List coloring with requests

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opracowane przez:

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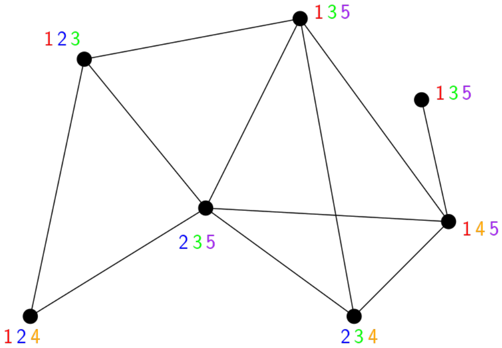

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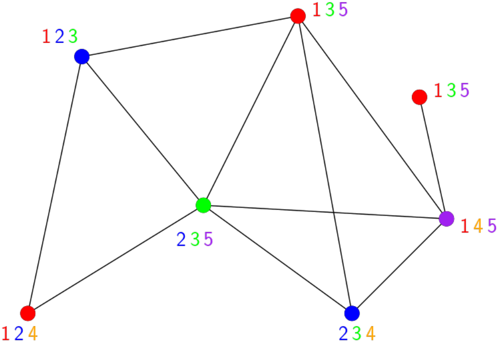

List coloring with request

A *request* for a graph G with a list assignment L is a function r with $\texttt{dom}(r) \subseteq V(G)$ such that $r(v) \in L(v)$ for all $v \in \texttt{dom}(r)$.

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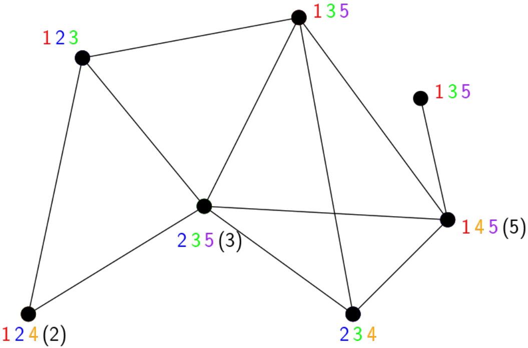

List coloring with request

A request is ϵ-*satisfiable* for $\epsilon > 0$ if there exists an L-coloring ϕ such that $\phi(v) = r(v)$ for at least $\epsilon|\mathtt{dom}(r)|$ vertices.

Graph G with a list assignment L is ϵ-*flexible* if every request is ϵ-satisfiable.

List coloring with request

A *weighted request* for a graph G with a list assignment L is a function w that to each pair v, c with $c \in L(v)$ assigns a nonnegative real number.

A request is ϵ-*satisfiable* for $\epsilon > 0$ if there exists an L-coloring ϕ such that

$$\sum_{v \in V(G)} w(v, \phi(v)) \geq \epsilon \sum_{v \in V(G), c \in L(v)} w(v, c).$$

Graph G with a list assignment L is *weighted* ϵ-*flexible* if every weighted request is ϵ-satisfiable.

Topics covered in the paper

In the paper, the following topics are discussed:

- Connection between weighted and unweighted requests
- Flexibility on degenerate graphs
- Satisfying one request on degenerate graphs.

Weighted and unweighted requests

Lemma 7

Let G be a graph with n vertices and L an assignment of lists of size d. If G and L are ϵ-flexible, then they are weighted $\frac{1}{d \log_{1/(1-\epsilon)} n}$-flexible.

Let w be a weighted request for G and L. For each $v \in V(G)$, let $c_v \in L(v)$ be a color such that $w(v, c_v) \geq w(v, c)$ for all $c \in L(v)$. Let $v_1, \ldots, v_n$ be an ordering of vertices of G such that $w(v_1, c_{v_1}) \geq w(v_2, c_{v_2}) \geq \ldots \geq w(v_n, c_{v_n})$.
Let $W = \sum_{i=1}^{n} w(v_i, c_{v_i})$ and $w(G, L) = \sum_{v \in V(G), c \in L(v)} w(v, c)$. Note that $W \geq w(G, L)/d$.

