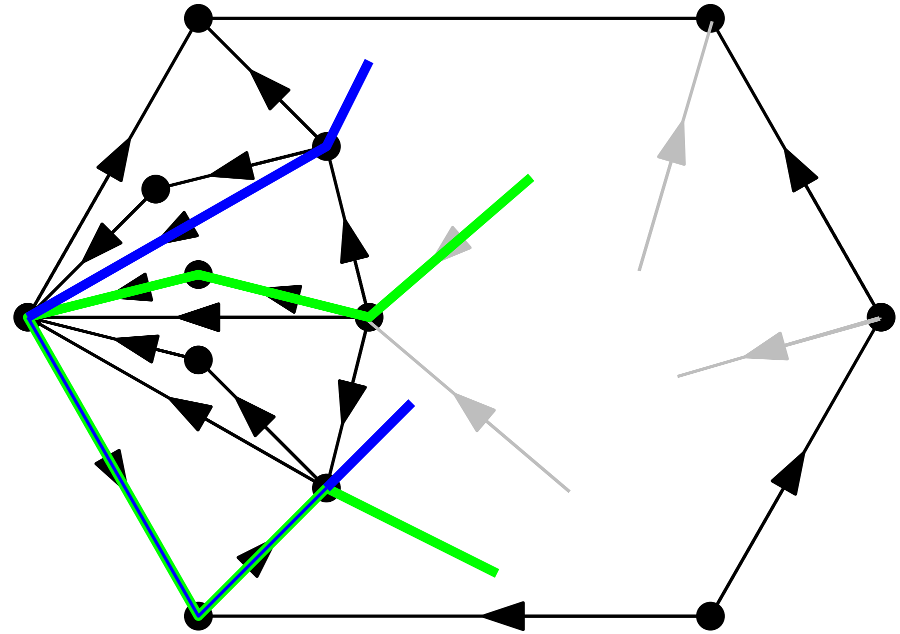
Alon-Tarsi number of planar graphs

A simple proof

Presentation by Łukasz Gniecki

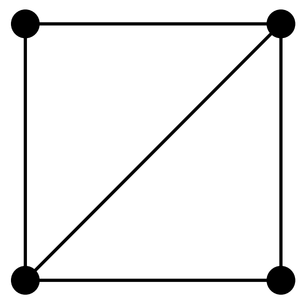


The Alon-Tarsi number of planar graphs – a simple proof
Yangyan Gu, Xuding Zhu

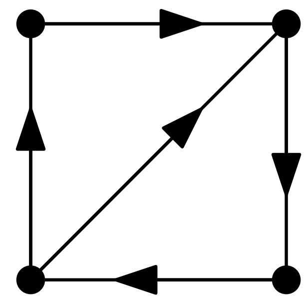
The Alon-Tarsi number

Let D be an orientation of G

G



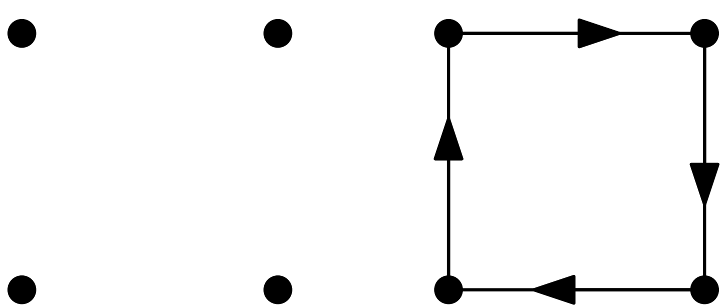
D



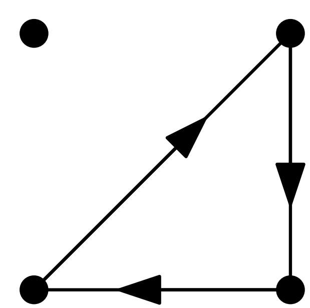
$H \subset E(D)$ is eulerian if $d_H^+(v) = d_H^-(v)$

H can be either even or odd (considering $|H|$)

Even



Odd



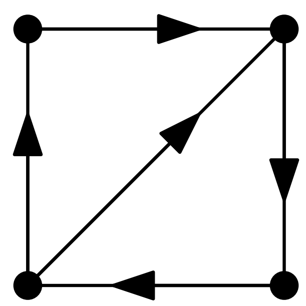
The Alon-Tarsi number

$\mathcal{E}_e(D) = \{H \subset E(D)$: H is eulerian and $|H|$ is even $\}$

$\mathcal{E}_o(D) = \{H \subset E(D)$: H is eulerian and $|H|$ is odd$\}$

diff$(D) = ||\mathcal{E}_e(D)| - |\mathcal{E}_o(D)||$

D is an AT orientation if diff$(D) \neq 0$

diff $\left(\right.$  $\left. \right) = 1$

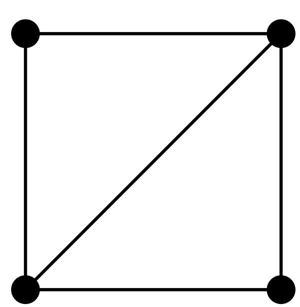
The Alon-Tarsi number

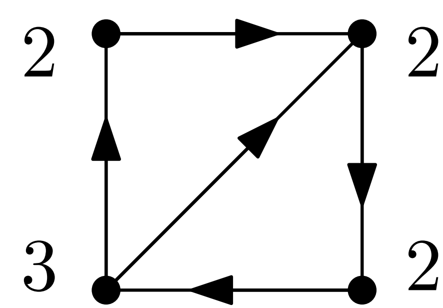
Let $f : V(G) \longrightarrow \mathbb{N}$

D is f-AT if it is AT, and $d_D^+(v) \leq f(v) - 1$

G is f-AT if it has an f-AT orientation D

The Alon-Tarsi number AT(G) is the minimum k such that G is f-AT where f is the constant function $f \equiv k$

AT $\Bigg($  $\Bigg)$ ≤ 3



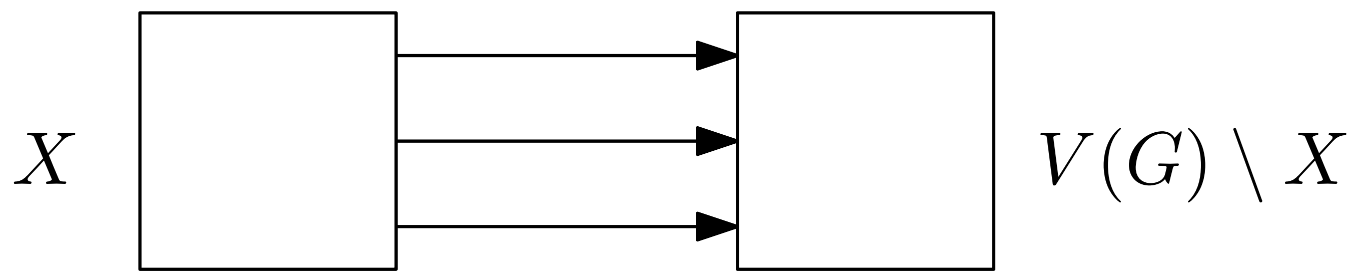
Helpful lemmas - product of diffs

Lemma 1

Let D be an orientation of G and $X \subset V(G)$

If all edges of D are oriented from X to $V(G) \setminus X$ then

$$\text{diff}(D) = \text{diff}(D[X]) \cdot \text{diff}(D[V(G) \setminus X])$$



$$|\mathcal{E}_e(D)| = |\mathcal{E}_e(X)||\mathcal{E}_e(V \setminus X)| + |\mathcal{E}_o(X)||\mathcal{E}_o(V \setminus X)|$$
$$|\mathcal{E}_o(D)| = |\mathcal{E}_e(X)||\mathcal{E}_o(V \setminus X)| + |\mathcal{E}_o(X)||\mathcal{E}_e(V \setminus X)|$$

$$\text{diff}(D) = ||\mathcal{E}_e(D)| - |\mathcal{E}_o(D)|| =$$
$$|\mathbf{(}|\mathcal{E}_e(X)| - |\mathcal{E}_o(X)|\mathbf{)}\ \mathbf{(}|\mathcal{E}_e(V \setminus X)| - |\mathcal{E}_o(V \setminus X)|\mathbf{)}| =$$
$$\text{diff}(X)\text{diff}(V \setminus X)$$

Helpful lemmas - edge removal

Let $f : V(G) \longrightarrow \mathbb{N}$.

