Another approach to non-repetitive colorings of graphs of bounded degree

Matthieu Rosenfeld, 2020

Presented by Katzper Michno, 25.01.2024.

Lovász local lemma

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Suppose $\mathbb{A} = \{A_1, \ldots, A_n\}$ is a set of events in an arbitrary probability space.
If $\forall_{i \in [n]}$:

- $\exists D_i \subseteq \mathbb{A}$ with $|D_i| \leq d$ such that A_i is mutually independent of $\mathbb{A} \setminus D_i$,
- $P(A_i) \leq p$,

and

$$ep(d+1) \leq 1$$

then

$$P\left(\bigcap_{i=1}^{n} \bar{A}_i\right) > 0$$

Local lemma intuition

Let $\mathbb{A} = \{A_1, \ldots, A_n\}$ be a set of **bad events** with **small probabilities** bounded by a constant p, i.e. $\forall_{i \in [n]} P(A_i) \leq p$.

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Create a graph $G = (\mathbb{A}, E)$ where A_i is mutually independent of $\mathbb{A} \setminus N(A_i)$.

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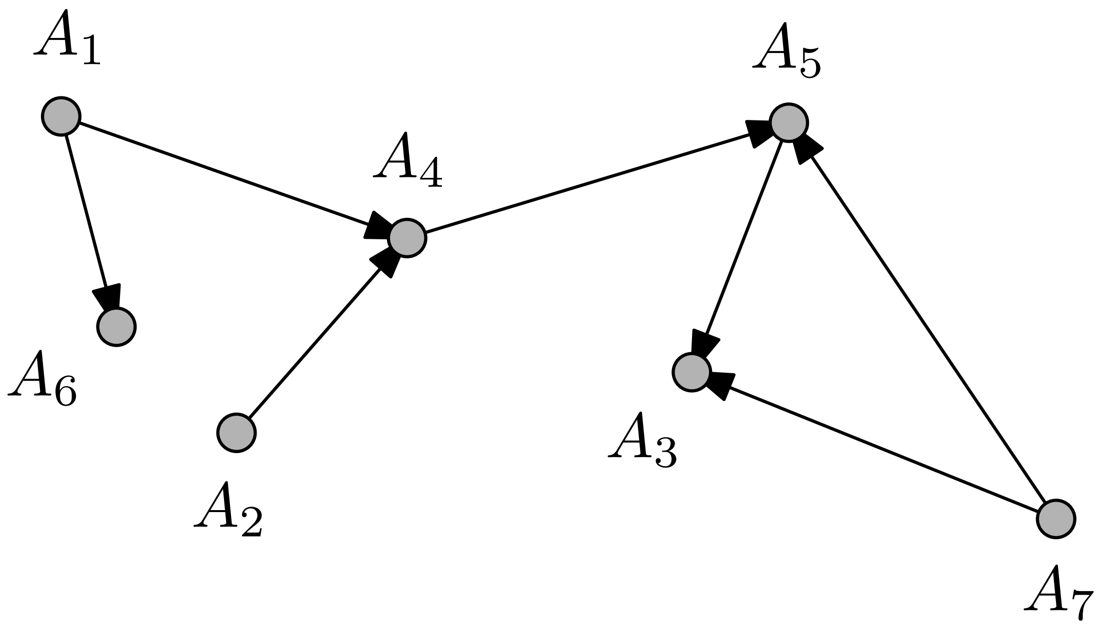

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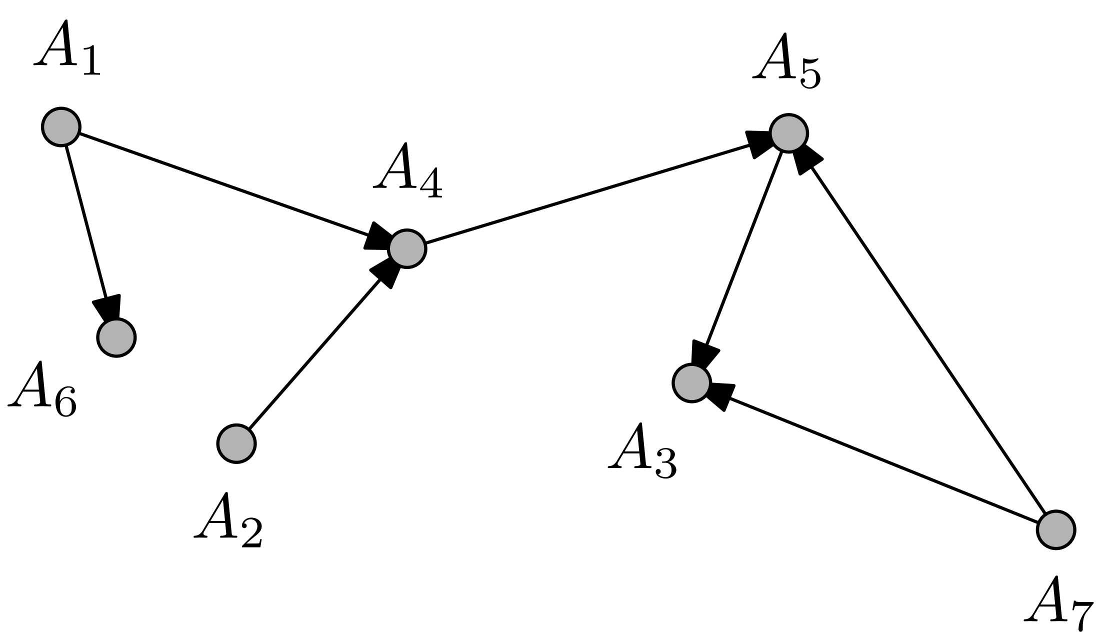

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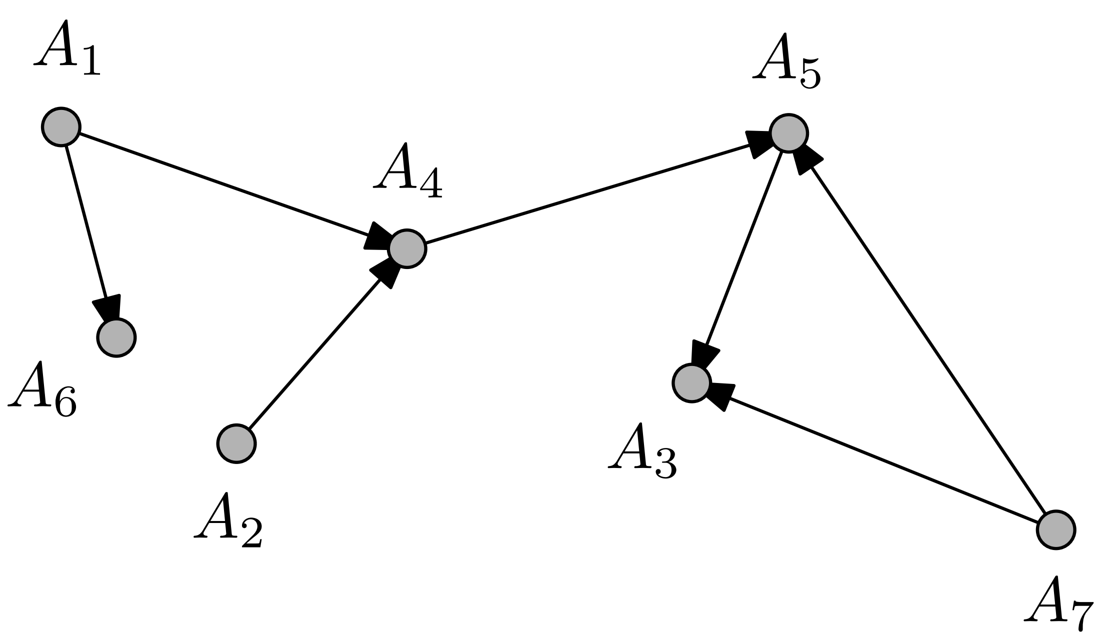

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Namely,

$$max_{i \in [n]}\, deg_{out}(A_i) \leq \frac{1}{ep} - 1$$

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Not only that, but also inspired a new proof technique - **entropy compression**.

