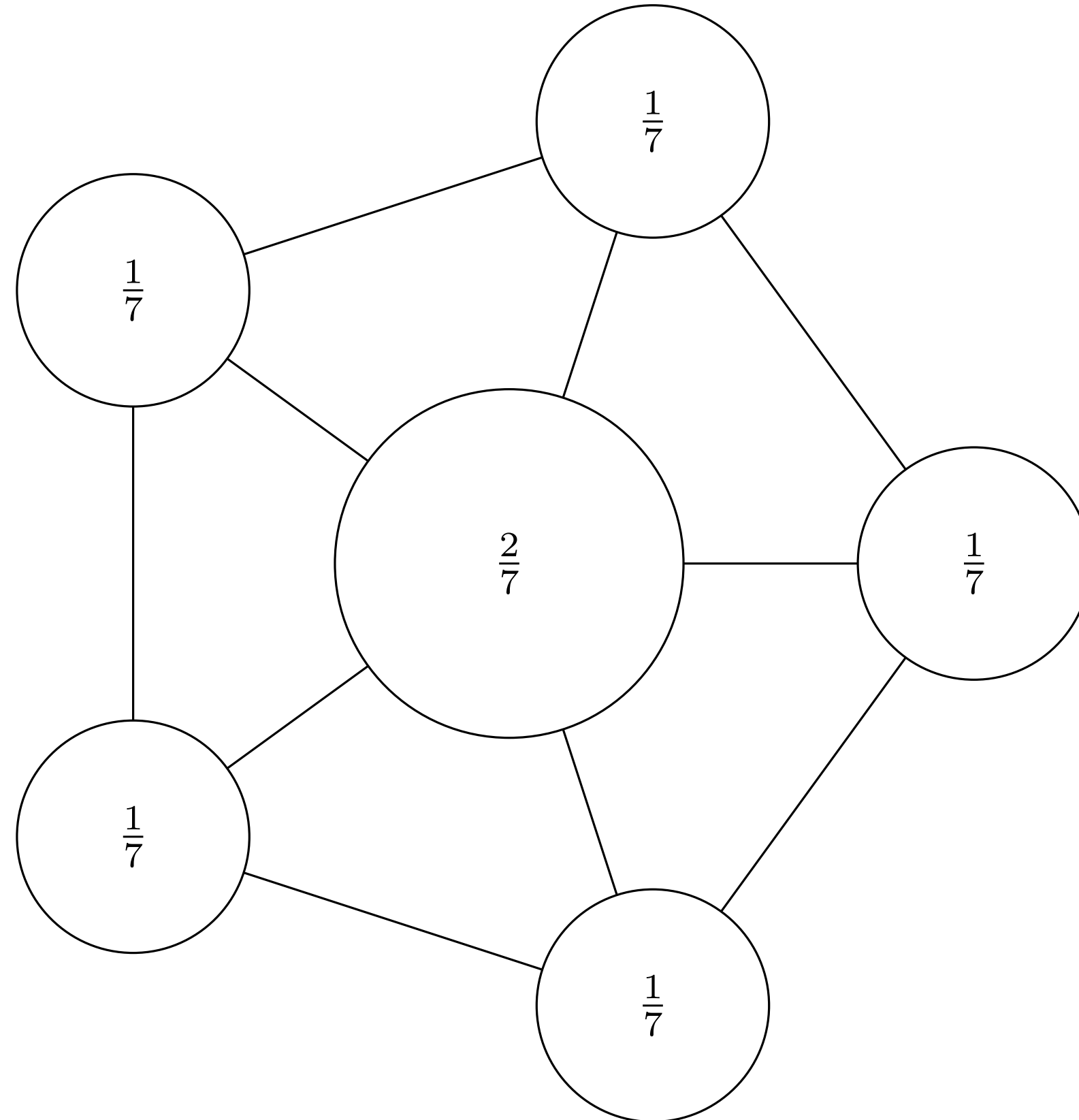
Counterexample



Counterexample



Counterexample



Thank you!