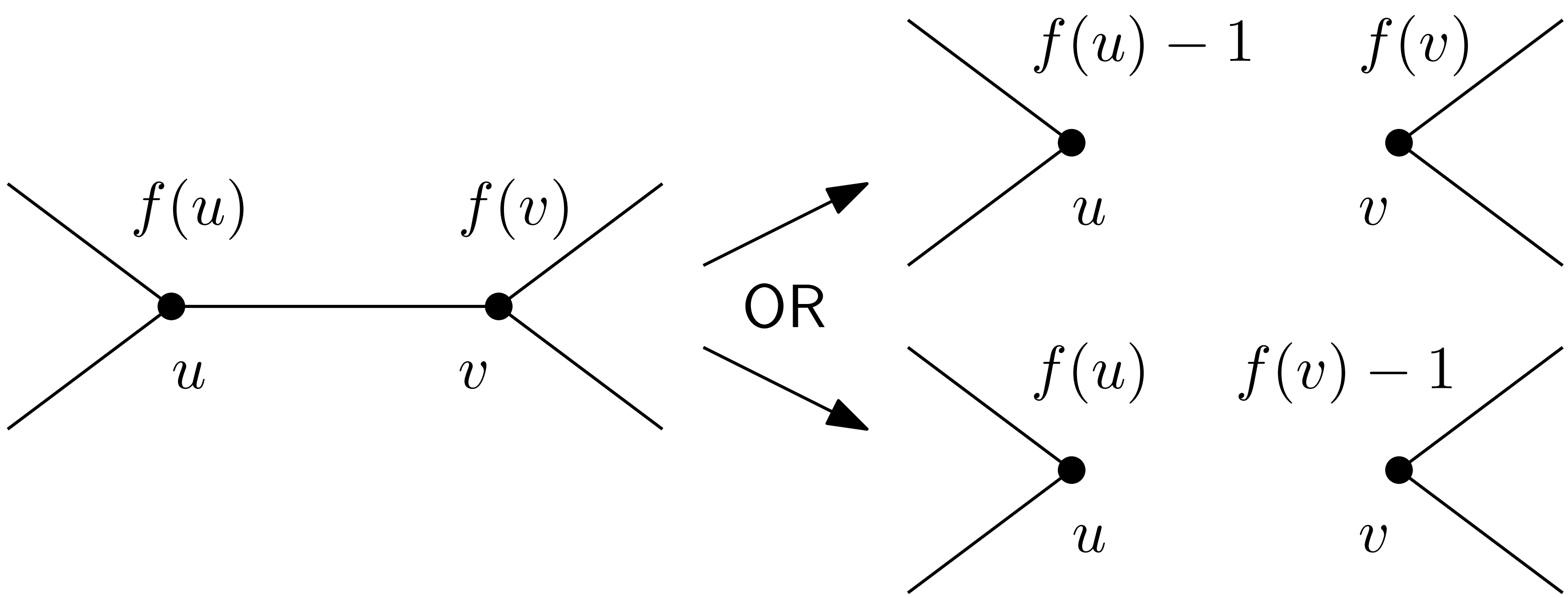
Define $f_{[u,-1]}$ as a function such that

$f_{[u,-1]}(v) = f(v)$ for $v \in V(G) \setminus \{u\}$

$f_{[u,-1]}(u) = f(u) - 1$.

Lemma 2

If G is f-AT and uv is an edge then $G - uv$ is $f_{[u,-1]}$-AT or $f_{[v,-1]}$-AT

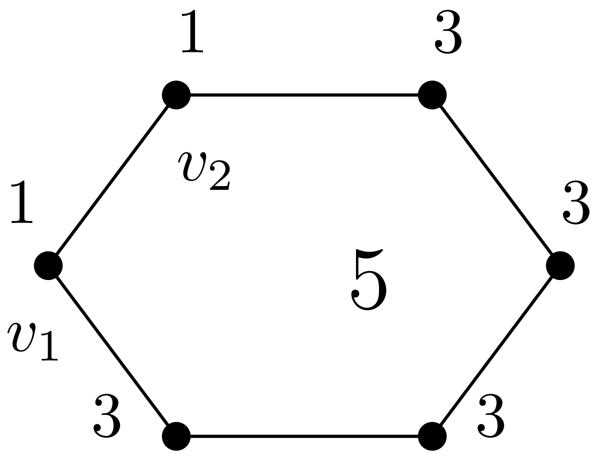


The AT number of planar graphs

Let G be a 2-connected planar graph and v_1v_2 a boundary edge.

Define f_{G,v_1,v_2}

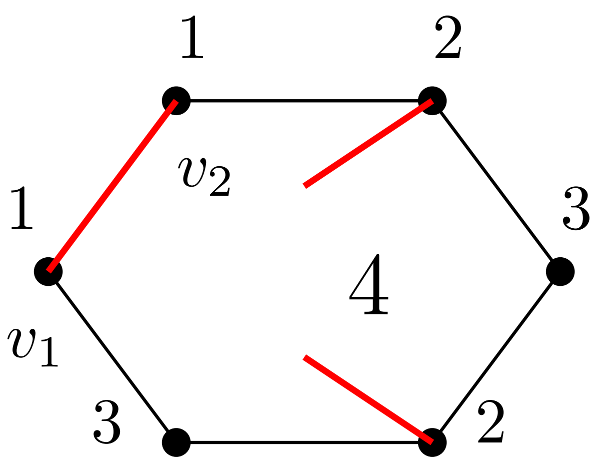
- 1 for v_1, v_2
- 3 for remaining boundary vertices
- 5 for inner vertices



Additionally let M be a matching which contains v_1v_2

Define $f_{G,v_1,v_2,M}$

- 1 for v_1, v_2
- $3 - d_M(v)$ for remaining boundary vertices
- 4 for inner vertices

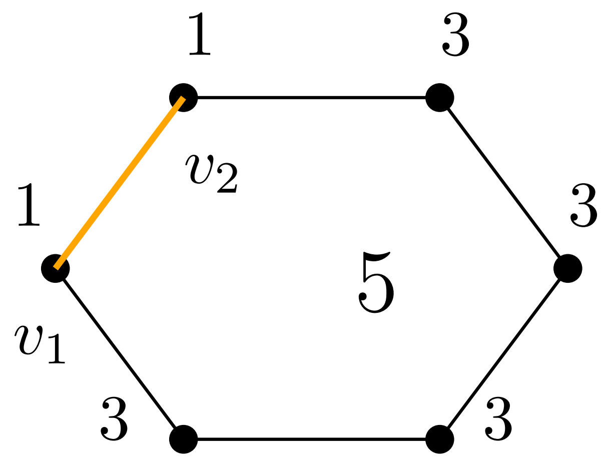


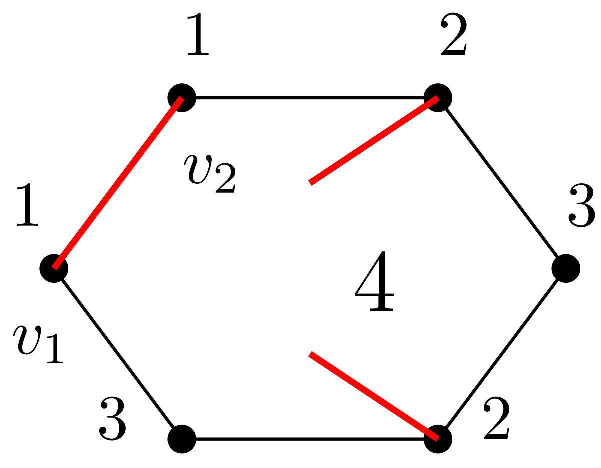
The AT number of planar graphs

Main Theorem

Let G be a 2-connected planar graph and v_1v_2 be a boundary edge.