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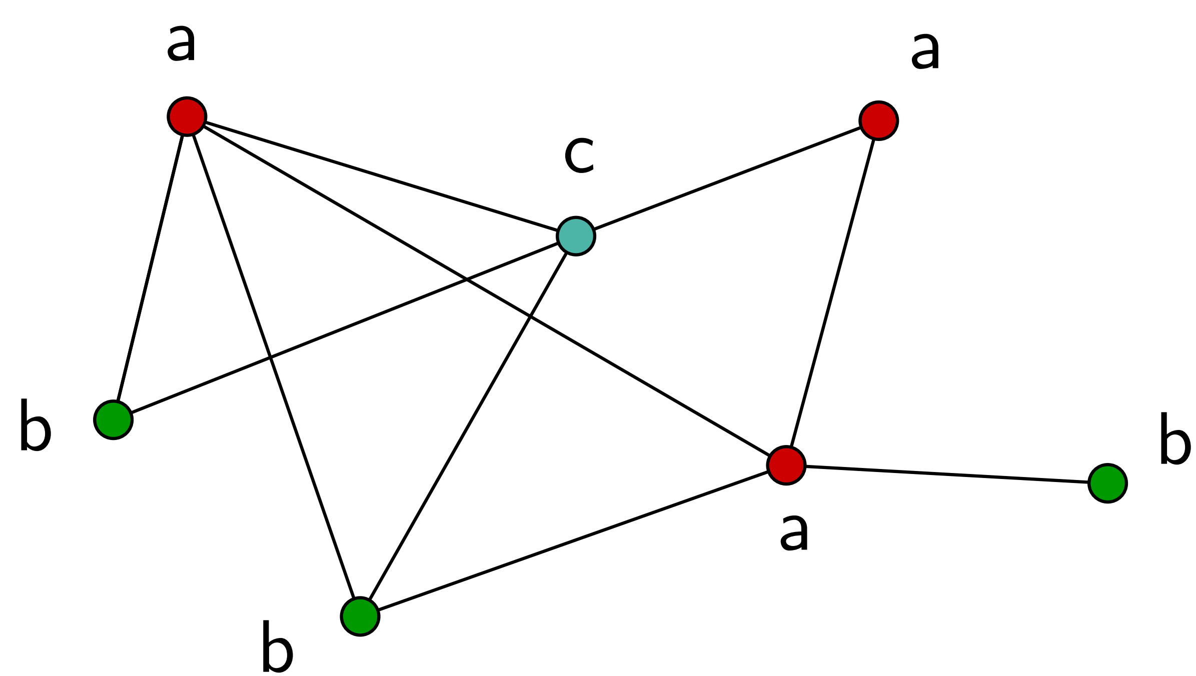

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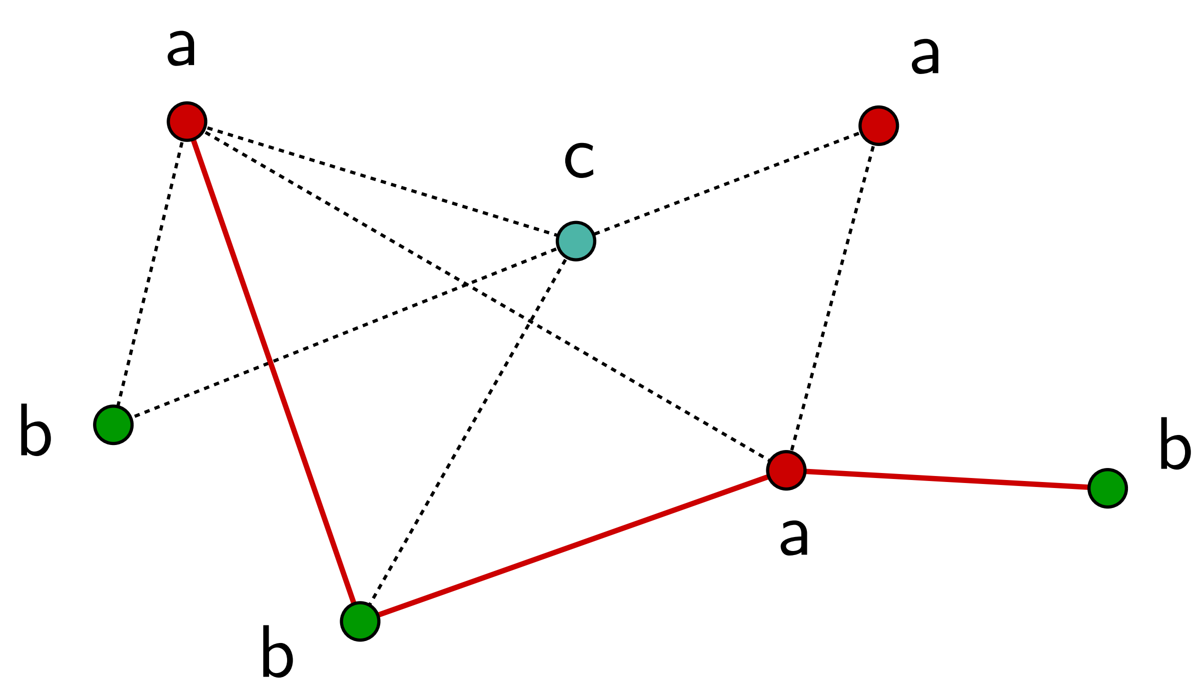

The **non-repetitive number** $\pi(G)$ is the minimal number of colors in a non-repetitive coloring. Similarly, the **non-repetitive index** and list versions are denoted as $\pi'(G)$, $\pi_{ch}(G)$, and $\pi'_{ch}(G)$.

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- $\pi_{ch}(L_n) \leq 4$ for every $n \in \mathbb{N}$ [M. Rosenfeld, 2020] using **a new unnamed technique...**.

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Algorithm 1: Choosing a nonrepetitive sequence from lists of size 4

$i \leftarrow 1$
while $i \leqslant n$ **do**
  $s_i \leftarrow$ random element of L_i
  if $s_1, \ldots, s_i$ is nonrepetitive **then**
    $i \leftarrow i + 1$
  else
    there is exactly one repetition, say $s_{i-2h+1}, \ldots, s_{i-h}, s_{i-h+1}, \ldots, s_i$
    $i \leftarrow i - h + 1$

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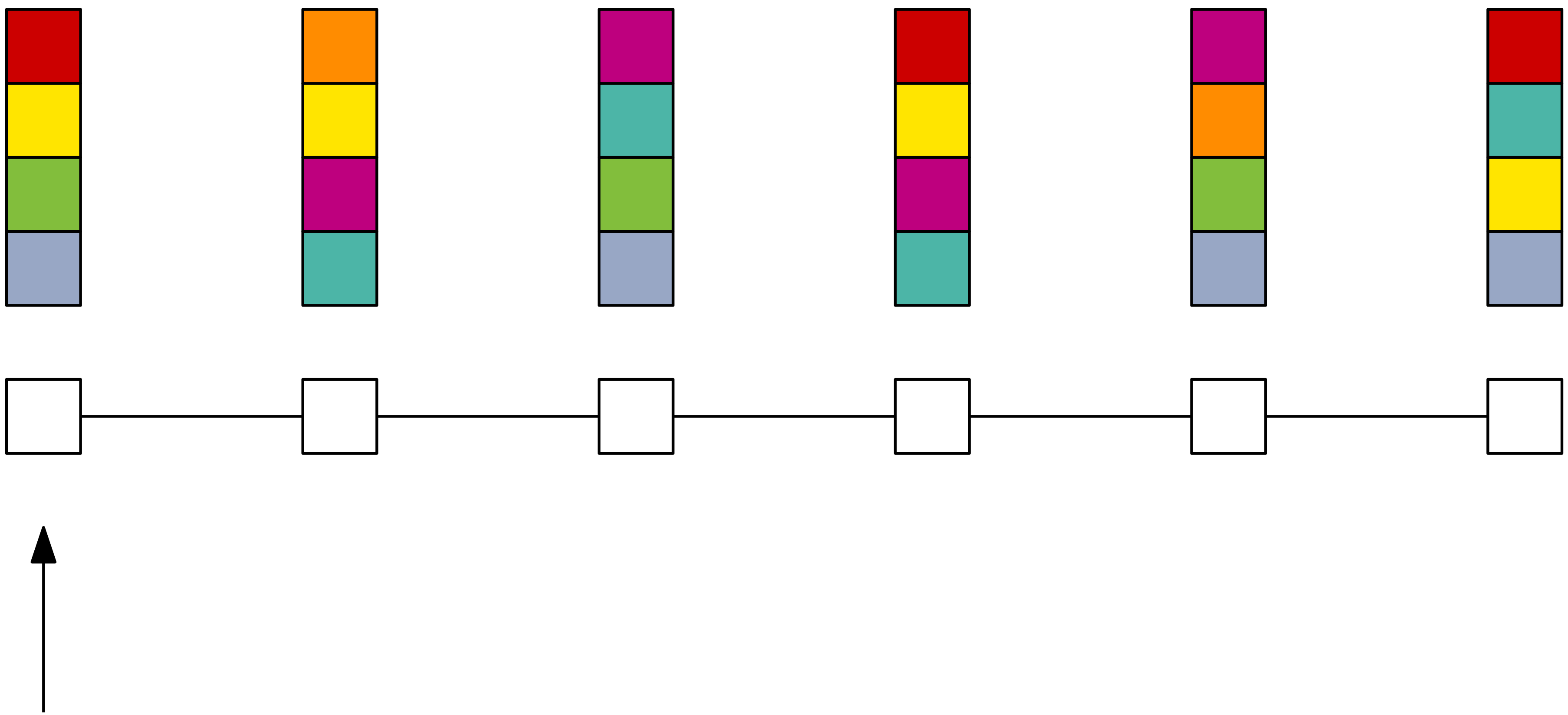