Weighted and unweighted requests

For $1 \leq k \leq n$, let r_k be the request with $dom(r_k) = \{v_1, \ldots, v_k\}$ and $r_k(v) = c_v$ for $v \in dom(r_k)$. Since G and L are ϵ-flexible, there exists an L-coloring ϕ_k such that a set M_k of requests are satisfied with $|M_k| \geq \epsilon k$. Let $n_0 = n$, $n_{i+1} = n_i - |M_i|$ and t - smallest index such that $n_t = 0$. (Note that $t \leq \log_{1/(1-\epsilon)} n$).

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We conclude that there exists i such that $w_i \geq W/k$, so by applying coloring ϕ_i to w we gain at least

$$\frac{w(G,L)}{dt} = \frac{w(G,L)}{d \log_{1/(1-\epsilon)} n},$$

showing that w is $\frac{1}{d \log_{1/(1-\epsilon)} n}$-satisfiable.

An example for $1/2$-satisfiable graph:

$$W = 37 \geq w(G, L)/d$$

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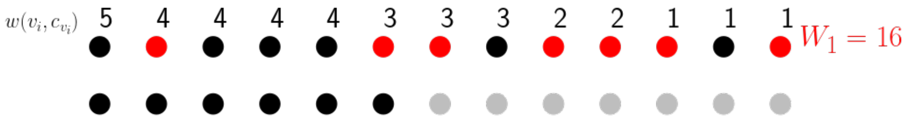

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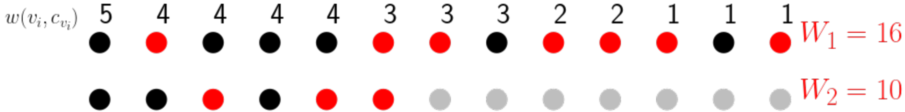

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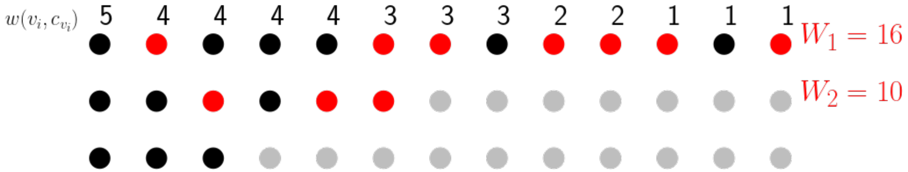

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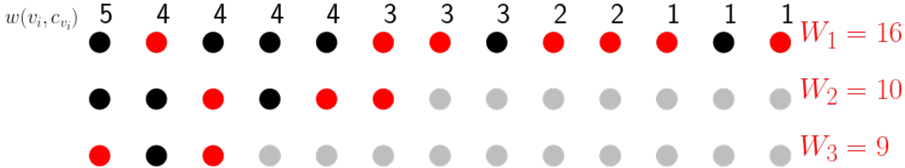

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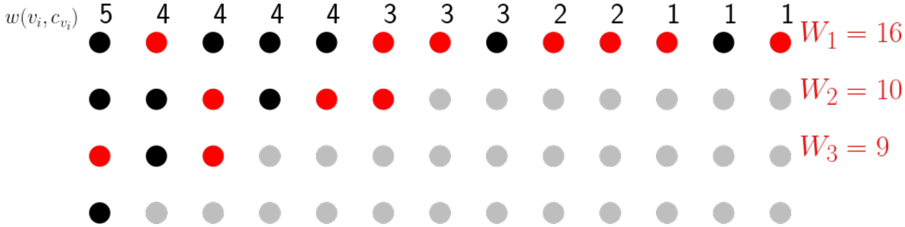