1. $G - v_1v_2$ is f_{G,v_1,v_2}-AT
2. G has a matching M which contains v_1v_2 such that $G - M$ is $f_{G,v_1,v_2,M}$-AT





Simple consequences:

AT(planar) ≤ 5

AT(planar - matching) ≤ 4

proof...

The AT number of planar graphs

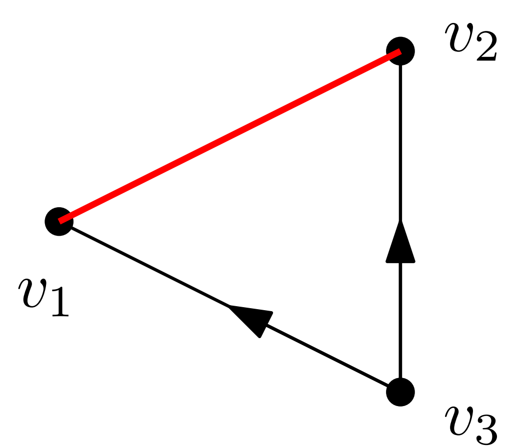
Induction on $|V(G)|$

Base

G is a triangle

Orient edges from v_3 to other vertices

Take matching M as $\{v_1v_2\}$



In the base case $f_{G,v_1,v_2,M} = f_{G,v_1,v_2} = \{(v_1,1),(v_2,1),(v_3,3)\}$

We see that $d^+(v) \leq f_{[G,v_1,v_2]}(v) - 1$

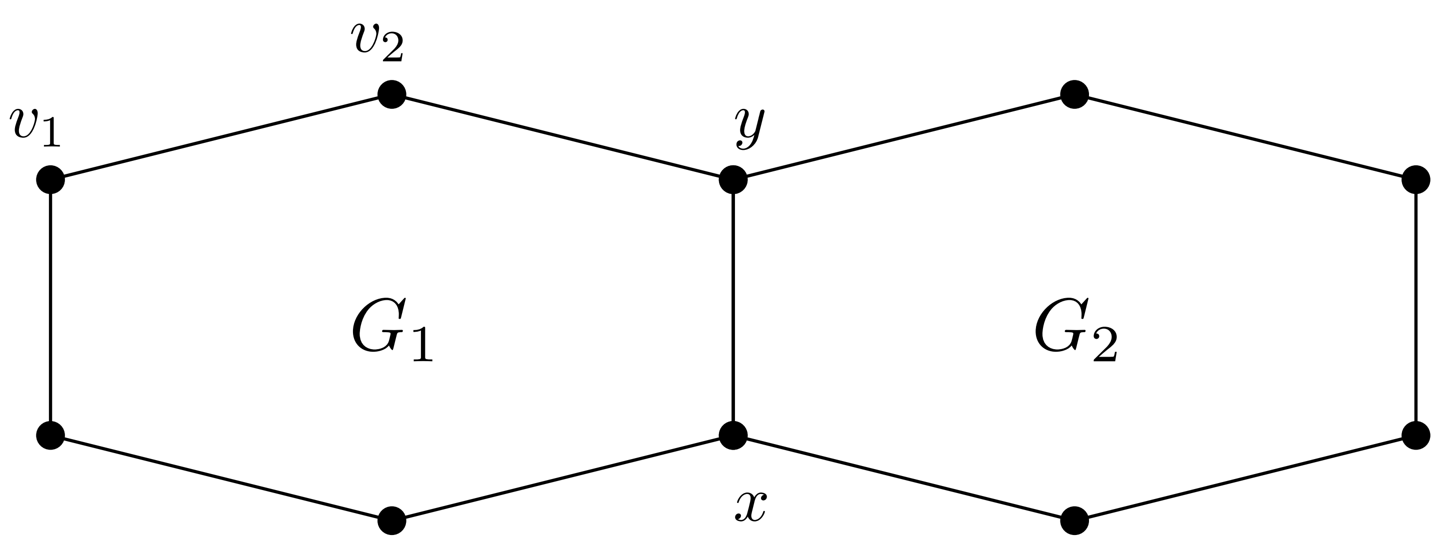
Also diff $= 1$

Step...

The AT number of planar graphs

Case 1

G has a chord xy



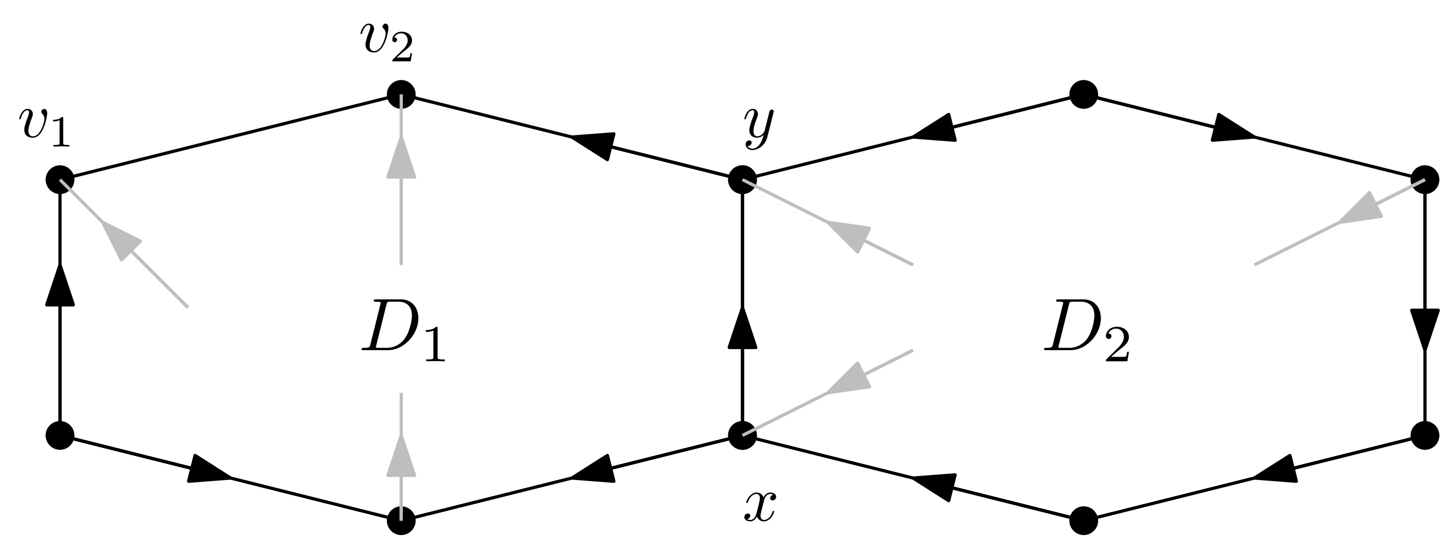
Let G_1 and G_2 be separated by the chord, both containing xy