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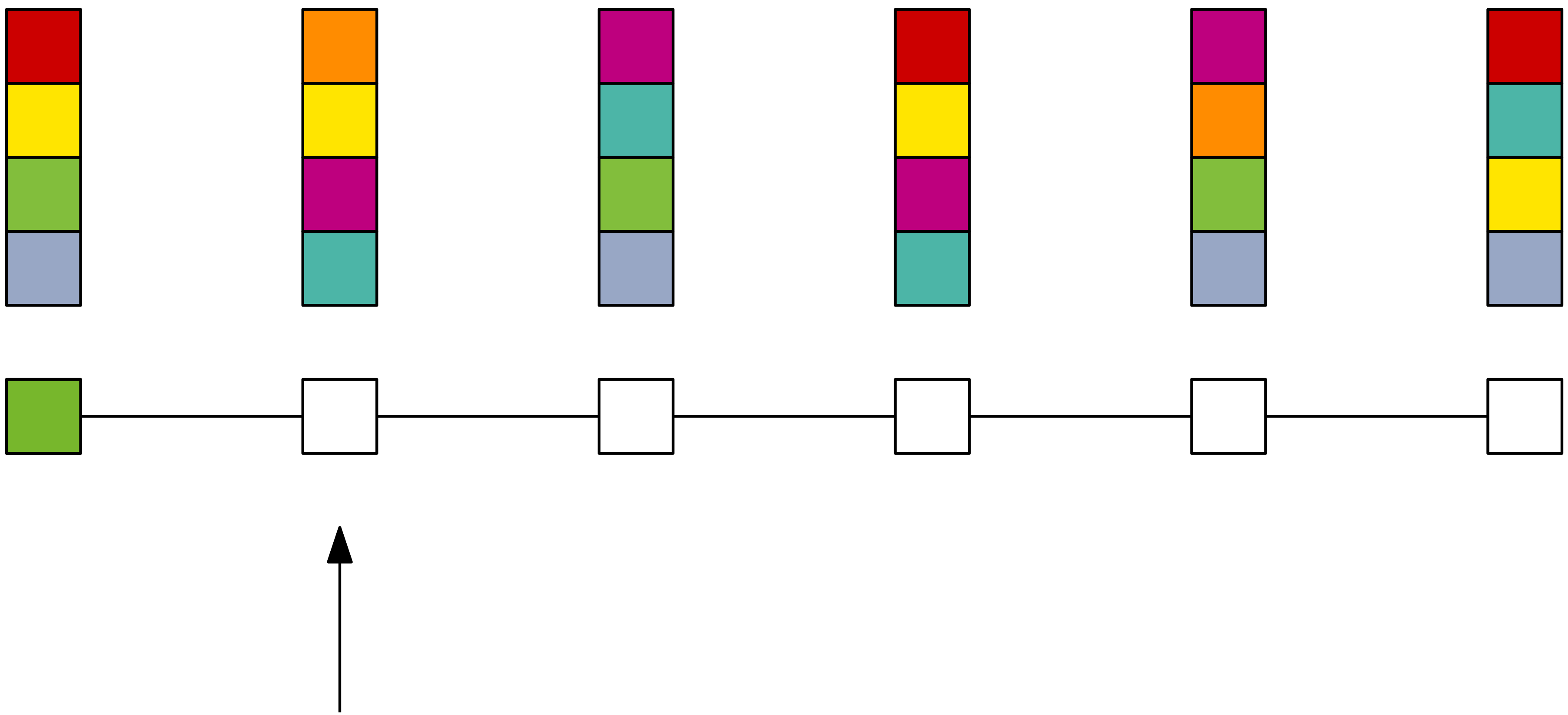

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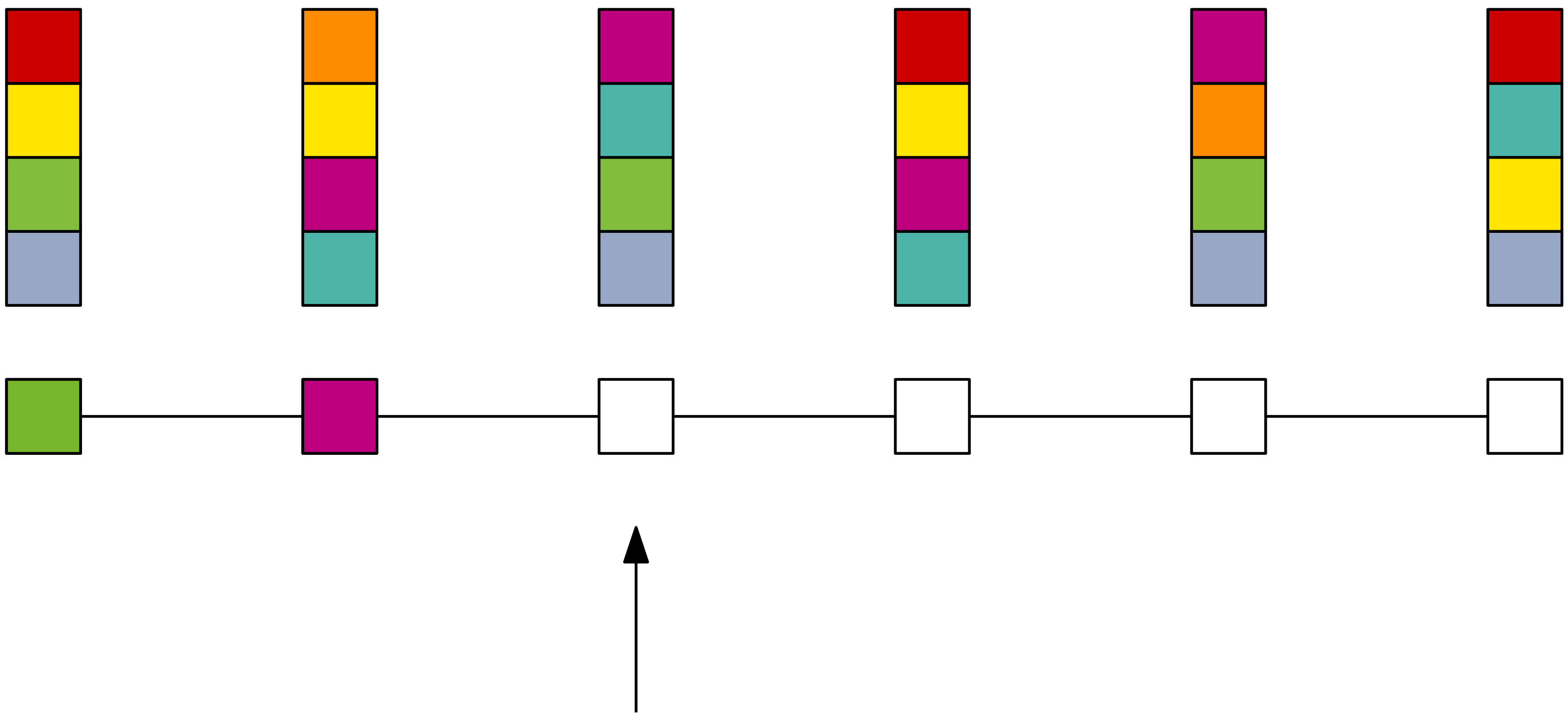

