Book embeddings of graphs

and why four pages are indeed necessary for planar graphs

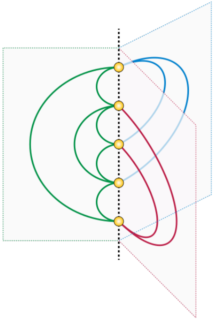
Filip Jasionowicz

Thursday 25th January, 2024

Book embedding of a graph

Definition (Book embedding)

- Vertices restricted to a line (*spine* of the book)
- Edges assigned to different half-planes delimited by the *spine* (*pages* of the book)
- No two edges on the same *page* cross



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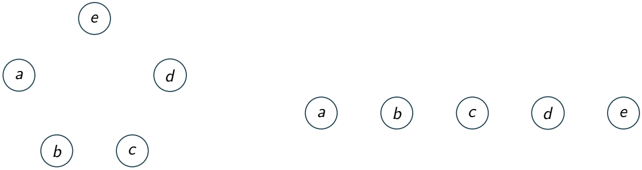
Book thickness $bt(G)$ of a graph G is a minimum number of pages required by any of its book embeddings.



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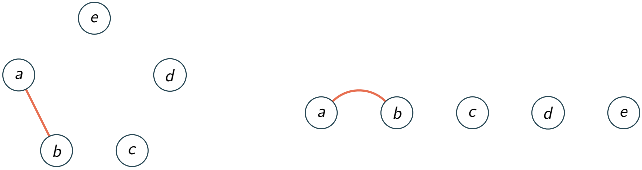

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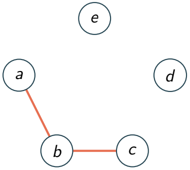

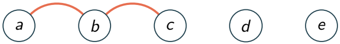

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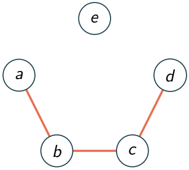

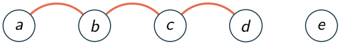

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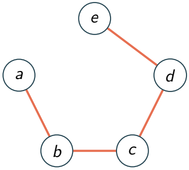

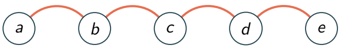

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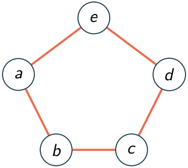

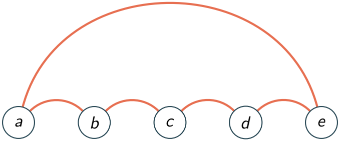

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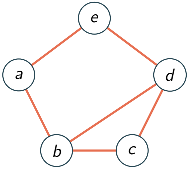

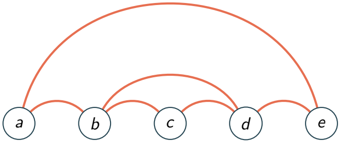

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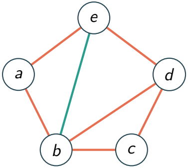

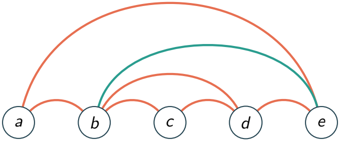

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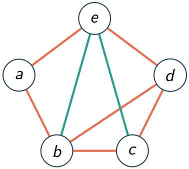

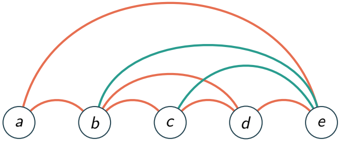

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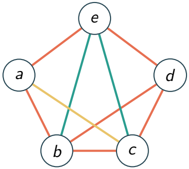

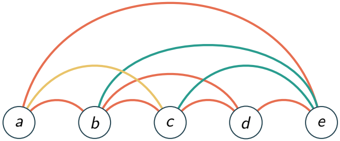

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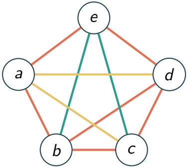

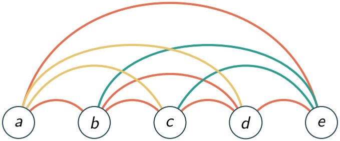

Outer planar

Theorem (Bernhart, Kainen, 1979)

For connected graph G:

1. $bt(G) = 0$ *if and only if* G *is a path*
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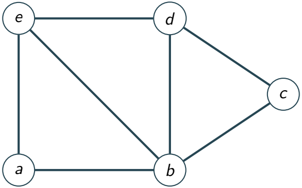

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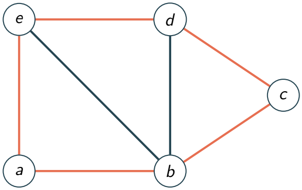

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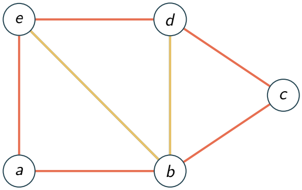

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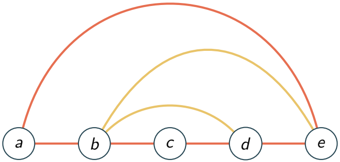

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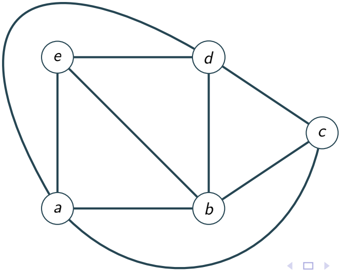