Apply inductive hypothesis on G_1, v_1v_2 and G_2, xy

The AT number of planar graphs

Case 1

G has a chord xy



Let D_1 be orientation of $G_1 - v_1v_2$, and D_2 orientation of $G_2 - xy$

Take $D = D_1 \cup D_2$

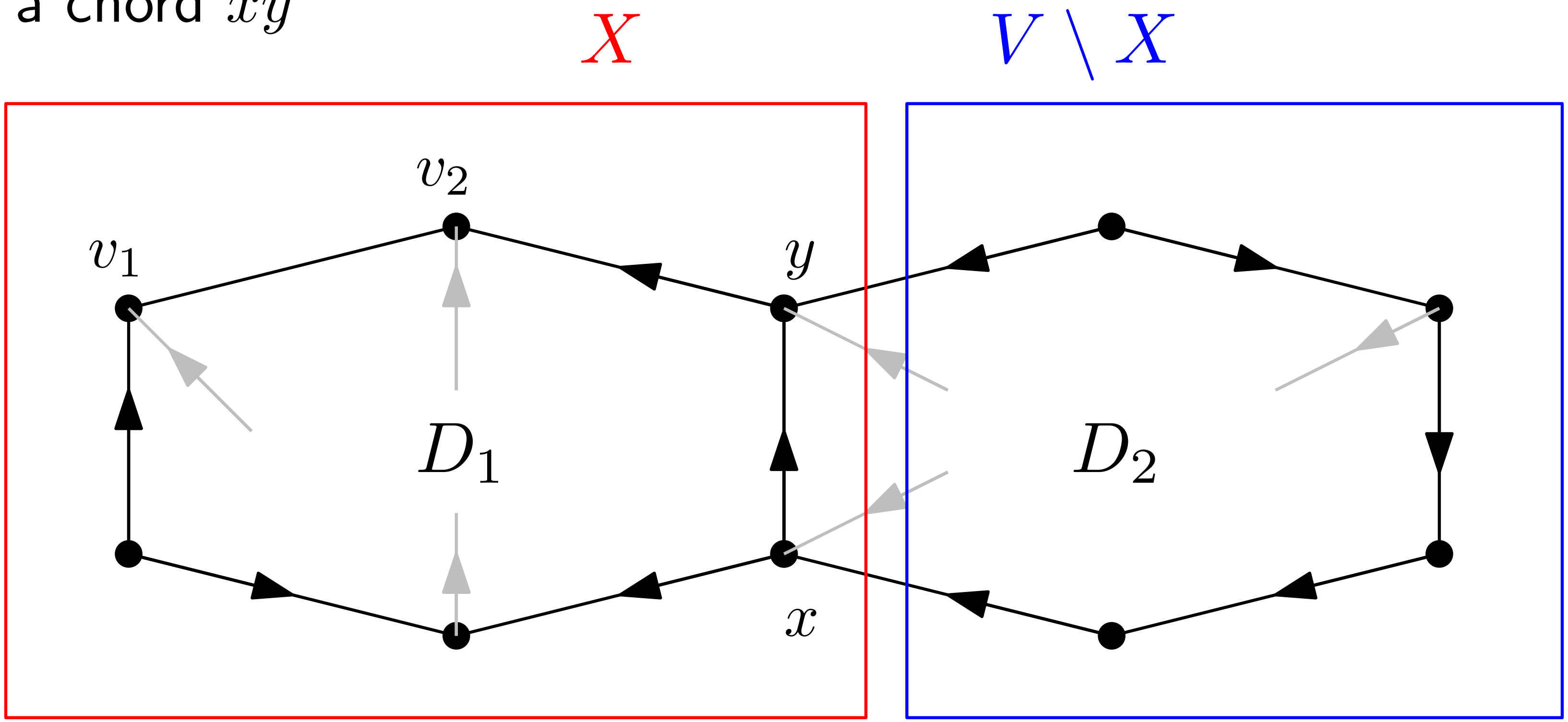
We need to show that $\text{diff}(D) \neq 0$

and that the out degrees in D are bounded by $f_{G,v_1,v_2} - 1$

The AT number of planar graphs

Case 1

G has a chord xy



From induction diff(D_1), diff$(D_2) \neq 0$

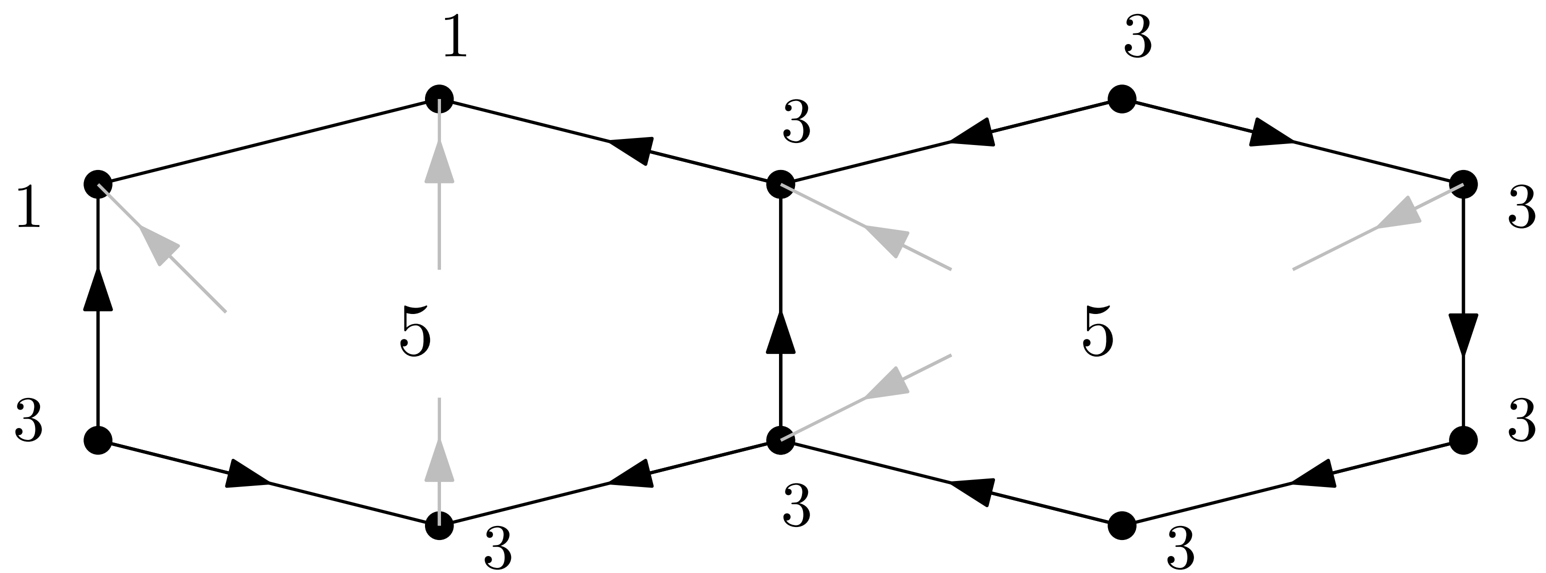
In D_2 all edges touching x, y point towards them

Applying **Lemma 1** on $X, V \setminus X$: diff$(D) =$ diff(D_1) diff$(D_2) \neq 0$

