

Constant-Factor Approximation of Domination Number

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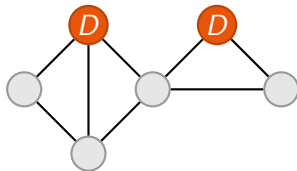
The Domination Problem

Definition: Dominating Set

For an undirected graph G , a set $D \subseteq V(G)$ is **dominating** if every vertex $v \in V(G) \setminus D$ has at least one neighbor in D .

Computational Complexity:

- Determining the minimal size $\text{dom}(G)$ is **NP-complete** (Karp).
- Approximating it within a factor better than $\mathcal{O}(\log |V(G)|)$ is NP-hard (Raz and Safra).
- **Objective:** Develop a constant-factor approximation algorithm strictly for structurally sparse graphs.



Example of a dominating set

Generalizing to Distance Domination

Definition: k -Dominating Set

A set $D \subseteq V(G)$ is **k -dominating** if the distance from any vertex $v \in V(G)$ to D is at most k . The minimum size of such a set is denoted as $\text{dom}_k(G)$.

- For $k = 1$, this is the standard domination problem.
- The parameter k represents the maximum allowable range (e.g., signal coverage, routing hops).
- Analyzing distance domination reveals deeper structural properties of the graph that are hidden in the $k = 1$ case.

Establishing a Lower Bound

To prove an approximation ratio, a strict lower bound is required.

Definition: d -Independent Set

A subset $A \subseteq V(G)$ is **d -independent** if the shortest path distance between each pair of vertices in A is strictly greater than d . The maximum size of such a set is $\alpha_d(G)$.

The Fundamental Inequality:

$$\text{dom}_k(G) \geq \alpha_{2k}(G) \quad (1)$$

- *Proof intuition:* Vertices in a $2k$ -independent set are spread so far apart that no single vertex can k -dominate more than one of them simultaneously.

Historical Context: Theorem 1

Previous work focused heavily on minor-closed graph classes.

Theorem 1 (Böhme and Mohar, 2003)

If G does not contain the complete bipartite graph $K_{q,r}$ as a minor, then:

$$\text{dom}(G) \leq (4r + (q - 1)(r + 1))\alpha_2(G) - 3r \quad (2)$$

Limitations of Theorem 1:

- Relies heavily on the presence or absence of specific minors.
- The proof mechanics are complex and tightly coupled to minor theory.
- Dvořák's goal was to simplify this by shifting focus to linear orderings.

Expanding the Scope: Theorems 2 & 3

Dvořák utilizes **arrangeability** ($\text{arr}(G)$) and 2-coloring number ($\text{col}_2(G)$).

Theorem 2 (Dvořák)

If G satisfies $\text{wcol}_2(G) \leq c$, then:

$$\text{dom}(G) \leq c^2 \alpha_2(G) \quad (3)$$

Theorem 3 (Dvořák)

Let G be a graph and d an integer. If $\delta(H) < d$ for every H such that $H \subseteq G$ or $\text{sd}_1(H) \subseteq G$, then:

$$\text{col}_2(G) \leq 4d^2(4d + 5) + 1 \quad (4)$$

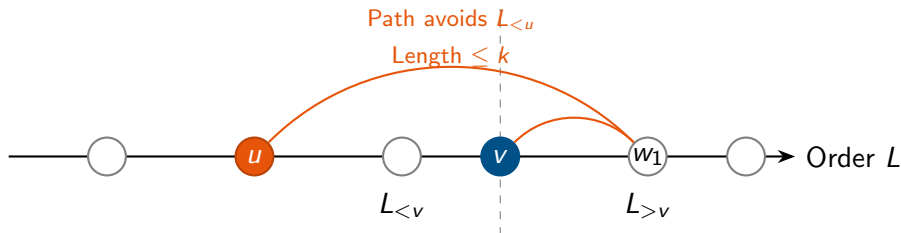
Significance: Theorem 3 proves that bounding the coloring number logically encompasses minor-closed classes, making Theorem 2 a direct generalization of Theorem 1.

The Core Tool: Weakly Accessible Sets

To handle $k > 1$, we generalize arrangeability using weak accessibility. Let L be a fixed linear ordering of $V(G)$.

Definition: Weakly k -Accessible Set ($Q_k(v)$)

A vertex u is weakly k -accessible from v if $u <_L v$ and there exists a path of length $\leq k$ between them where all internal vertices are $>_L u$.



Weak Coloring Number and Bounded Expansion

Definition: Weak k -Coloring Number ($\text{wcol}_k(G)$)

The minimum, over all possible linear orderings $L \in \Pi(G)$, of $\max_{v \in V(G)} |Q_k(v)| + 1$.

Bounded Expansion Classes:

- A graph class has *bounded expansion* if for every integer $k \geq 1$, there exists a constant c_k such that $\text{wcol}_k(G) \leq c_k$ for all graphs in the class.
- **Examples:** Planar graphs, graphs of bounded genus, classes excluding a topological minor.
- **Intuition:** In these networks, a "good" linear ordering L always exists where no vertex looks left and sees too many weakly accessible neighbors.

Theorem 4: The General Bound

This is the central optimization result of the paper.

Theorem 4

If $1 \leq m \leq 2k + 1$ and G satisfies $\text{wcol}_m(G) \leq c$, then:

$$\text{dom}_k(G) \leq c^2 \alpha_m(G) \quad (5)$$

Mathematical Implications:

- The approximation ratio c^2 is strictly constant, independent of $|V(G)|$.
- The bound naturally scales for any distance k .
- **The Parity Benefit:** The theorem holds for $m = 2k + 1$, which provides a strictly tighter bound than evaluating at $m = 2k$.