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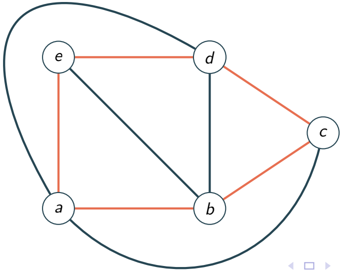

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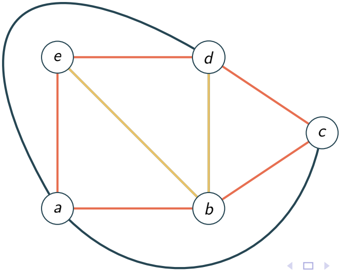

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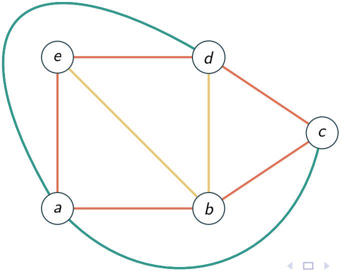

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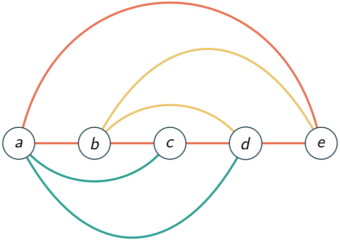

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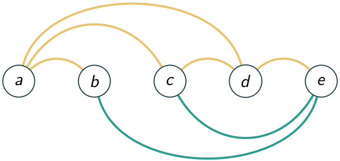

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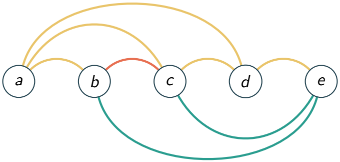

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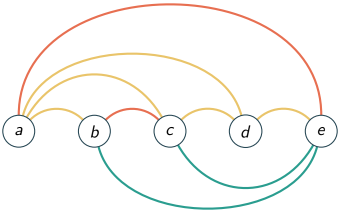

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$bt(K_n) = \left\lceil \frac{n}{2} \right\rceil$

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For graph G with n vertices and m edges $bt(G) \geq \left\lceil \frac{m-n}{n-3} \right\rceil$.

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An outerplanar graph on n vertices has at most $2n - 3$ edges so for q-page embedding of graph G with n vertices and m edges:

$$m \leq n + q(n-3)$$

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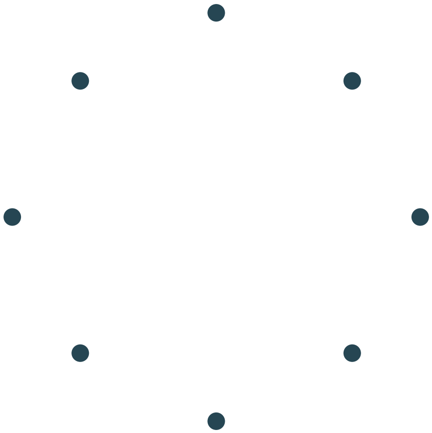

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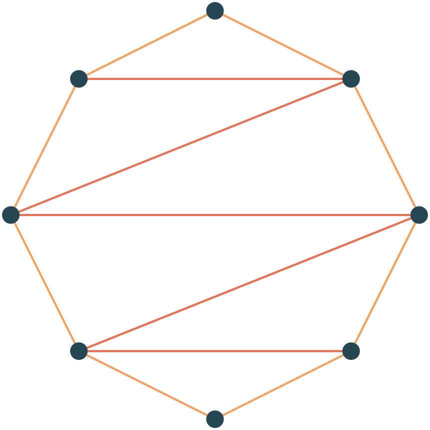

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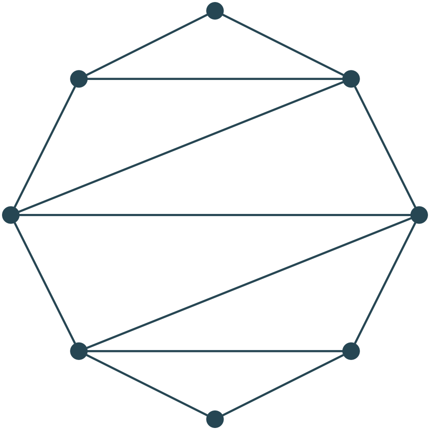

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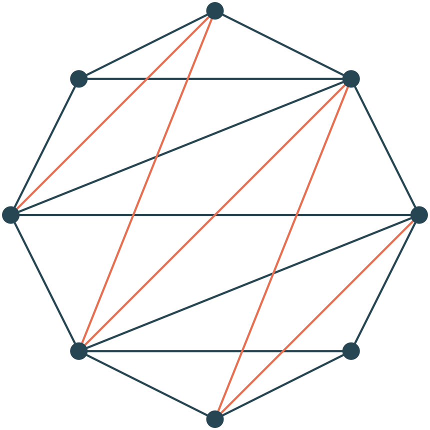

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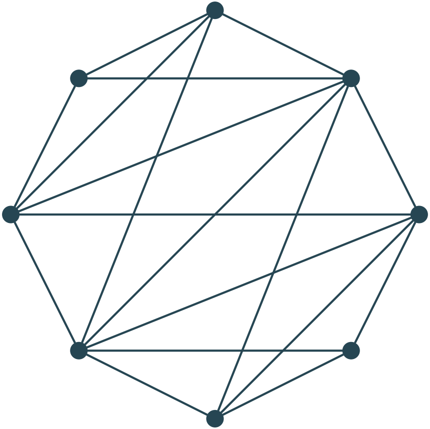

