

Classical simulation of quantum CSP strategies

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DOI:10.1109/LICS65433.2025.00035

May 27, 2026

Motivating example: Graph k -colorability problem

Graph k -colorability:

- Assign one of k colors to graph vertices, i.e. $c : V \rightarrow [k]$ such that $c(u) \neq c(v)$ for $(u, v) \in E$;
- Decision problem **COL _{k}** : given a graph G , is it k -colorable?
- **COL₂** is in P complexity class, while **COL _{≥ 3}** in NPC.

Motivating example: Graph k -colorability problem

Graph k -colorability:

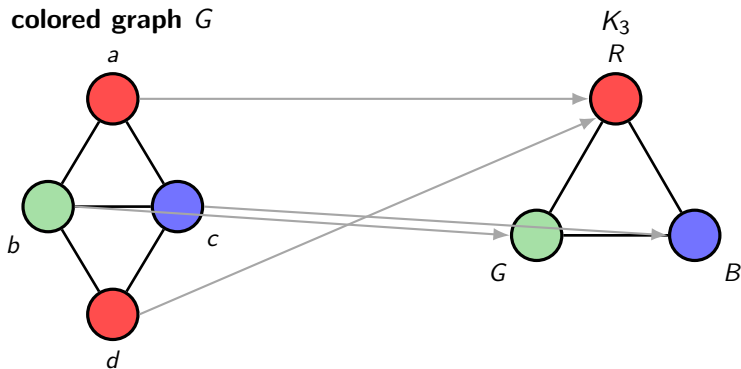
- Assign one of k colors to graph vertices, i.e. $c : V \rightarrow [k]$ such that $c(u) \neq c(v)$ for $(u, v) \in E$;
- Decision problem **COL $_k$** : given a graph G , is it k -colorable?
- **COL $_2$** is in P complexity class, while **COL $_{\geq 3}$** in NPC.

The same problem can be restated as a graph homomorphism problem:

Definition (Graph colorability)

Graph $G \in \mathbf{COL}_k$ iff there is a graph homomorphism $G \rightarrow K_k$.

Colorability as homomorphism problem - example



Important notice

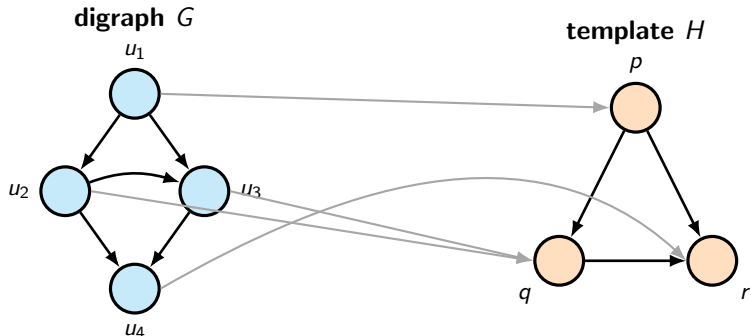
From now on, when we say "graph", we really mean a directed graph (digraph).

Graph colorability: generalization

For a fixed graph H , called a template, we define

Definition (Homomorphism problem)

Graph $G \in \mathbf{HP}_H$ iff there is a graph homomorphism $G \rightarrow H$.



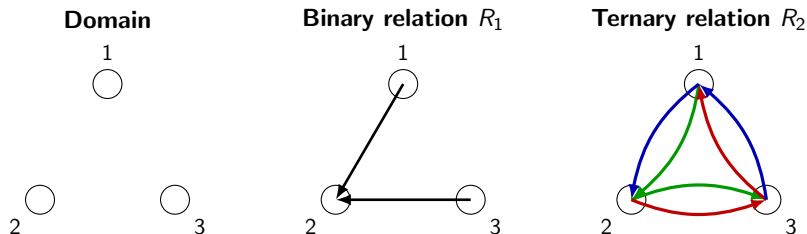
Definition (Relational signature)

A relational signature σ is a finite set of relational symbols, each associated with a positive integer number called *arity*. We denote $\text{ar}(R)$ the arity of $R \in \sigma$.