The AT number of planar graphs

Case 1

G has a chord xy



From induction D_1 aligns with f_{G_1,v_1,v_2} and D_2 aligns with $f_{G_2,x,y}$

Out degree of x, y is only influenced by D_1

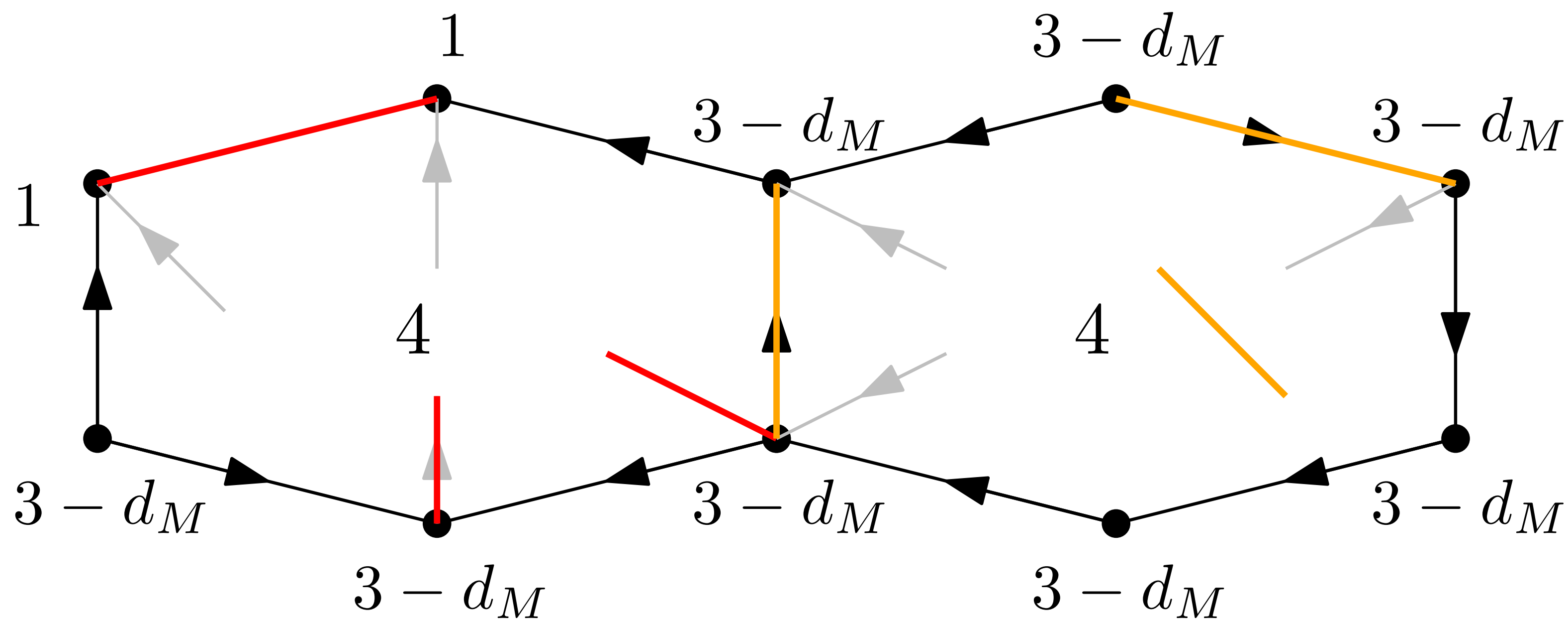
Out degrees in D are bounded by $f_{G,v_1,v_2} - 1$

$G - v_1v_2$ is f_{G,v_1,v_2}-AT

The AT number of planar graphs

Case 1

G has a chord xy



Matching: from induction we get M_1, M_2

$v_1v_2 \in M_1, xy \in M_2$

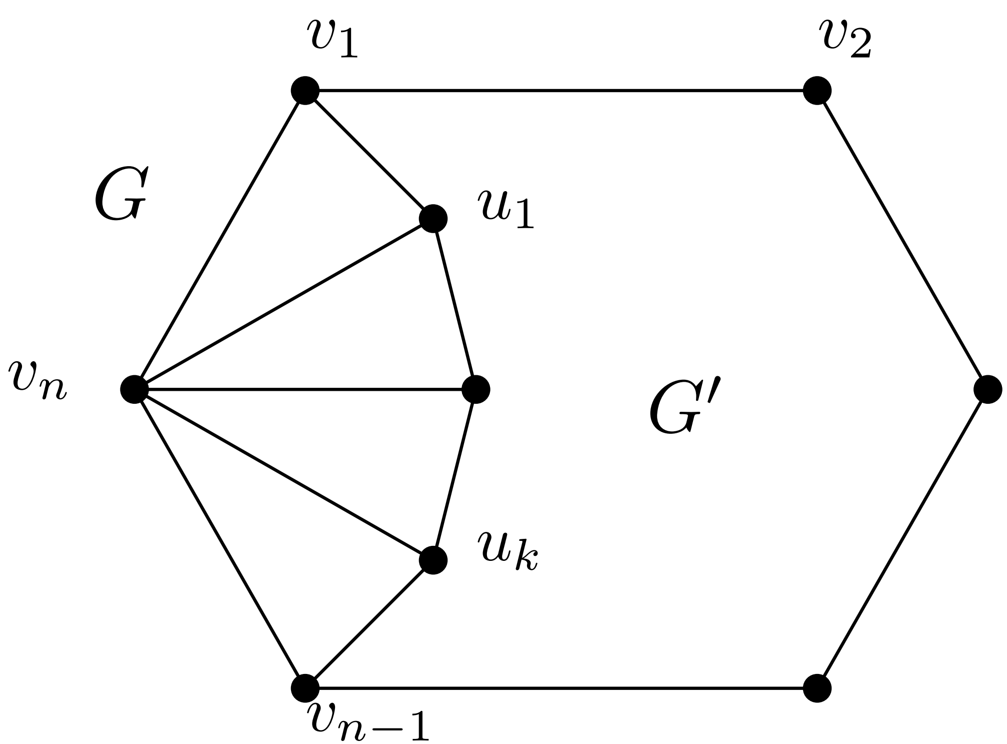
We pick $M = M_1 \cup (M_2 \setminus \{xy\})$

$G - M$ is $f_{G,v_1,v_2,M}$-AT

The AT number of planar graphs

Case 2

G has no chord



Let $v_1, ..., v_n$, $n \geq 3$ be the boundary

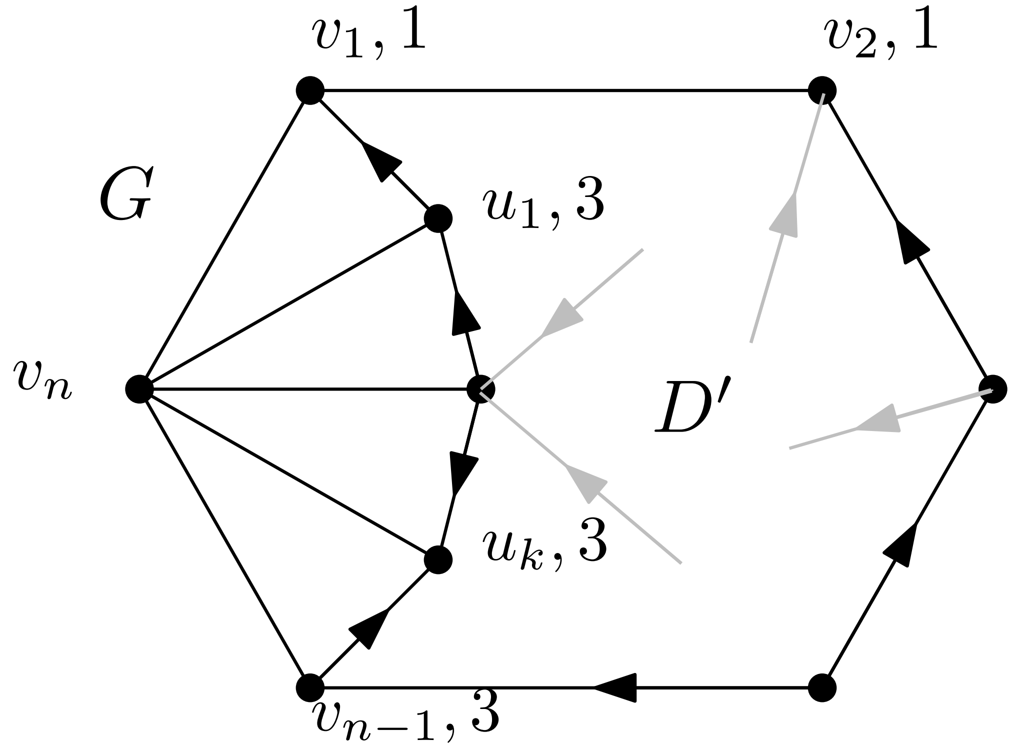
Let $u_1, ..., u_k$ be the neighbors of v_n other than v_1, v_{n-1}