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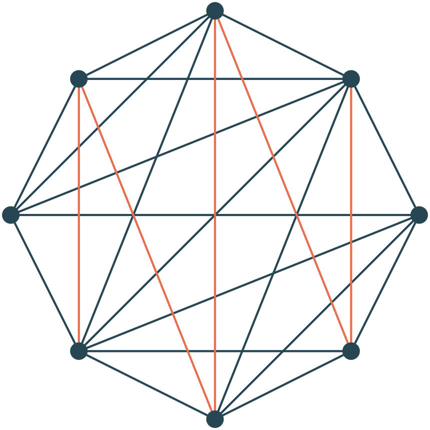

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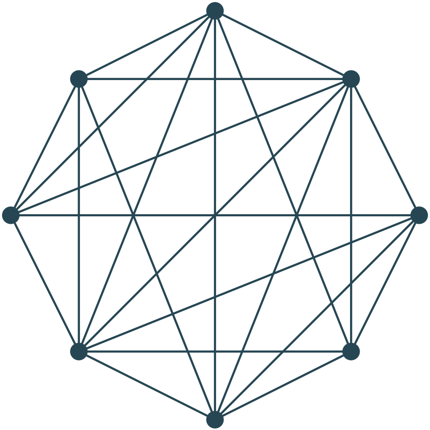

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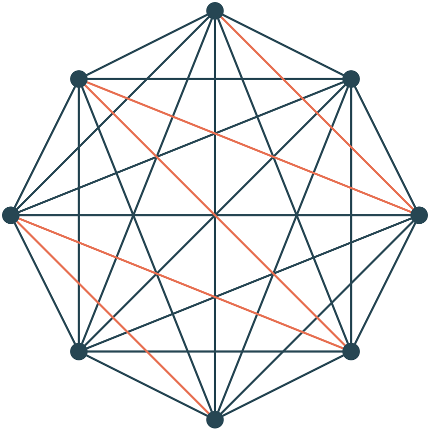

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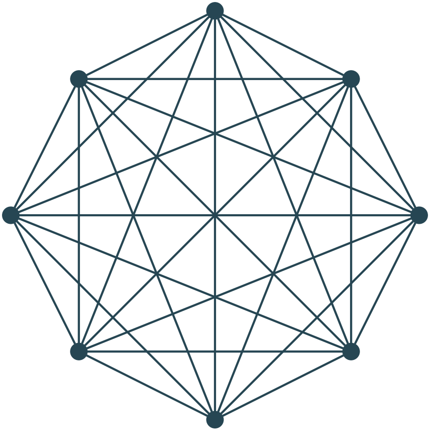

Planar graphs - upper bounds

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Theorem (Yannakakis, 1989)

For planar graph G: $bt(G) \leq 4$.

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There exist a maximal non-hamiltonian planar graph G.

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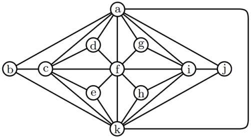

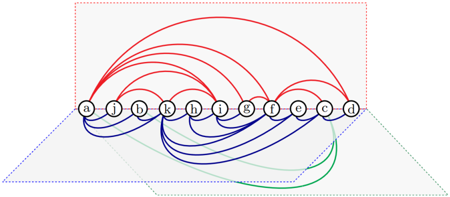

Main theorem

Theorem (Bekos et al., 2020)

There exist a planar graph G such that G: $bt(G) \geq 4$.

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A 3-page book embedding of a graph does not contain:

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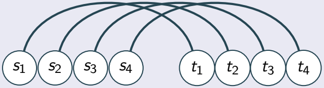

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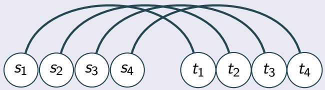
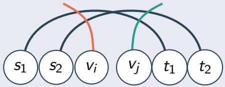


2. *A pair of crossing edges that both cross two edges assigned to two different pages:*



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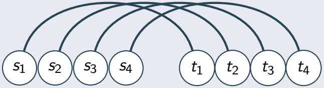
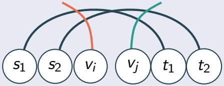
Lemma

A 3-page book embedding of a graph does not contain:

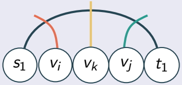
1. *A 4-twist:*



2. *A pair of crossing edges that both cross two edges assigned to two different pages:*



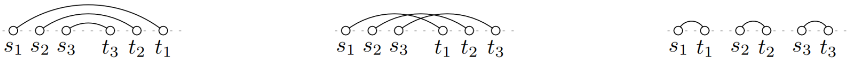
3. *An edge that cross three edges assigned to three different pages:*



Prerequisites

Definition

For set F of independent edges (s_i, t_i) with $s_i \prec t_i$ we define *rainbow*, *twist* and *necklace* as follows:



Prerequisites

Lemma (Erdös, Szekeres, 1935)

Given $a, b \in \mathbb{N}$ *every sequence of distinct real numbers of length at least* $ab + 1$ *contains a monotonically increasing subsequence of length* $a + 1$ *or a monotonically decreasing subsequence of length* $b + 1$.