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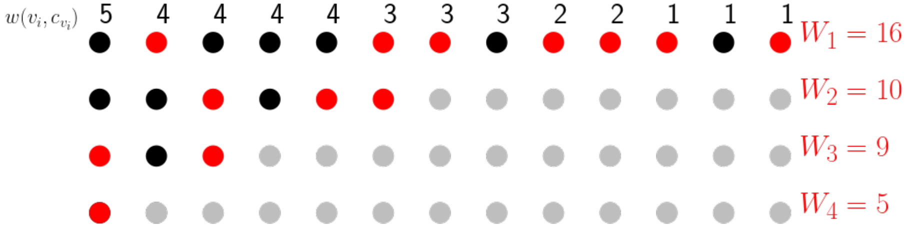

$$\Sigma W_i \geq W$$

$$W_1 \geq W/4$$

Weighted and unweighted requests

Now we want to show that logarithmic factor cannot be improved - we will construct graph G with $O(4^k)$ vertices and assignment L of lists of size 3 which are $1/6$-flexible but not weighted ϵ-flexible for any $\epsilon > 1/k$.

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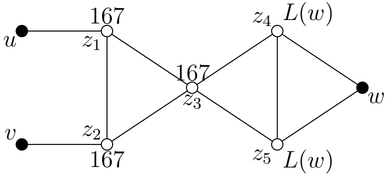
Now we want to show that logarithmic factor cannot be improved - we will construct graph G with $O(4^k)$ vertices and assignment L of lists of size 3 which are $1/6$-flexible but not weighted ϵ-flexible for any $\epsilon > 1/k$.
All the lists will be either $\{1,2,3\}$, $\{1,4,5\}$ or $\{1,6,7\}$. First, we will construct an auxiliary gadget, connecting pair of vertices (u,v) with w (u and v can be the same vertex):



Note that a proper coloring ϕ of the graph without the gadget is a proper coloring of G with a gadget unless $\phi(u) = \phi(v) = 1 \neq \phi(w)$.

Weighted and unweighted requests

The graph constructed as follows:

- There are nonnegative integers $s_1, \dots s_n$ and t
- There is a set S of n vertices $v_1, \dots v_n$, nt vertices $x_{i,j}$ and a vertex y

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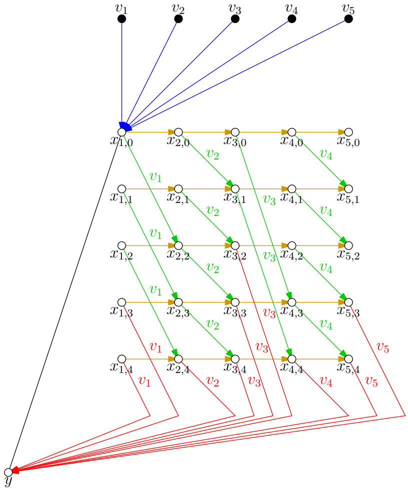
The graph constructed as follows:

- There are nonnegative integers $s_1, \ldots s_n$ and t
- There is a set S of n vertices $v_1, \ldots v_n$, nt vertices $x_{i,j}$ and a vertex y
- There is a gadget between:
  (a) (v_i, v_i) and $x_{1,0}$ for $1 \leq i \leq n$,
  (b) $(x_{i,j}, x_{i,j})$ and $x_{i+1,j}$ for $1 \leq i \leq n-1$ and $1 \leq j \leq t$,
  (c) $(v_i, x_{i,j})$ and $x_{i+1,j+s_i}$ for $1 \leq i \leq n-1$ and $1 \leq j \leq t - s_i$,
  (d) $(v_i, x_{i,j})$ and y for $1 \leq i \leq n$ and $t - s_i + 1 \leq j \leq t$
- There is a single edge between $x_{1,0}$ and y.

Weighted and unweighted requests



Weighted and unweighted requests

Lemma

Any weighted request on G, L such that $w(v_i, 1) = 0$ for each i is $1/4$-satisfiable.