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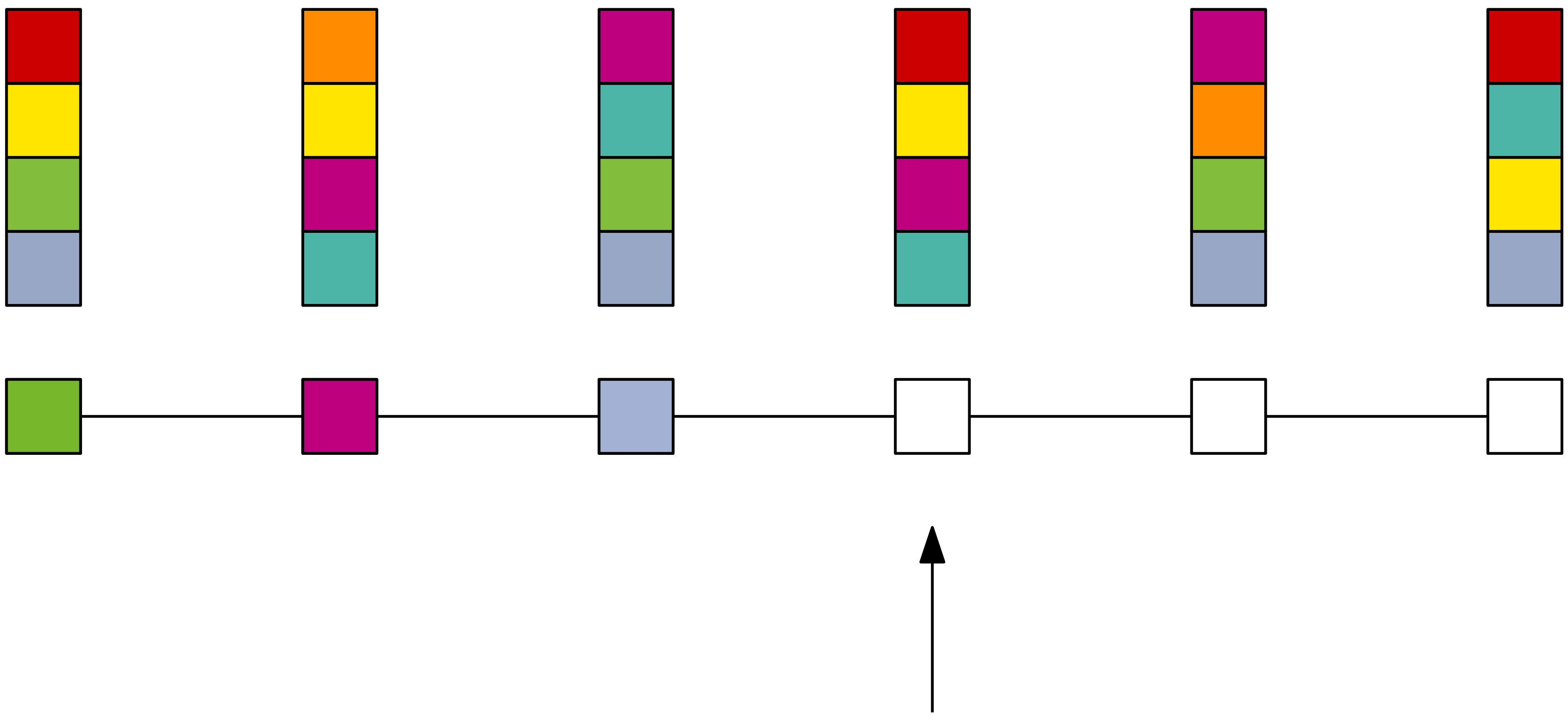

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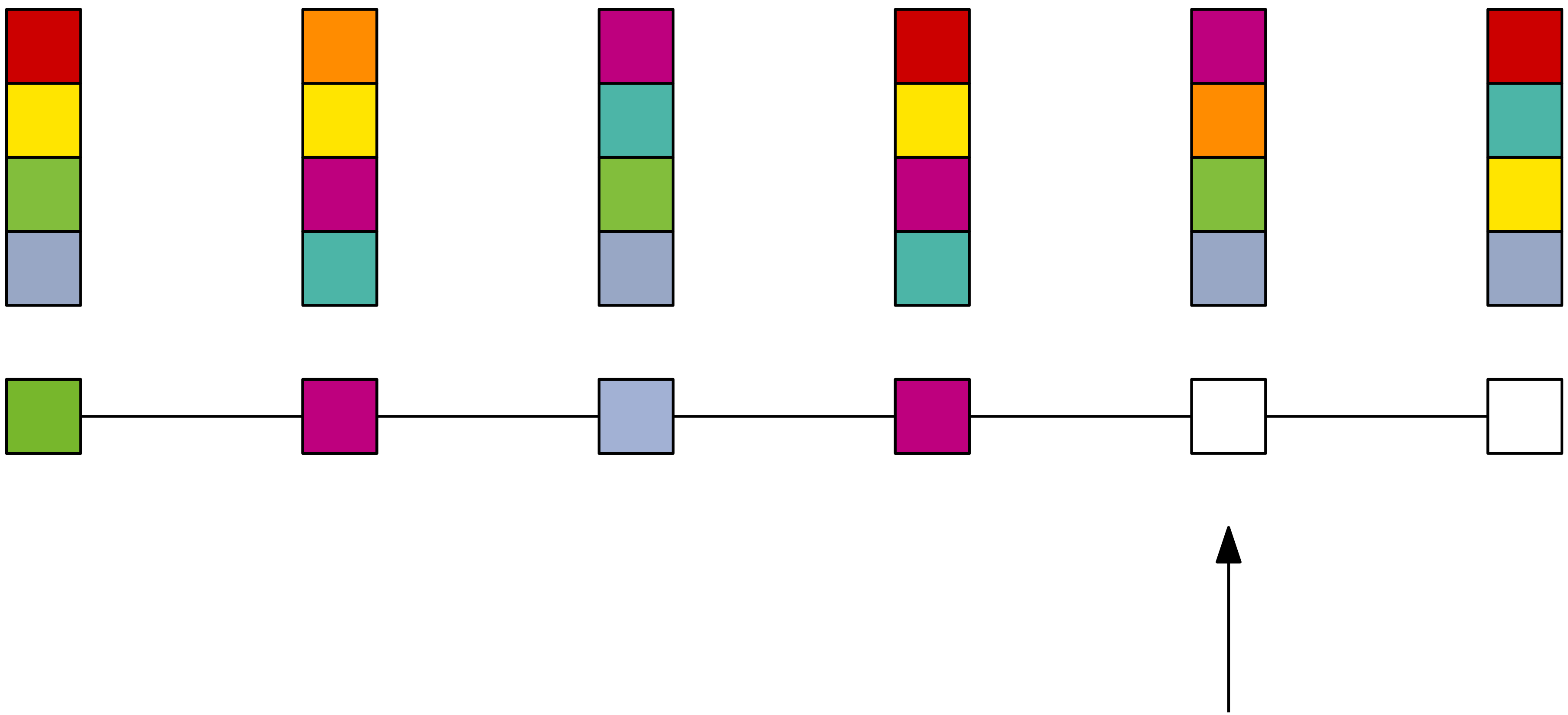