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For every vertex ordering $\prec$ *of a graph with* $m = r^3$ *distinct edges one can identify* r *edges that form either* r*-rainbow or* r*-twist or* r*-necklace*

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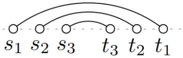
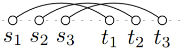
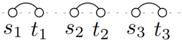
For every vertex ordering $\prec$ of a graph with $m = r^3$ distinct edges one can identify r edges that form either r-rainbow or r-twist or r-necklace

$$(s_1, t_1), (s_2, t_2), (s_3, t_3), \ldots, (s_m, t_m)$$

Take m edges (assume $s_i < t_i$) and sort them by the s_i coordinate and apply the previous lemma to t_i with $a := r^2$ and $b := r - 1$







Gadget

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For $k \geq 2$, define Q_k as:

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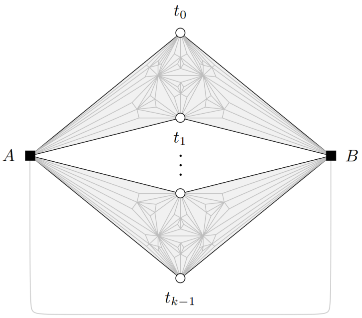

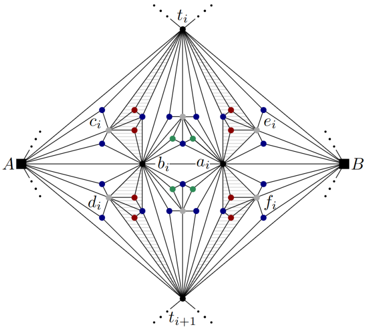

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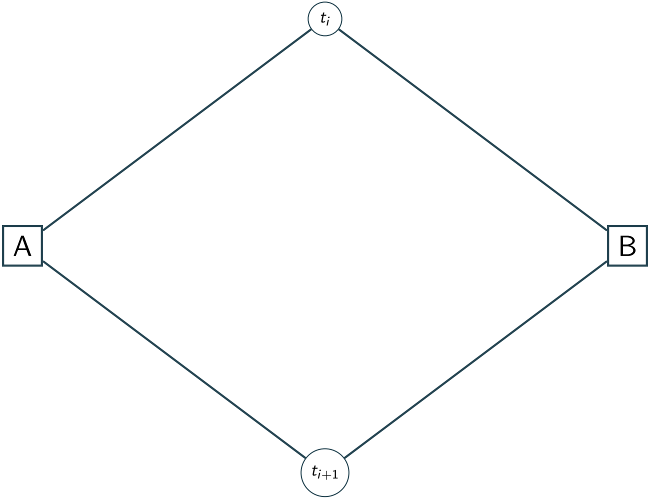

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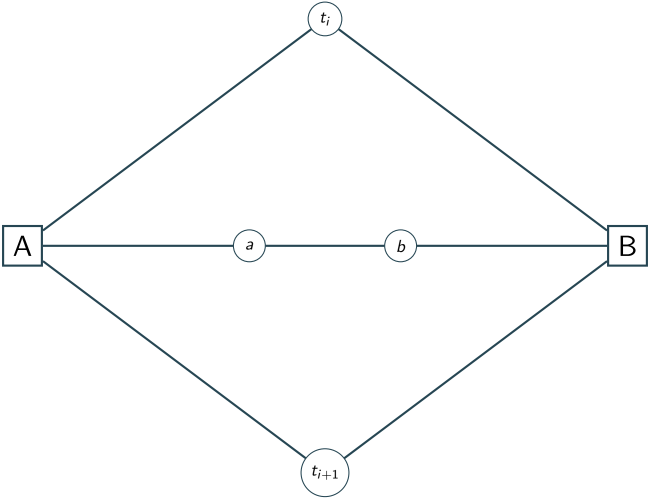

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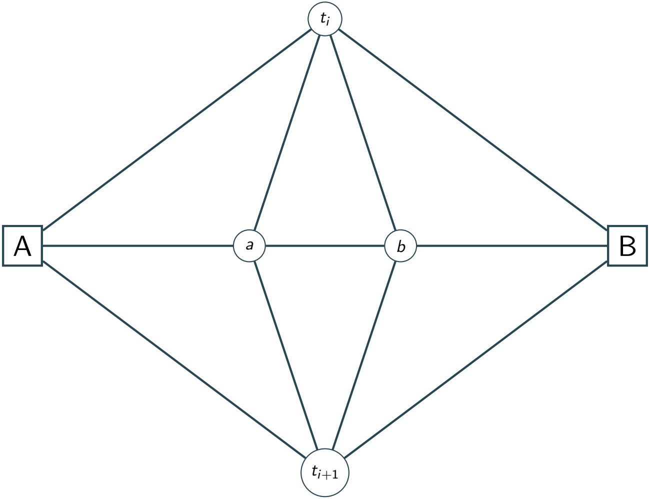

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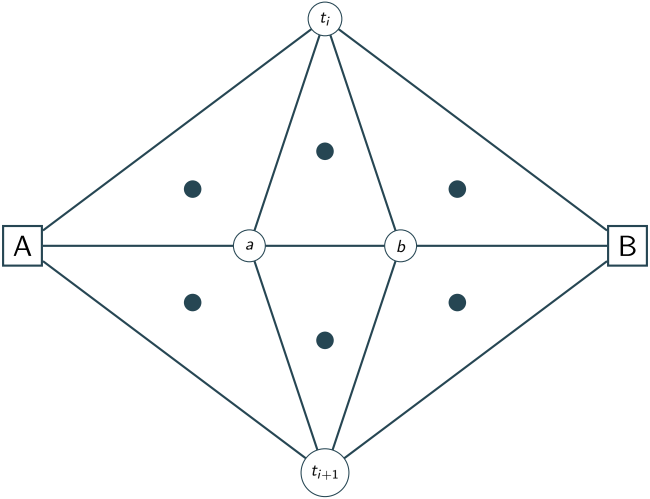