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Consider colorings $\phi_1 \dots \phi_4$ described below:

Coloring	S	$x_{i,j}$	y	(a), (b), (c) z_1, z_2, z_3, z_4, z_5	(d) z_1, z_2, z_3, z_4, z_5
ϕ_1	2	1	4	$6, 7, 1, 4, 5$	$1, 6, 7, 1, 5$
ϕ_2	3	1	5	$7, 6, 1, 5, 4$	$1, 7, 6, 4, 1$
ϕ_3	3	4	1	$1, 6, 7, 1, 5$	$6, 1, 7, 5, 4$
ϕ_4	3	5	1	$7, 1, 6, 4, 1$	$7, 6, 1, 4, 5$

See that for each $v, c \in L(v)$ except for $v \in S, c = 1$ there exists at least one of the four colorings ϕ_i such that $\phi_i(v) = c$, so at least one of them gains $1/4$ of the total weight.

Weighted and unweighted requests

Lemma

For set $R \subseteq \{1, \ldots n\}$ all of the v_i for $i \in R$ can be colored by 1 if and only if $\sum_{i \in R} s_i \leq t$.

This lemma follows from the construction of the graph.

Weighted and unweighted requests

Now we construct such graph for $n = 2^k - 1$, $t = 2^{k-1}$ and s_i equal to the largest power of two such that $is_i \leq 2^k - 1$ (that is $s_1 = 2^{k-1}$, $s_2 = s_3 = 2^{k-2}, \ldots, s_{2^{k-1}} = \ldots = s_n = 1$).
For any request r, consider its restrictions r_1, r_2 with $\mathtt{dom}(r_1) = \{v \in S \cap \mathtt{dom}(r) : r(v) = 1\}$ and $\mathtt{dom}(r_2) = \mathtt{dom}(r) \setminus \mathtt{dom}(r_1)$.
Due to the previous lemmas, r_1 is $1/4$-satisfiable and r_2 is $1/2$-satisfiable (because we can take R - larger half of indices i such that $v_i \in \mathtt{dom}(r_2)$ and $\sum_{i \in R} s_i \leq 2^{k-1}$), so altogether r is $1/6$-satisfiable.

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However, if we consider the weighted request w such that $w(v_i, 1) = s_i$ for $1 \leq i \leq n$ we have $w(G, L) = k2^{k-1}$ and we know that every L-coloring ϕ of G satisfies

$$\sum_{1 \leq i \leq n, \phi(v_i) = 1} s_i \leq 2^{k-1},$$

so w is not ϵ-satisfiable for any $\epsilon > 1/k$.

Flexibility of degenerate graphs

Graph G is d-degenerate if every subgraph of G has vertex of degree at most d.
Graph G is weakly-d-degenerate if every subgraph of G has either vertex of degree at most d or connected set of $d+1$ vertices of degree $d+1$.

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Theorem 2

For every nonnegative integer d there exists $\epsilon \geq 0$ such that every weakly d-degenerate graph with an assignment of lists of size $d+2$ is weighted ϵ-flexible.

Flexibility of degenerate graphs

Lemma

If there exists a probability distribution on L-colorings ϕ of G such that for every $v \in V(G)$ and $c \in L(v)$, $\mathrm{P}[\phi(v) = c] \geq \epsilon$, then G with L is weighted ϵ-flexible.

Lemma

If there exists a probability distribution on L-colorings ϕ of G such that for every $v \in V(G)$ and $c \in L(v)$, $\mathrm{P}[\phi(v) = c] \geq \epsilon$, then G with L is weighted ϵ-flexible.

This lemma allows us to prove the theorem above - for weakly-d-degenerate graph G let P be singleton $\{w\}$ if G contains vertex w od degree at most d or the set of $d+1$ connected vertices of degree $d+1$ otherwise. We choose random coloring of G by firstly choosing random coloring of $G \setminus P$ inductively and then choosing coloring of P uniformly at random from all the possible proper colorings left. One can show that in this way, for every $v \in V(G)$ and $c \in L(v)$, $\mathrm{P}[\phi(v) = c] \geq (\frac{1}{d+2})^{(d+1)^2}$.