We apply the induction hypothesis on $G' = G - v_n$

The AT number of planar graphs

Case 2

G has no chord

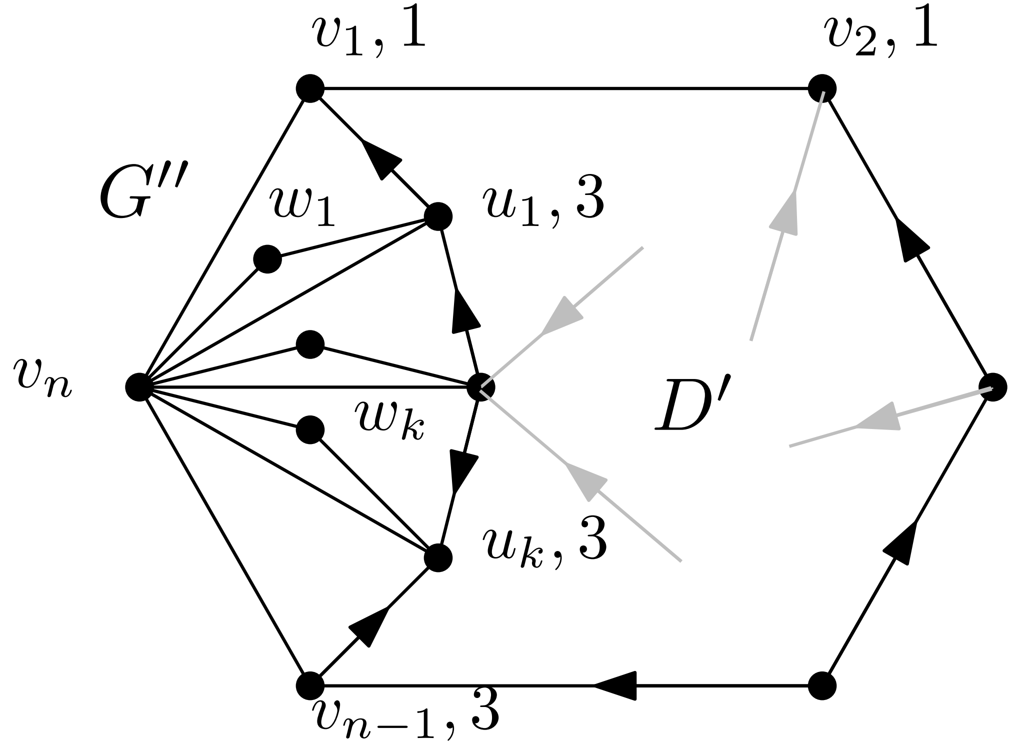


and obtain an orientation D' of $G' - v_1 v_2$
with diff $\neq 0$, aligning with f_{G', v_1, v_2}

The AT number of planar graphs

Case 2

G has no chord

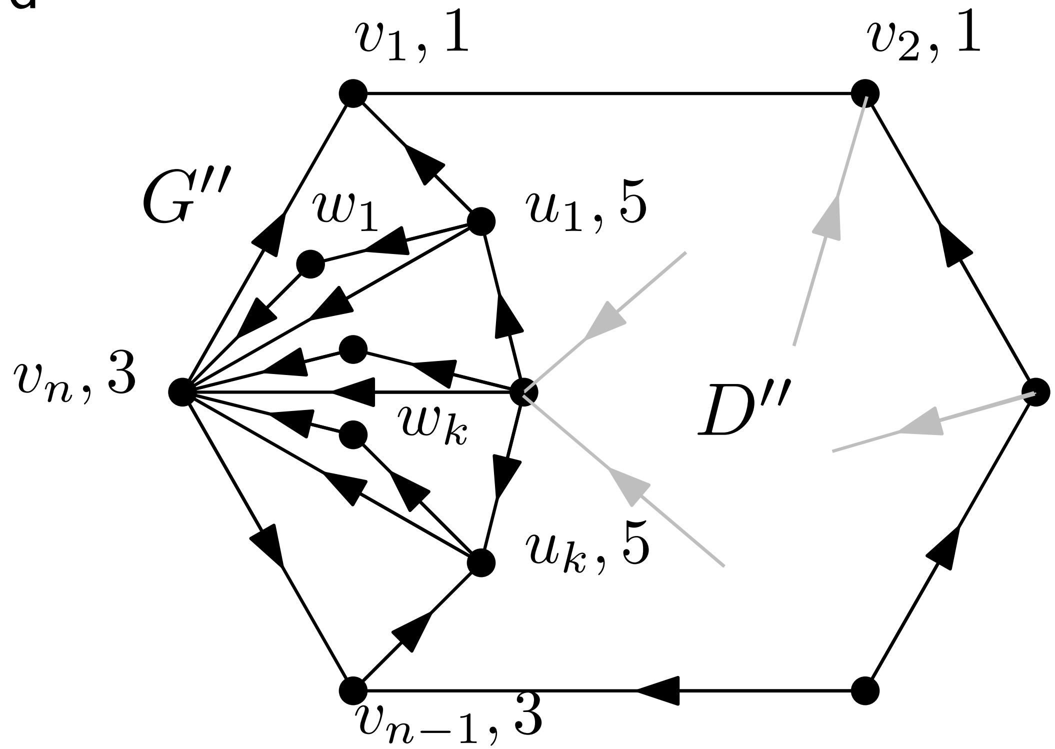


We add additional vertices $w_1, ..., w_k$ to obtain G''

The AT number of planar graphs

Case 2

G has no chord



We create D'' by orienting the remaining edges of $G'' - v_1 v_2$

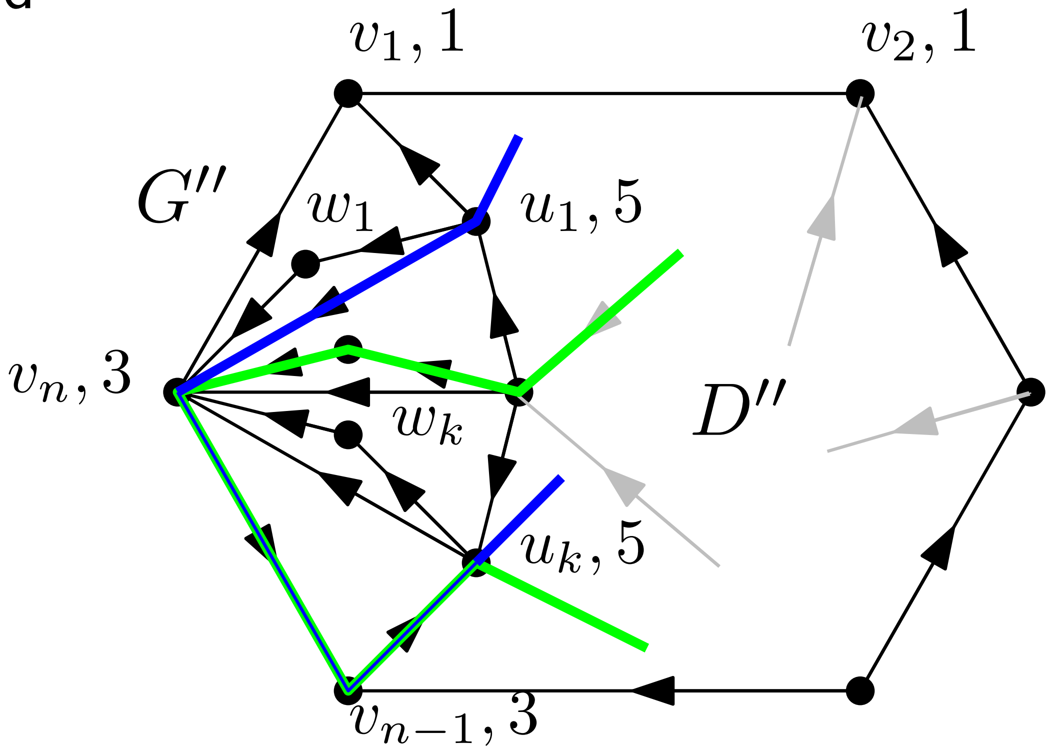
Each eulerian subset of D' remains eulerian in D''

There are also some new eulerian subsets...