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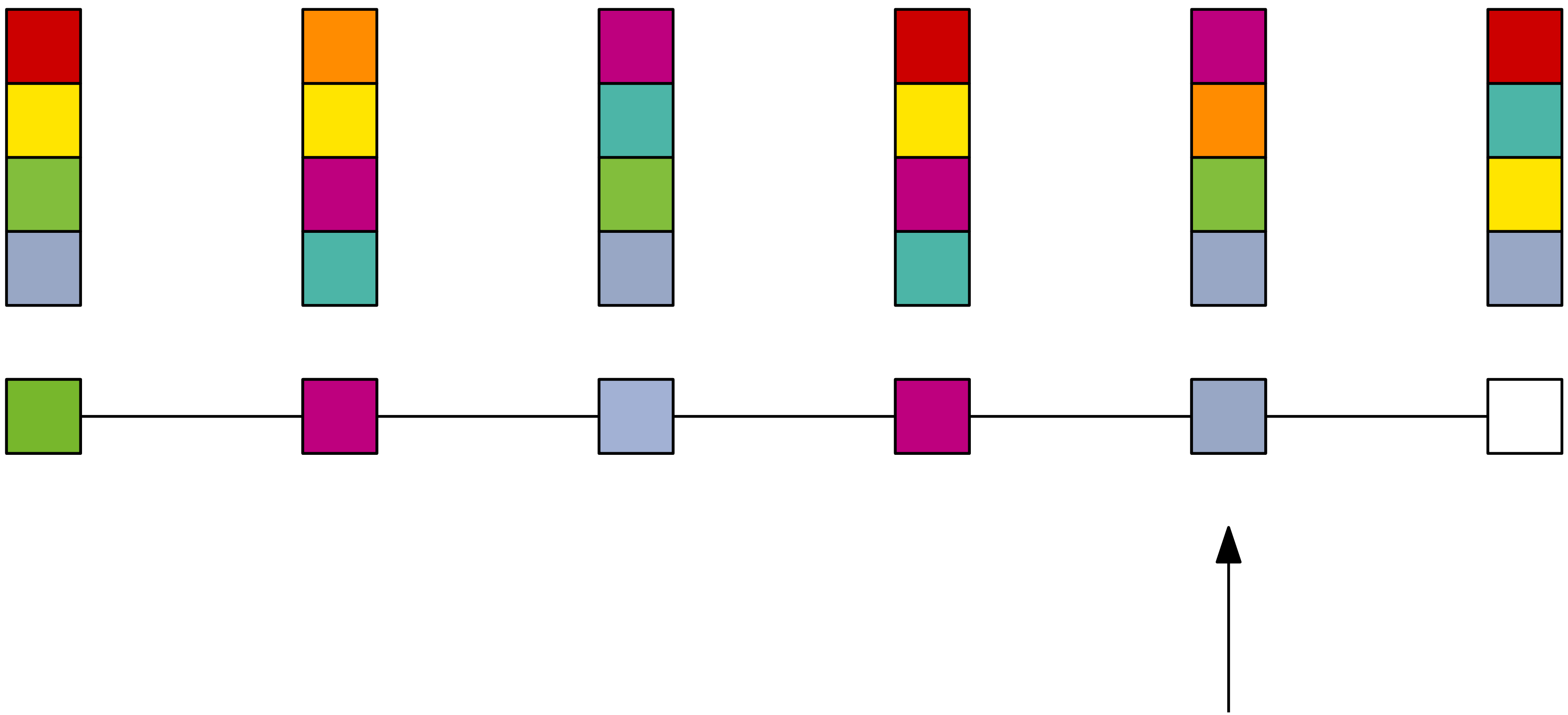

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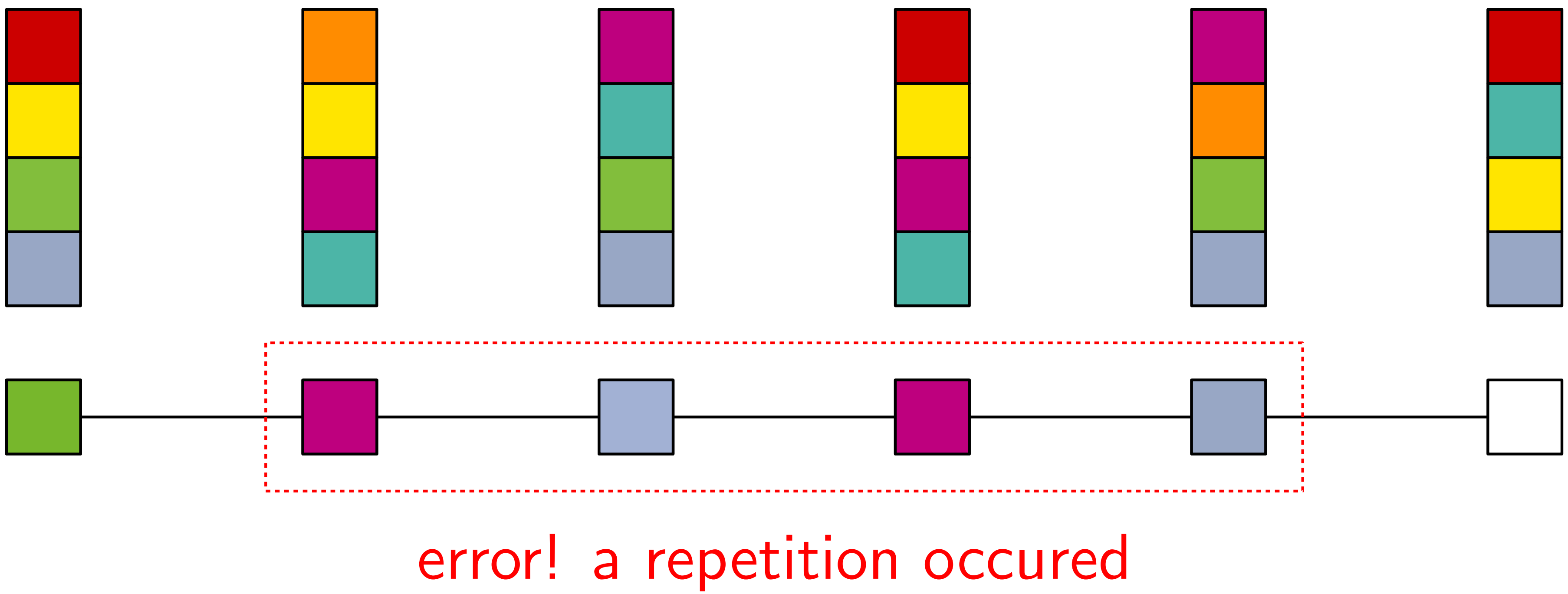

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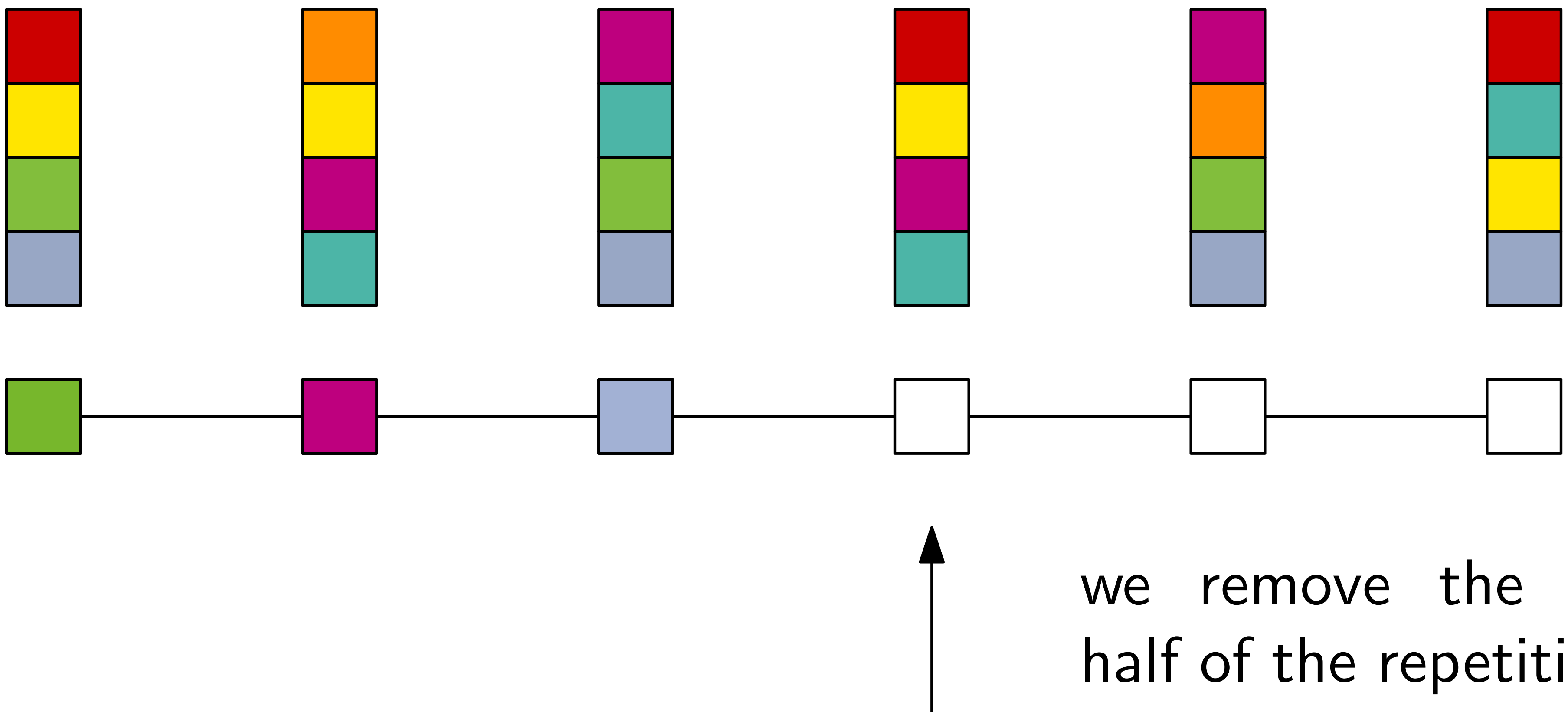

we remove the second half of the repetition and continue...