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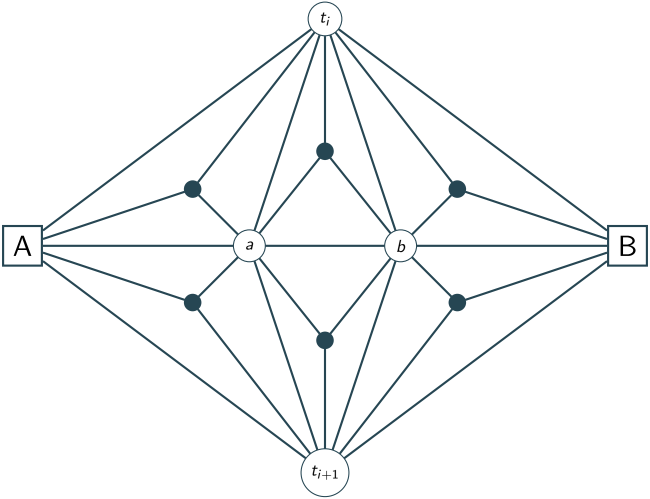

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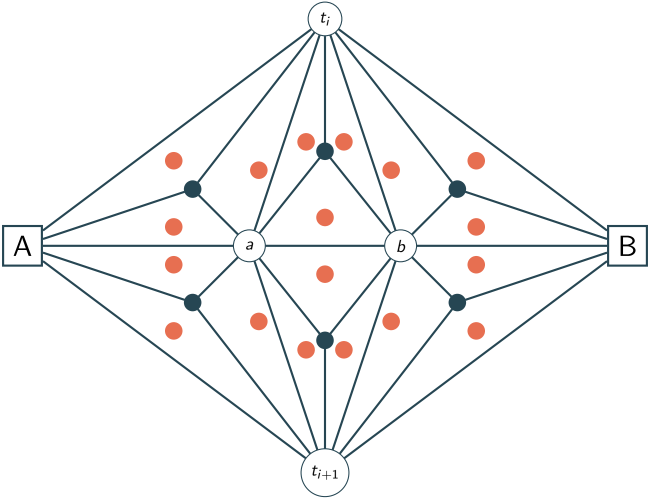

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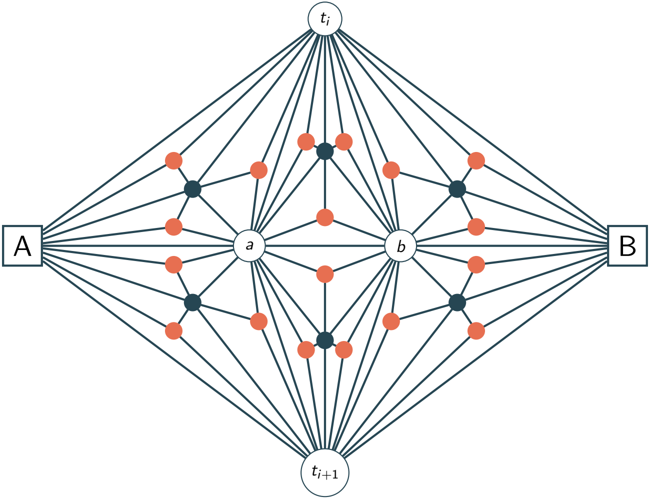

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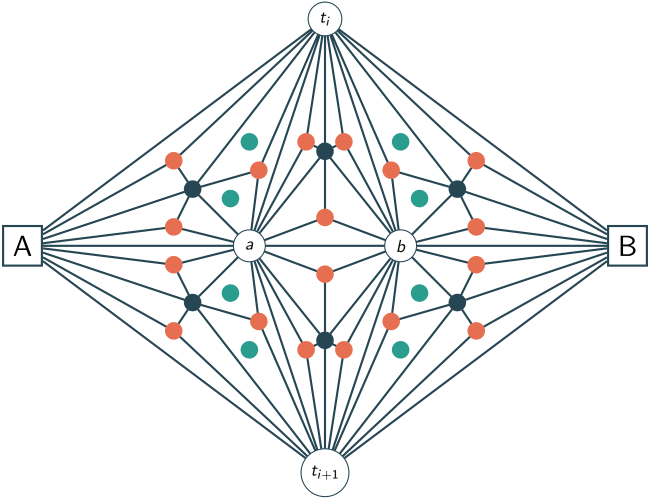

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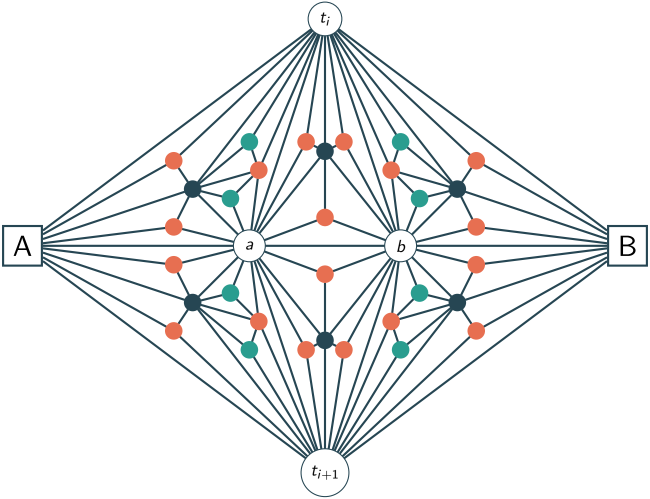