The AT number of planar graphs

Case 2

G has no chord



Every new subset has to go through v_n and then v_{n-1}

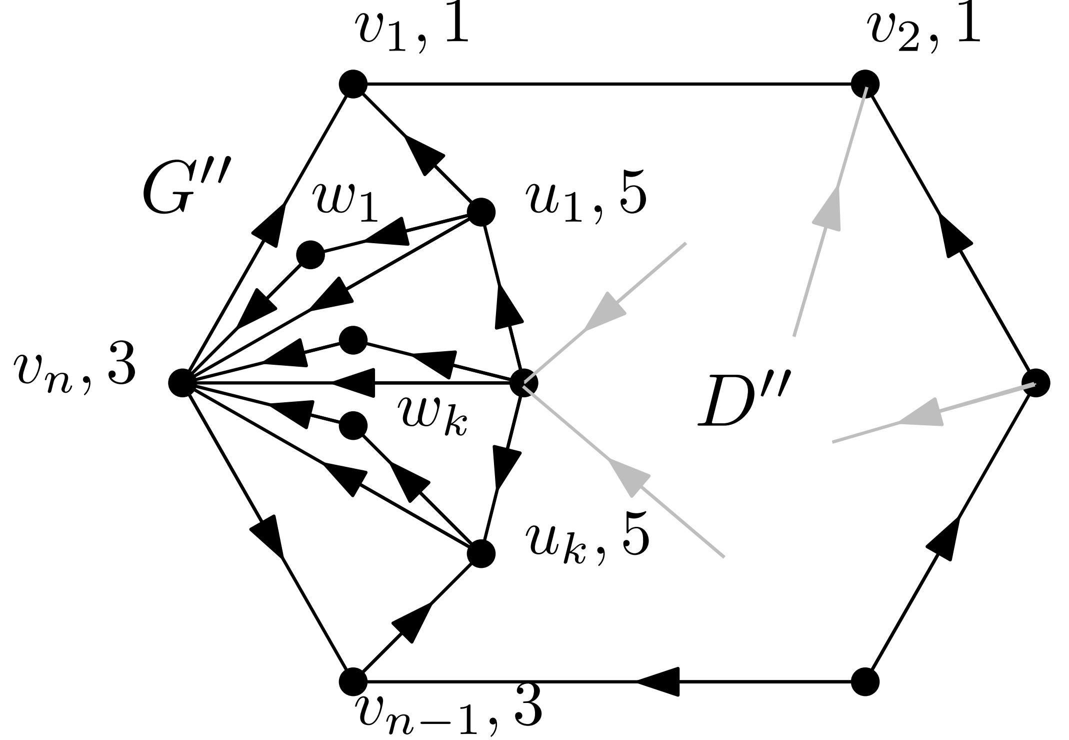
Observation: we added the same number of even eulerian subsets as odd eulerian subsets