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The key idea is that **we can always recover the evaluation from the log**, and the other way around; **every evaluation corresponds to a unique log**.

But the number of possible evaluations is exactly 4^M, while one can show that the number of different logs is $o(4^M)$ (the number of sequences $(d_1, \ldots, d_2)$ can be estimated using Catalan numbers).

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Lemma. Let L be a list assignment of a path P such that all lists are of size 4. Let C_n be the number of non-repetitive colorings of the first n vertices of P that respect L. Then for any integer $n < |P|$, we have

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Note that this lemma implies something more than the entropy compression argument - that the number of non-repetitive colorings of a path of length k from lists of 4 colours is at least 2^k!

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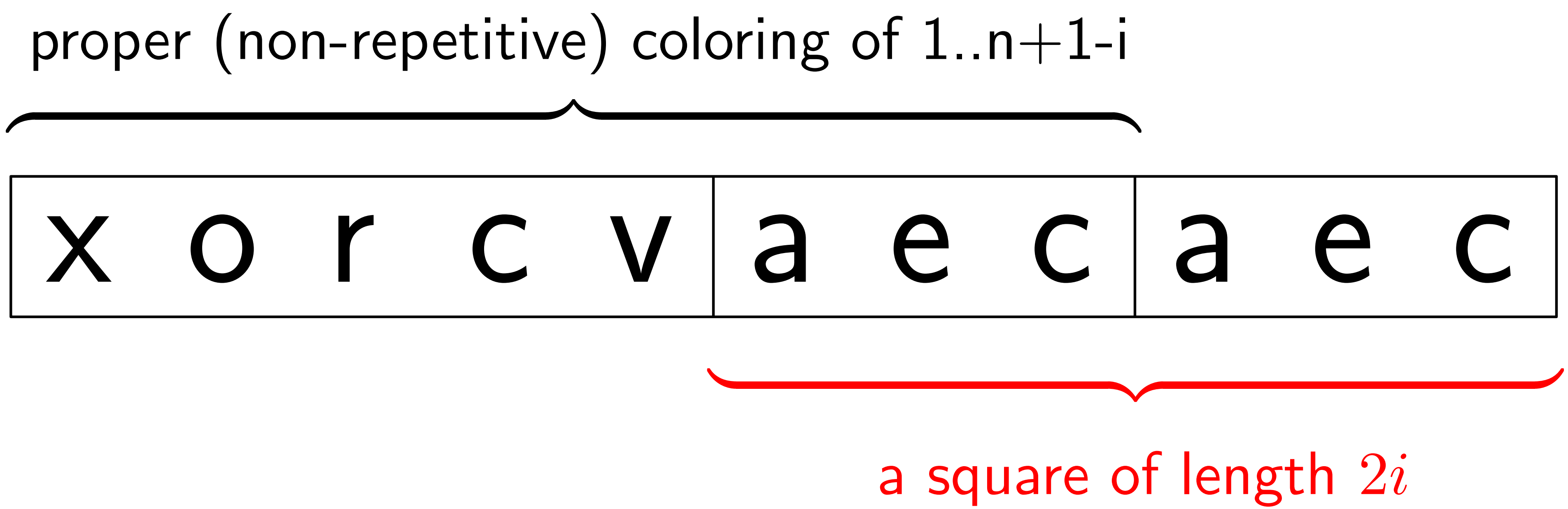

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$$\begin{aligned} C_{n+1} &\geq 4C_n - |F| \\ &\geq 4C_n - \sum_{i=1}^{n/2} 2^{1-i} C_n \\ &\geq 2C_n. \end{aligned}$$

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For example, in the previous lemma we had $\gamma = 4$ and $\alpha = 2$.

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And so the main issue will be to upper bound $|F|$, just like we did in the previous lemma.

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And we want the coefficients (a_i) to be small, in the previous lemma we had $a_i = 1$. If they are small enough, everything should be fine.

Example: $\pi_{ch}(G)$ for bounded $\Delta(G)$

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Theorem. For every graph G with maximum degree $\Delta \geq 1$, we have

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Again - we prove something stronger, let

$$\delta = \frac{3}{2^{\frac{2}{3}}} + 2^{\frac{2}{3}}\Delta^{-\frac{1}{3}} + \Delta^{-\frac{2}{3}}$$

$$\gamma = \Delta(\Delta - 1)(1 + \delta + \Delta^{-\frac{1}{3}}) + 1$$

$$\alpha = \Delta(\Delta + 1)(1 + 2^{\frac{1}{3}}\Delta^{-\frac{1}{3}}).$$

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Lemma. Let $\Delta \geq 2$ and G be a graph of maximal degree at most Δ and L be a list assignment of G. Suppose each list is of size at least γ, then for any vertex v of G we have

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It's obvious that this is indeed a stronger statement, but let's believe it for the sake of not being too technical today.

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Again, let F be the set of repetitive colorings of G respecting L which induce a non-repetitive coloring on $G \setminus \{v\}$.

We write $F = \bigcup_{i \geq 1} F_i$, where F_i is the set of colorings for F that contain a path of length $2i$ inducing a square.

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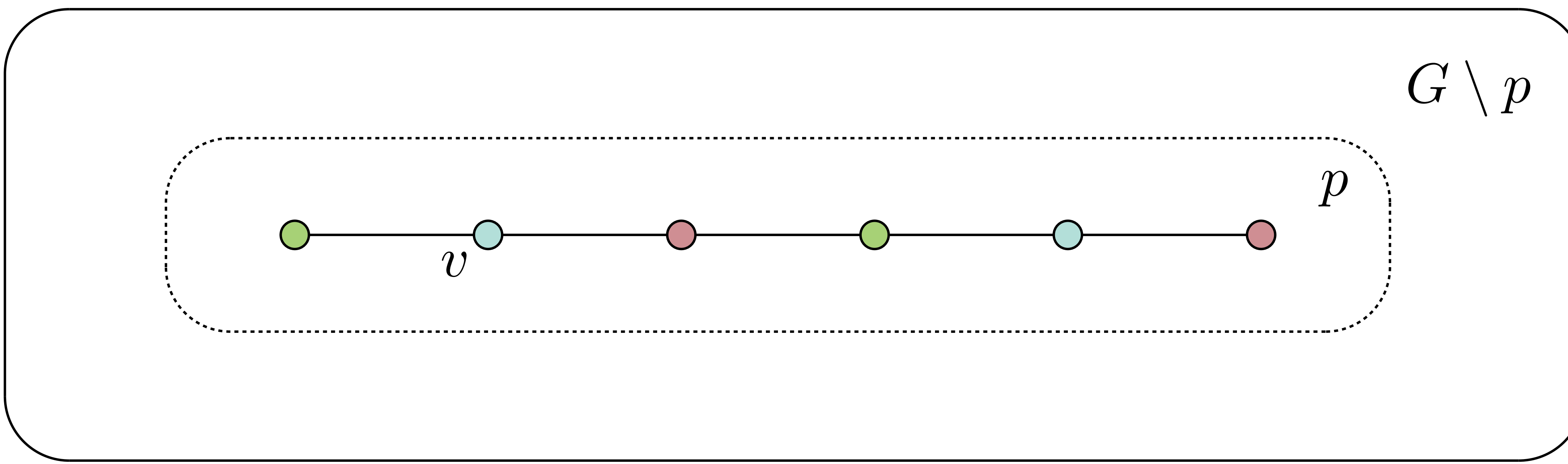