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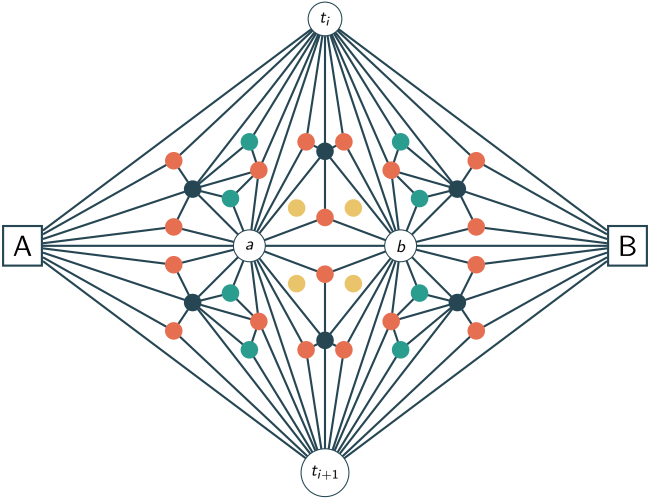

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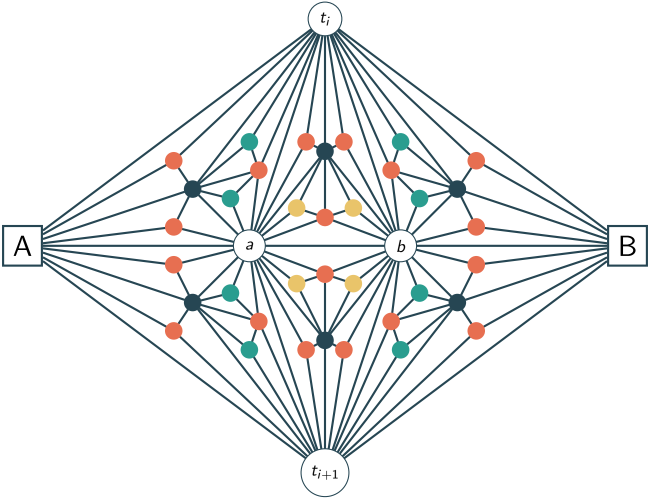

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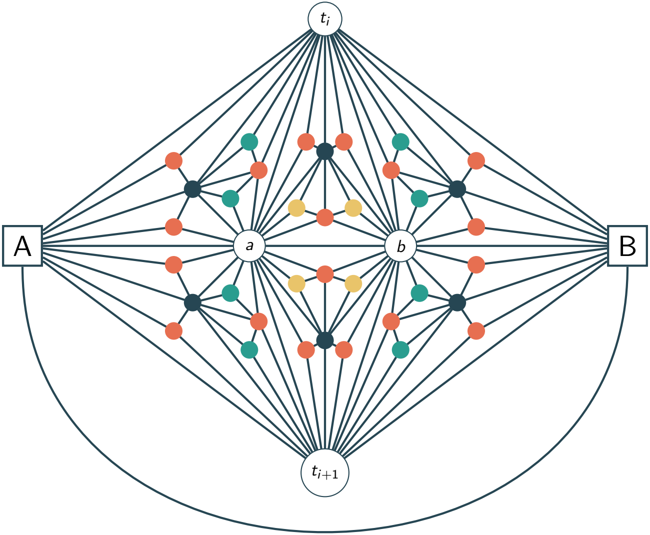

Gadget properties

Fact

Graph Q_k with $k \geq 7$ does not admit an embedding in a book with three pages: $\mathcal{B}lue$, $\mathcal{R}ed$, $\mathcal{G}reen$ under following restrictions:

1. *A and B are consecutive in the ordering*
2. *All edges (A, t_i) are on the $\mathcal{B}lue$ page*
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Graph Q_k with $k \geq 10$ does not admit an embedding in a book with three pages: $\mathcal{B}lue$, $\mathcal{R}ed$, $\mathcal{G}reen$ under following restrictions:

1. *All vertices t_i are on the same side of edge (A, B)*
2. *All edges (A, t_i) are on the $\mathcal{B}lue$ page*
3. *All edges (t_i, B) are on the $\mathcal{R}ed$ page*

Some definitions

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$Q := Q_{10}$

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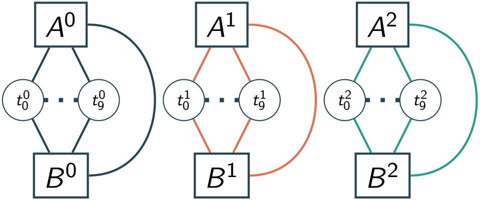

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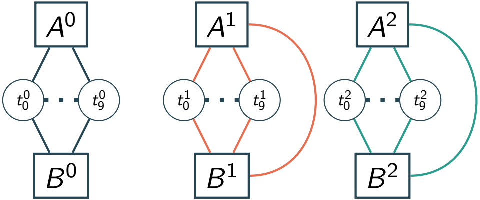

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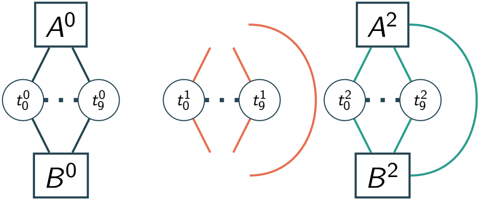

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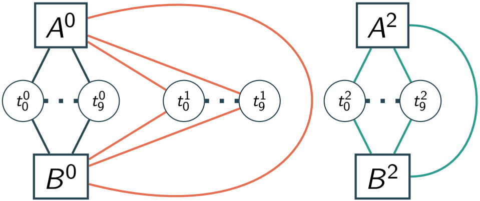