Algorithm 1: The Greedy Strategy

Given the optimal ordering L , the algorithm is highly efficient.

- 1: **Input:** Graph G , ordering L where $|Q_m(v)| < c$
 - 2: $R \leftarrow V(G)$
 - 3: $D \leftarrow \emptyset$
 - 4: $A' \leftarrow \emptyset$
 - 5: **while** $R \neq \emptyset$ **do**
 - 6: $v \leftarrow$ the **first** vertex in R according to L
 - 7: $A' \leftarrow A' \cup \{v\}$
 - 8: $D \leftarrow D \cup \{v\} \cup Q_m(v)$
 - 9: Remove all w from R where $\text{dist}(w, \{v\} \cup Q_m(v)) \leq k$
 - 10: **end while**
 - 11: **Return** D, A'
- ▷ Vertices remaining to be covered
 - ▷ The output k -dominating set
 - ▷ Intermediate greedy set

How do we keep the time complexity at $\mathcal{O}(n)$?

- Constantly checking distances to remove nodes from R (line 8) could ruin our linear time.
- **The Trick:** Dvořák maintains a value $p(v) = \min(k + 1, \text{dist}(v, D))$ for every node.
- When a new node is added to D , we set its $p(v) = 0$.
- We then recursively propagate this update to its neighbors, **only** if it improves their current $p(w)$ score ($p(u) > p(w) + 1$).
- This strict rule limits the total number of updates per node, ensuring the greedy sweep runs precisely in $\mathcal{O}(c(k + 1)n)$.

The Preprocessing Challenge: Finding the Order

There is a catch: Algorithm 1 needs the optimal ordering L on the input. But finding an ordering that minimizes $wcol_m$ is likely **NP-complete**.

The Solution: m -Admissibility as a Proxy

- Instead of weak coloring, Dvořák optimizes an easier parameter called **m -admissibility** (adm_m).
- **Lemma 5** proves that if a graph has a bounded m -admissibility (say, c), its weak coloring number is automatically bounded by $(c(c-1)^{m-1} + 1)^m$.
- This means optimizing admissibility is good enough to guarantee our constant-factor approximation.

Algorithm 2: Constructing the Order

We build the ordering **backward** (from position n down to 1):

- Put all vertices into a set S .
- Find a vertex $v_i \in S$ that minimizes $b_m(S, v_i)$.
- $b_m(S, v_i)$ is the maximum number of paths of length $\leq m$ from v_i back to other nodes in S (intersecting only at v_i).
- Extract v_i from S and place it at the current end of the sequence.
- **Intuition:** We repeatedly strip away the nodes that have the fewest "return paths" to the remaining graph, pushing the most loosely connected nodes to the back of the line.

Fast Preprocessing via First-Order Logic

- Evaluating the number of paths for admissibility is still NP-complete for large m (≥ 5).
- **Theorem 6 (The Ultimate Shortcut):** Dvořák utilizes his prior work on First-Order (FO) logic data structures.
- In graphs with bounded expansion, we can encode the complex path-counting problem as a pure FO logic query.
- These specialized structures can answer FO queries in **constant time** $\mathcal{O}(1)$.
- **Result:** The entire preprocessing phase is compressed down to **strict linear time** $\mathcal{O}(n)$.

Proof Framework: The Auxiliary Graph H

To prove the approximation ratio, we must extract a true m -independent set A from our greedy set A' .

Constructing the Auxiliary Graph H :

- **Vertices:** $V(H) = A'$.
- **Edges:** Connect $u, v \in V(H)$ if their distance in the original graph G is $\leq m$.

The Key Lemma: Dvořák proves that the standard coloring number of this auxiliary graph is bounded:

$$\text{col}_1(H) \leq c \tag{6}$$

Proof Framework: The Pigeonhole Principle

Extracting the Independent Set:

- Since $\text{col}_1(H) \leq c$, the graph H can be properly colored using exactly c colors.
- A proper coloring guarantees that no two connected vertices share the same color.
- By the Pigeonhole Principle, the largest color class must contain at least $|A'|/c$ vertices.
- Let this largest color class be A . Because no two vertices in A share an edge in H , their distance in G must be strictly greater than m . Thus, A is our **m -independent set**.

Concluding the Bound:

$$|D| \leq c|A'| \leq c(c|A|) = c^2|A|$$

Why evaluate at $2k + 1$ instead of $2k$?

The Counter-example (G_n):

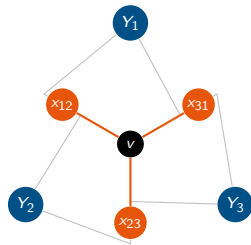
- Crown-like graphs: $sd_{2k-1}(K_n)$ plus a central vertex v .
- These graphs are sparse (exclude $K_{3,3}$ minor).

The Efficiency Gap:

- **Baseline:** $\alpha_{2k}(G_n) = 2$ (fixed).
- **Required:** $\text{dom}_k(G_n) \geq n/2$ (unbounded).

Conclusion:

- At $m = 2k$, the approximation ratio $\rightarrow \infty$ as n grows.
- $m = 2k + 1$ is the **tightest possible** bound to ensure a constant factor.



Set Y (Clique nodes)

Set X (Central path nodes)

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