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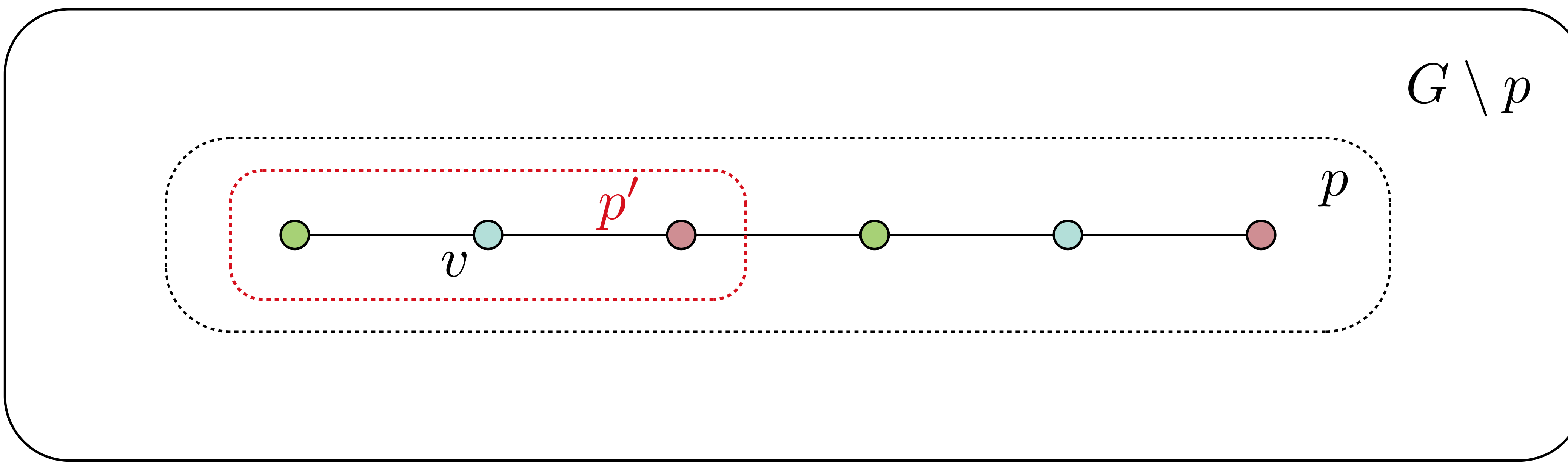

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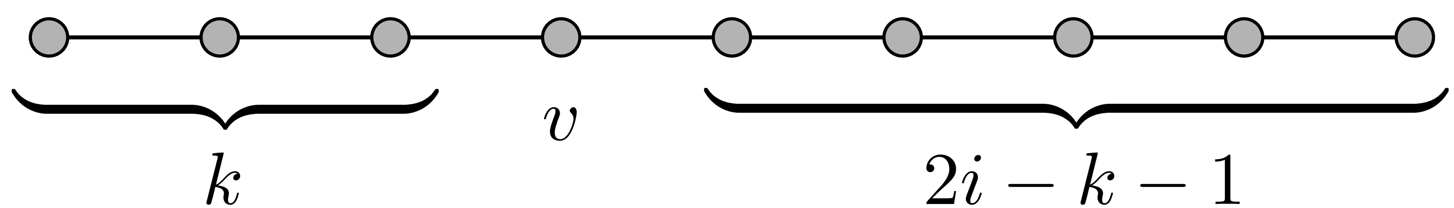

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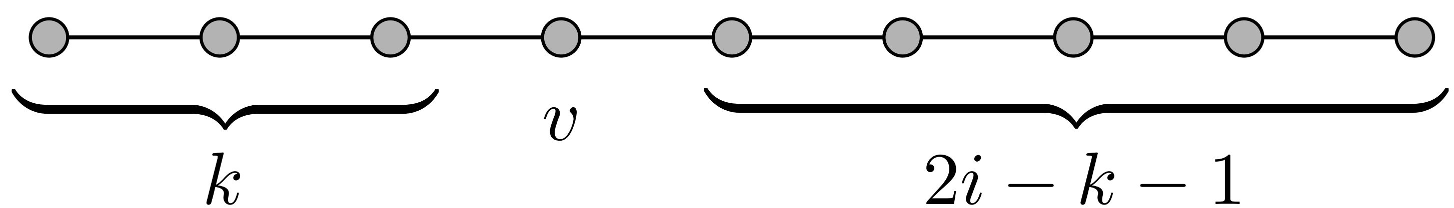

If we want to construct such path, we can first find the part of length k and then the second. Do it vertex after vertex. Note that in the first move we have at most Δ possibilities and for each next move at most $\Delta - 1$.

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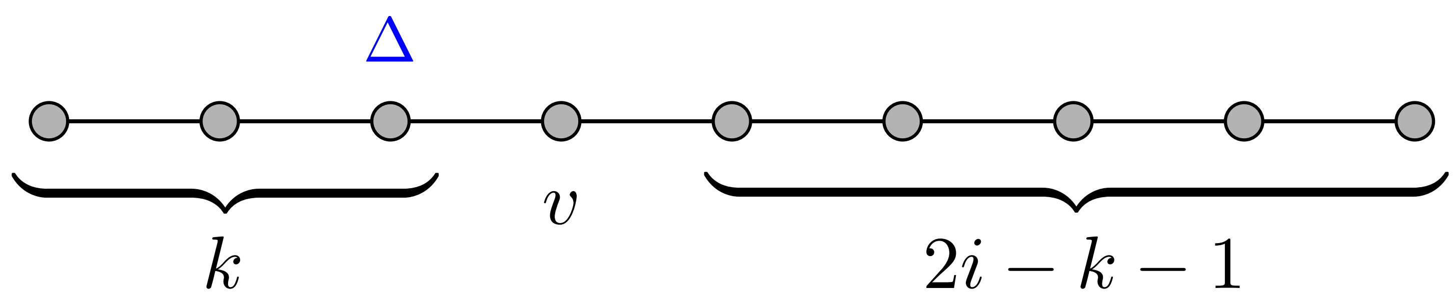

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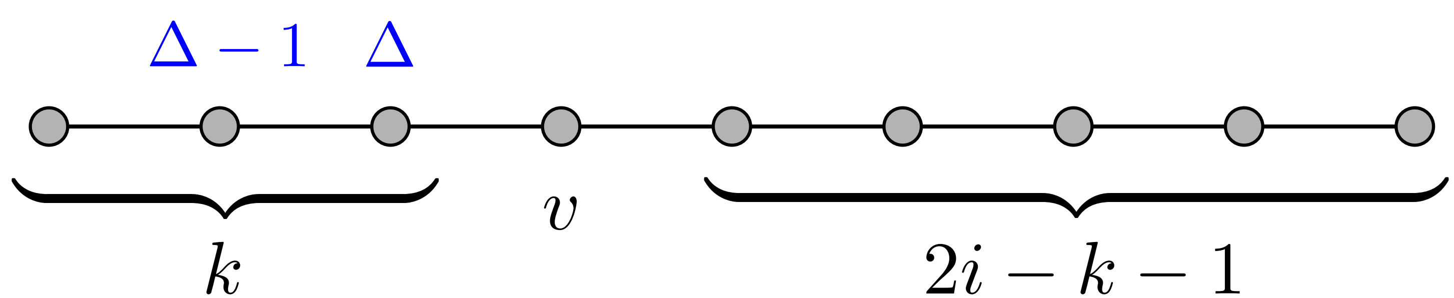