Thus, diff(D'') = diff$(D') \neq 0$

The AT number of planar graphs

Case 2

G has no chord

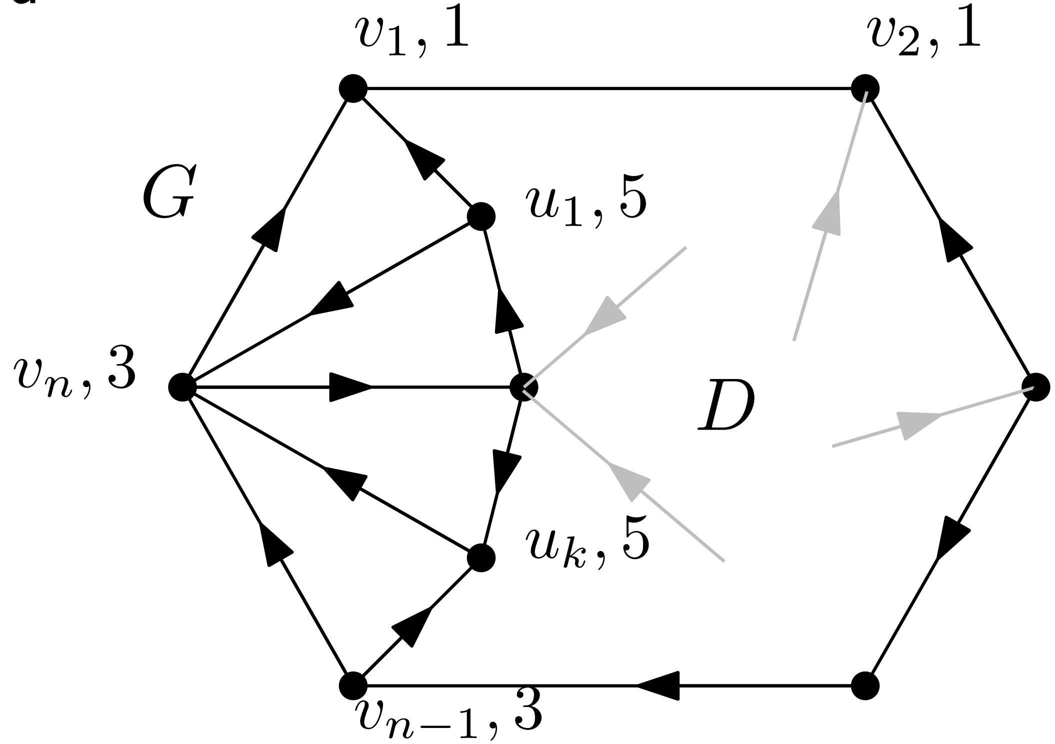


The given G'' will be f_{G'',v_1,v_2}-AT

The AT number of planar graphs

Case 2

G has no chord



Remove each w_i along with edges $u_i w_i$ and $w_i v_n$ to obtain G

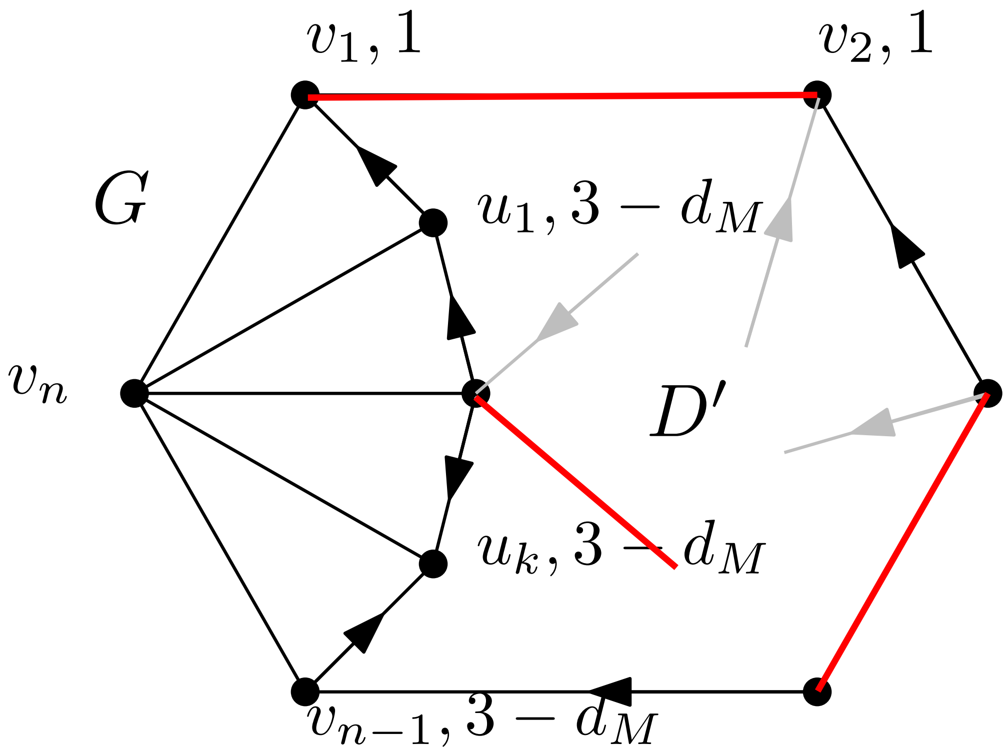
when removing edges, use **Lemma 2** which may alter the orientation but diff $\neq 0$ is preserved, and out degrees not increased

Conclusion: G is f_{G,v_1,v_2}-AT

The AT number of planar graphs

Case 2

G has no chord

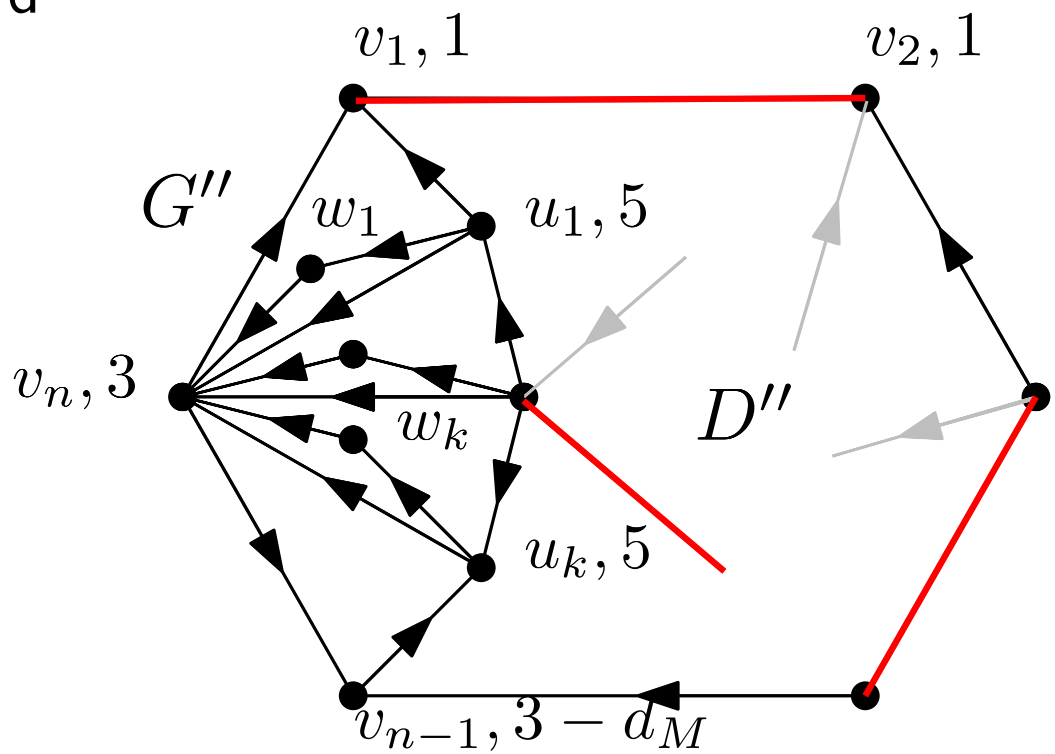


For the matching, the induction gives matching M' of G' such that $G' - M'$ is $f_{G',v_1,v_2,M'}$-AT satisfied by an orientation D' of $G' - M'$

The AT number of planar graphs

Case 2

G has no chord

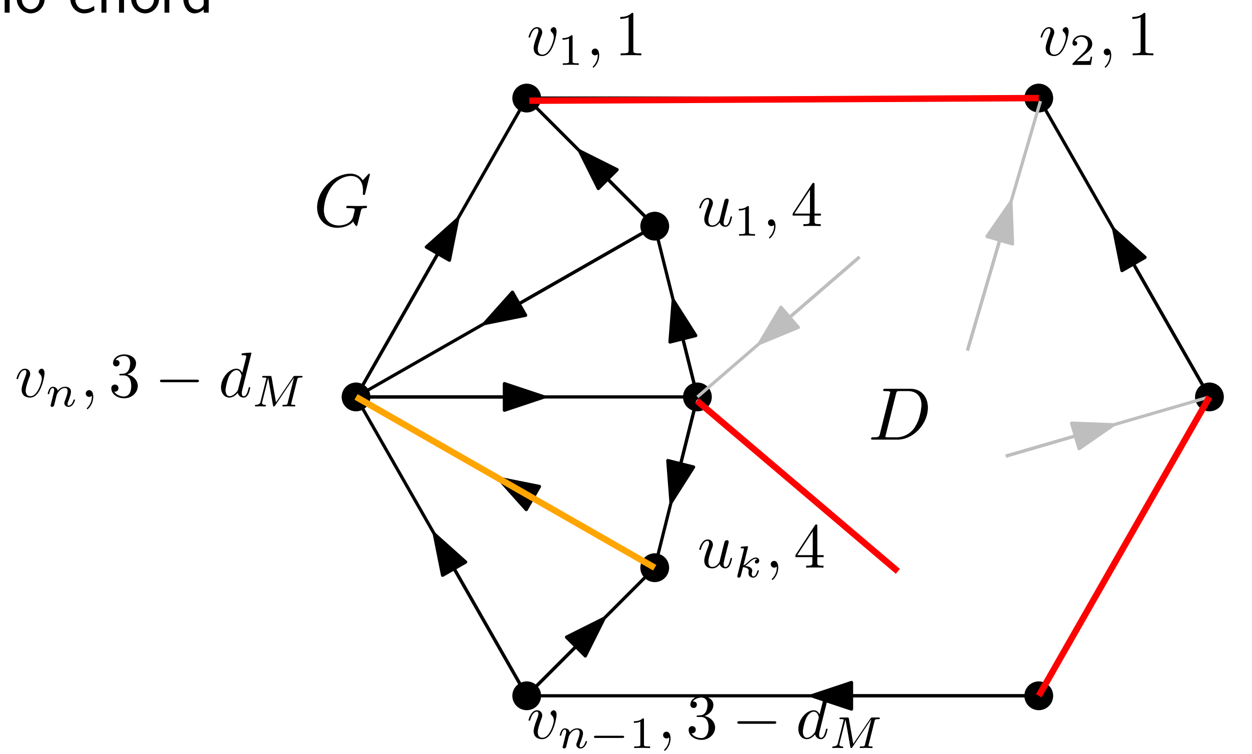


We do the same trick of adding vertices w_i along with edges $u_i w_i$ and $w_i v_n$ and orienting the remaining edges as on the picture

The AT number of planar graphs

Case 2

G has no chord



Removal of w_i preserves diff

but additionally lowers the function on each u_i

except it can happen **at most** once that v_n gets the decrease

We possibly need to help the unlucky u_i by including $u_i v_n$ in the matching