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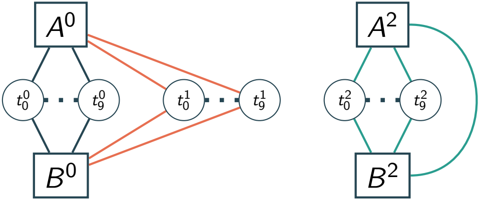

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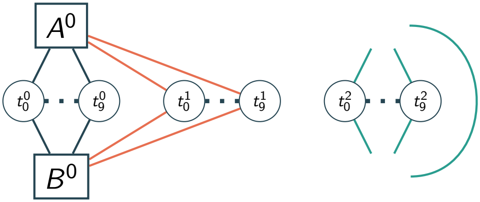

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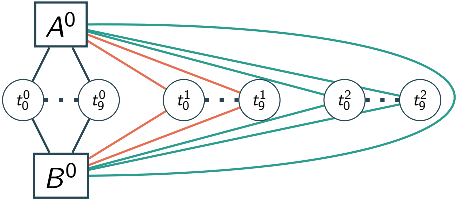

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Number of different possible three page book embeddings of Q is finite and bounded by $Q_n! \cdot 3^{Q_m}$.

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For $N \geq \left(\kappa^{\binom{Q_n}{2}}\right)^3 \cdot Q_n! \cdot 3^{Q_m}$ one can identify κ copies of Q in G_N such that they are all 3-page embedded in the same way and for each pair of vertices of Q the corresponding pairs of vertices in this κ copies form a κ-rainbow or κ-twist κ-necklace

Proof idea

Definition (Final graph)

Define G as G_N with edges between vertices a_i and b_i in each Q replaced by copies of G_N (identify A in G_N with a_i and B in G_N with b_i).

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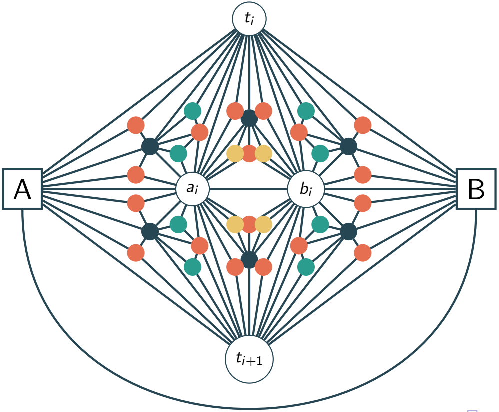

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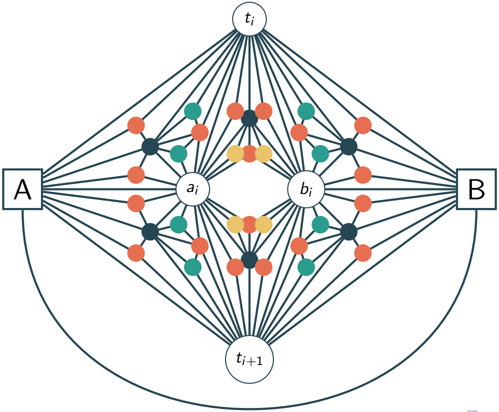

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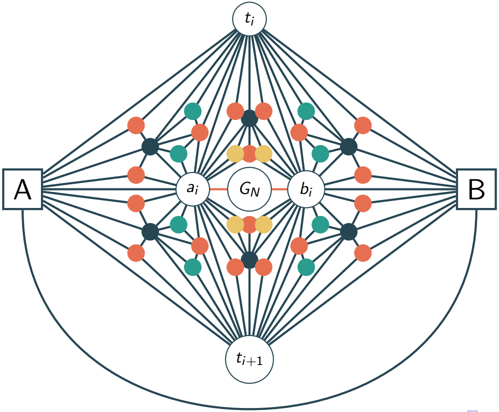

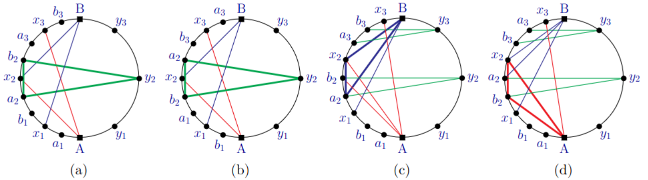

Figure 5: Illustrations for (a) FP2.1, (b) FP2.2, (c) FP2.3, and (d) FP2.4.

FP2.3 $[A \ldots a_1 \ldots b_1 \ldots x_1 \ldots a_2 \ldots b_2 \ldots x_2 \ldots a_3 \ldots b_3 \ldots x_3 \ldots B \ldots y_3 \ldots y_2 \ldots y_1]$

Refer to Fig. 5c. Since edges (A, x_3), (x_1, B) and (a_2, y_2) form a 3-twist, we can assume that $(A, x_i) \in \mathcal{Red}$, $(a_i, y_i) \in \mathcal{Green}$ and $(x_i, B) \in \mathcal{Blue}$, which implies that $(b_i, y_i) \in \mathcal{Green}$, $(a_i, B) \in \mathcal{Blue}$, $(A, b_i) \in \mathcal{Red}$ and $(a_i, x_i) \in \mathcal{Blue}$. Consider now vertex s_2^{Bax} of G_N that was introduced due to the stellation of face $\langle B, a_2, x_2 \rangle$ in G_N. Due to edge (a_2, s_2^{Bax}), vertex s_2^{Bax} cannot be in $[x_2 \ldots y_2]$. Analogously, vertex s_2^{Bax} cannot be in $[y_2 \ldots a_2]$, due to edge (x_2, s_2^{Bax}). Finally, vertex s_2^{Bax} cannot be in $[a_2 \ldots x_2]$, due to edge (B, s_2^{Bax}). Hence, there is no feasible placement of s_2^{Bax} in $\mathcal{E}$; a contradiction.