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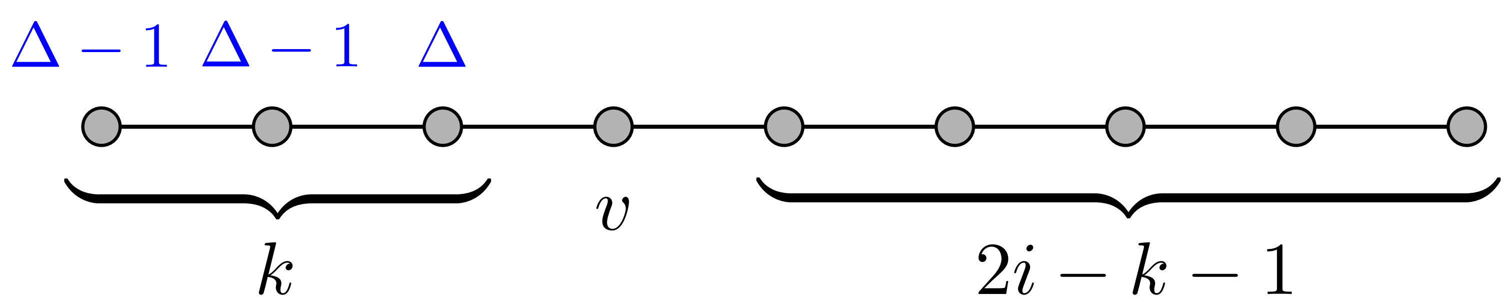

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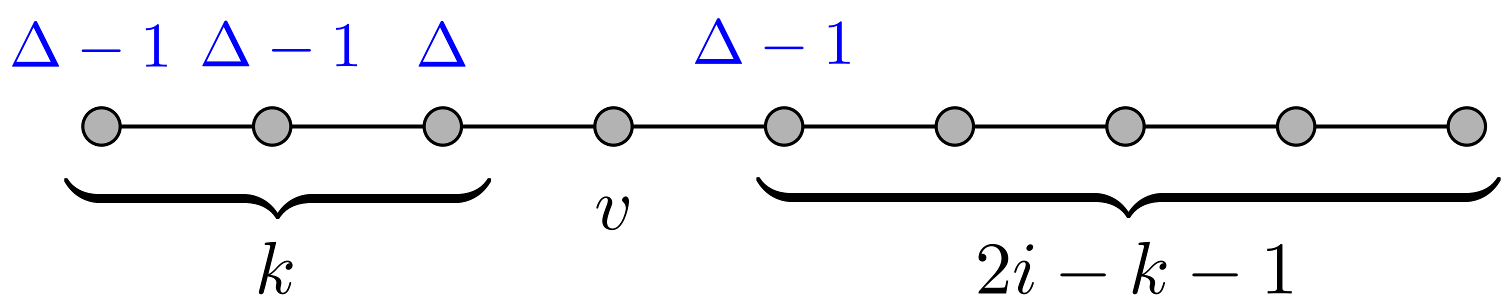

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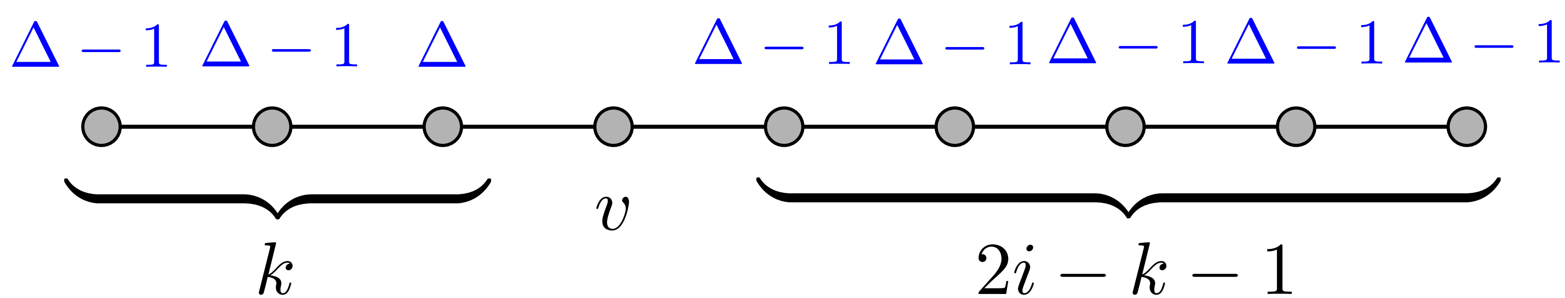

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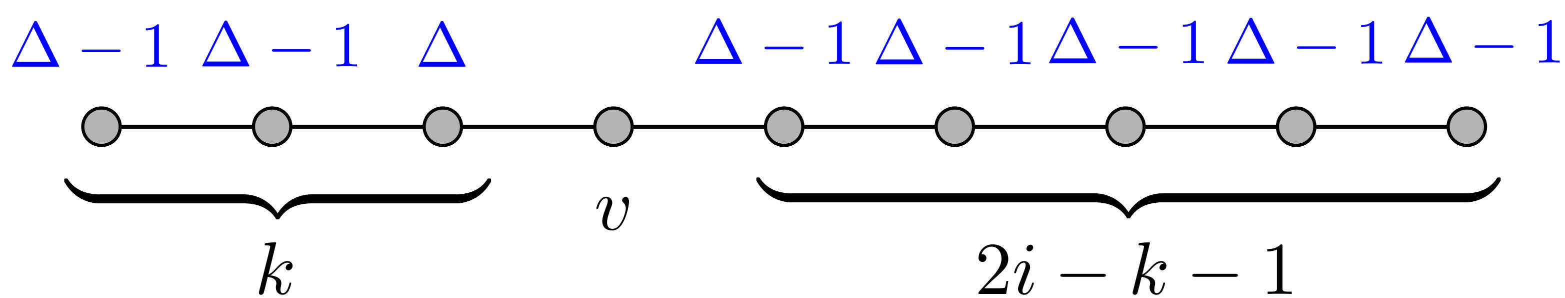

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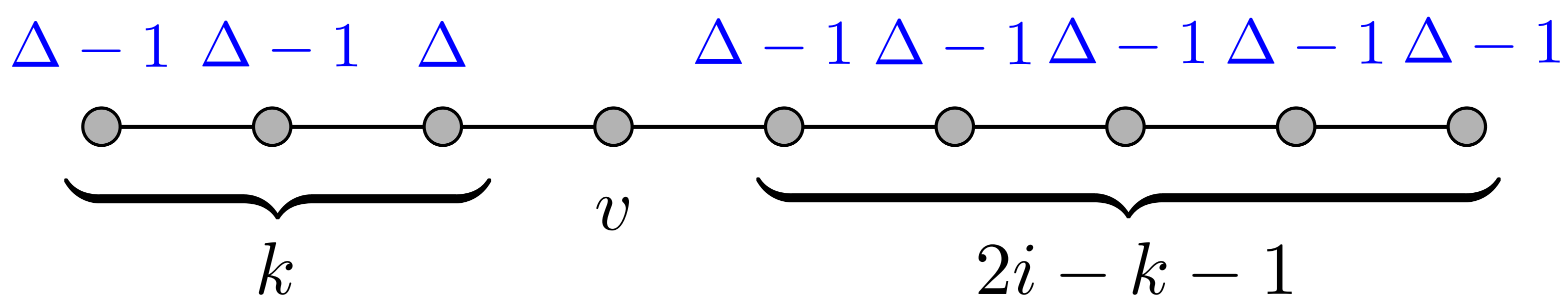

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Turns out this is enough to complete the proof after some calculations.

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Again, there are also list versions denoted as $\pi_{T_w ch}(G)$ and $\pi_{Tch}(G)$.

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new bound	previously known best bound
$\pi_{ch}(G) \leq \Delta^2 + \frac{3}{2^{\frac{2}{3}}}\Delta^{\frac{5}{3}} + 2^{\frac{2}{3}}\Delta^{\frac{4}{3}}$	$\pi_{ch}(G) \leq \Delta^2 + \frac{3}{2^{\frac{2}{3}}}\Delta^{\frac{5}{3}} + 2^{\frac{2}{3}}\Delta^{\frac{4}{3}} + 2\Delta + \mathcal{O}(\Delta^{\frac{2}{3}})$
$\pi_{T_w ch}(G) \leq 6\Delta$	???
$\pi_{T_w ch}(G) \leq \lceil 4.25\Delta \rceil$ for $\Delta \geq 300$	???
$\pi_{Tch}(G) \leq \Delta^2 + \frac{3}{2^{\frac{1}{3}}}\Delta^{\frac{5}{3}} + 8\Delta^{\frac{4}{3}} + 1$	$\pi_{Tch}(G) \leq \Delta^2 + 2^{\frac{4}{3}}\Delta^{\frac{5}{3}} + \mathcal{O}(\Delta^{\frac{4}{3}})$

The end

Thank you!