Flexibility of degenerate graphs

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Theorem 4

For every integer $d \geq 2$ and $d - 1$-choosable graph G with average degree at most d there exists $\epsilon \geq 0$ such that G with an assignment of lists of size at least d is weighted ϵ-flexible.

Theorem 5

For every nonnegative integer d and graph G with average degree less than $d + 1 + 2/(d + 4)$there exists $\epsilon \geq 0$ such that G with an assignment of lists of size $d + 2$ is weighted ϵ-flexible.

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It is a open problem if Theorem 2 works for assignment of lists of size $d+1$.

Satisfying one request on degenerate graphs

Theorem 6

Let $d \geq 2$ be an integer such that $d+1$ is a prime. If r is a request for a d-degenerate graph with an assignment of lists of size at least $d+1$ and $|\mathtt{dom}(r)| = 1$, then r is 1-satisfiable.

Satisfying one request on degenerate graphs

Theorem 6

Let $d \geq 2$ be an integer such that $d+1$ is a prime. If r is a request for a d-degenerate graph with an assignment of lists of size at least $d+1$ and $|\mathtt{dom}(r)| = 1$, then r is 1-satisfiable.

To prove this theorem, we will use Alon's Combinatorial Nullstellensatz:

Alon's Combinatorial Nullstellensatz

Let G be a graph with vertex set $(v_1, v_2, \ldots, v_n)$. The graph polynomial is defined as $p_G(x_1, x_2, \ldots, x_n) = \prod_{v_i v_j \in E(G),\, i<j} (x_j - x_i)$. Suppose that the coefficient of $x_1^{d_1} \ldots x_n^{d_n}$ in $p_G(x_1, x_2, \ldots, x_n)$ is non-zero. Let L be a list assignment for G such that $|L(v_i)| \geq d_i + 1$ for $i = 1, 2, \ldots, n$. Then G is L-colorable.

Satisfying one request on degenerate graphs

First, we prove a simple lemma about permutations. Let S_d be the set of all permutations of the set $\{1, \dots d\}$ and S_d^0 be the set of all permutations of the set $\{0, \dots d-1\}$. Then the following holds:

Lemma

Let d be a positive integer and let $r : \{1, \dots, d\} \to \mathbb{N}$ be a function such that $\sum_{i=1}^{d} r(i) = d$. For $\pi \in S_d^0$, $\pi + r \in S_d$ if and only if

$$\pi(t) = \sum_{j:\pi(j)<\pi(t)} r(j)$$

for every $t \in \{1, \dots, d\}$ such that $r(t) > 0$.

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for every $t \in \{1, \dots, d\}$ such that $r(t) > 0$.

π	3	0	4	1	5	2
r	2	2	0	0	1	1
$\pi + r$	5	2	4	1	6	3

Satisfying one request on degenerate graphs

This lemma allows us to prove the following:

Lemma

Let $d \geq 2$ be an integer such that $d+1$ is prime. Let G be a d-degenerate graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. Let $r : \{1, \ldots, d\} \to \mathbb{N}$ satisfy $\sum_{i=1}^{d} r(i) = d$. Let L be a list assignment for G such that $|L(v_i)| \geq d + 1 - r(i)$ for $1 \leq i \leq n$.
Let $h = \prod_{i=1}^{n} x_i^{r(i)}$. Then there exists a permutation $\sigma \in S_d$ such that coefficient of the term $\frac{1}{h} \prod_{i=1}^{d} x_i^{\sigma(i)} \prod_{i=d+1}^{n} x_i^{d}$ in p_G is non-zero.

This lemma with Combinatorial Nullstellensatz directly proves that if d, G, r, L are defined as above, then G is L-corolable, which is a generalization of Theorem 6 (it suffices to choose $r(i) = d$ for some i).