FP2.4 $[A \ldots b_1 \ldots a_1 \ldots x_1 \ldots b_2 \ldots a_2 \ldots x_2 \ldots b_3 \ldots a_3 \ldots x_3 \ldots B \ldots y_3 \ldots y_2 \ldots y_1]$

Refer to Fig. 5d. Since edges (A, x_3), (x_1, B) and (a_2, y_2) form a 3-twist, we can assume that $(A, x_i) \in \mathcal{Red}$, $(a_i, y_i) \in \mathcal{Green}$ and $(x_i, B) \in \mathcal{Blue}$. Hence, $(a_i, B), (x_i, B) \in \mathcal{Blue}$, $(b_i, y_i) \in \mathcal{Green}$ and $(A, b_i), (b_i, x_i) \in \mathcal{Red}$. It is not hard to see that there is no feasible placement for vertex s_2^{Abx} of G_N introduced due to the stellation of face $\langle A, b_2, x_2 \rangle$ in $\mathcal{E}$.

Trick...

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To perform the trick you need:

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1. *28-core 2.4 GHz Intel Xeon E5-2680 machine with 256GB RAM*

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4. *candidate graph*
5. *approx. 48 hours*

SAT formulation

Theorem (Bekos et al., 2015)

For graph G and $p \in \mathbb{N}$ there exist a SAT formula $\mathcal{F}(G, p)$ such that G has embedding on p pages if and only if $\mathcal{F}(G, p)$ is satisfiable. In addition, $\mathcal{F}(G, p)$ has $O(n^2 + m^2 + pm)$ variables and $O(n^3 + m^2)$ clauses.

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And a few rules...

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1. asymmetry: $\sigma(v_i, v_j) \leftrightarrow \neg\sigma(v_j, v_i)$

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3. at least one page per edge: $\phi_1(e_i) \lor \phi_2(e_i) \lor ... \lor \phi_p(e_i)$
4. both edges on the same page:
   $((\phi_1(e_i) \land \phi_1(e_j)) \lor ... \lor (\phi_p(e_i) \land \phi_p(e_j))) \rightarrow \chi(e_i, e_j)$

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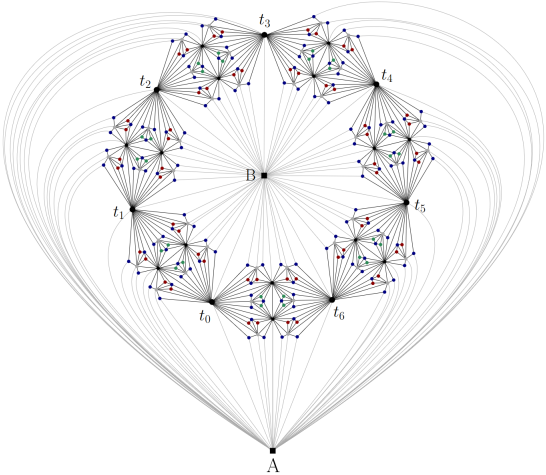
5. no crossings on one page:

SAT formulation - rules

1. asymmetry: $\sigma(v_i, v_j) \leftrightarrow \neg\sigma(v_j, v_i)$
2. transitivity: $\sigma(v_i, v_j) \wedge \sigma(v_j, v_k) \rightarrow \sigma(v_i, v_k)$
3. at least one page per edge: $\phi_1(e_i) \vee \phi_2(e_i) \vee \ldots \vee \phi_p(e_i)$
4. both edges on the same page: $((\phi_1(e_i) \wedge \phi_1(e_j)) \vee \ldots \vee (\phi_p(e_i) \wedge \phi_p(e_j))) \rightarrow \chi(e_i, e_j)$
5. no crossings on one page:

$$
\begin{aligned}
&\chi((v_i, v_j), (v_k, v_\ell)) \rightarrow \\
&\quad \neg(\sigma(v_i, v_k) \wedge \sigma(v_k, v_j) \wedge \sigma(v_j, v_\ell)) \wedge \neg(\sigma(v_i, v_\ell) \wedge \sigma(v_\ell, v_j) \wedge \sigma(v_j, v_k)) \\
&\wedge \neg(\sigma(v_j, v_k) \wedge \sigma(v_k, v_i) \wedge \sigma(v_i, v_\ell)) \quad \wedge \neg(\sigma(v_j, v_\ell) \wedge \sigma(v_\ell, v_i) \wedge \sigma(v_i, v_k)) \\
&\wedge \neg(\sigma(v_k, v_i) \wedge \sigma(v_i, v_\ell) \wedge \sigma(v_\ell, v_j)) \quad \wedge \neg(\sigma(v_k, v_j) \wedge \sigma(v_j, v_\ell) \wedge \sigma(v_\ell, v_i)) \\
&\wedge \neg(\sigma(v_\ell, v_i) \wedge \sigma(v_i, v_k) \wedge \sigma(v_k, v_j)) \quad \wedge \neg(\sigma(v_\ell, v_j) \wedge \sigma(v_j, v_k) \wedge \sigma(v_k, v_i))
\end{aligned}
$$

Candidate graph



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