Definition (Relational structure)

A **relational structure** $\mathbb{X}$ with the signature σ consists of a domain (or base set, element set) X with a relation $R^{\mathbb{X}} \subseteq X^{\text{ar}(R)}$ associated with each $R \in \sigma$.

Relational structures - example



Relational structure with domain $\{1, 2, 3\}$ and $\sigma = \{R_1, R_2\}$;
 $\text{ar}(R_1) = 2$; $\text{ar}(R_2) = 3$

Constraint Satisfaction Problem

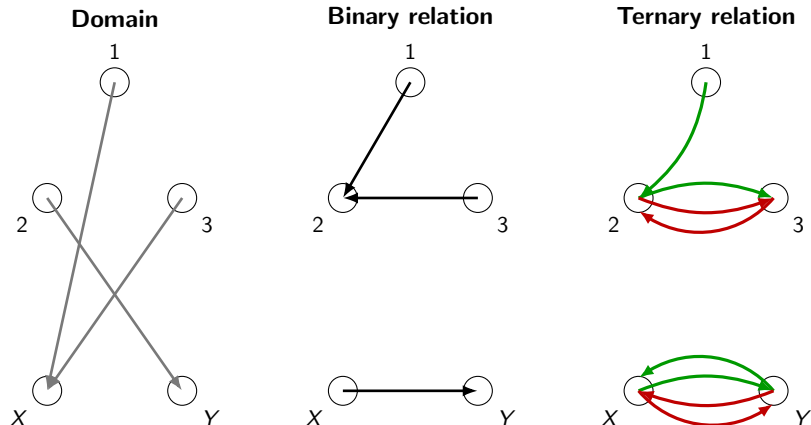
Definition (Relational structures homomorphism)

For relational structures $\mathbb{X}$ and $\mathbb{Y}$ with the same signature σ , we say that $f : X \rightarrow Y$ is a relational structures homomorphism, if $\forall R \in \sigma \forall (x_1, \dots, x_{\text{ar}(R)}) \in R^{\mathbb{X}}$ we have $(f(x_1), \dots, f(x_{\text{ar}(R)})) \in R^{\mathbb{Y}}$.

Definition (Constraint Satisfaction Problem)

Relational structure $\mathbb{X} \in \mathbf{CSP}(\mathbb{Y})$ iff $\mathbb{X}$ has the same signature as $\mathbb{Y}$ and there is a relational structures homomorphism $\mathbb{X} \rightarrow \mathbb{Y}$.

CSP - example



$$Y = \{0, 1\}$$

- $R^Y = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$: 1-in-3 positive SAT
- $R^Y = \{0, 1\}^3 \setminus \{(0, 0, 0), (1, 1, 1)\}$: not-all-equal SAT (NAE)
- $R_{a,b,c} = \{0, 1\}^3 \setminus \{(a, b, c)\}$, $\mathbb{Y} = (Y, R_{000}, R_{001}, R_{011}, R_{111})$ - 3-SAT
- k -rainbow coloring of uniform hypergraphs
- 2-SAT - NL-complete, in P
- $R^Y = \{(0, \dots, 0)\}$ - loop; $R^Y = \emptyset$: (almost) trivial problems
- ...

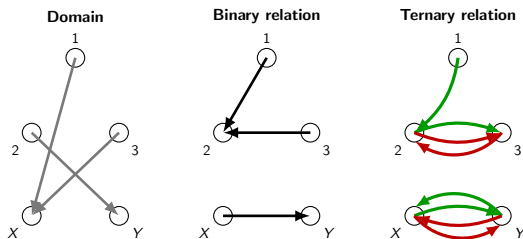
Is there a template, such that the corresponding CSP is NP-intermediate?

Two prover games

- Input: compatible relational structures $\mathbb{X}$ and $\mathbb{Y}$;
- Strategizing: Alice and Bob strategize;
- Separation: Alice and Bob are separated and cannot communicate;
- Inquiry: Referee sends a relation symbol R and a tuple $(x_1, \dots, x_{\text{ar}(R)}) \in R^{\mathbb{X}}$ to Alice, and a domain element $x \in X$ to Bob;
- Response: Alice responds with a tuple $(y_1, \dots, y_{\text{ar}(R)}) \in R^{\mathbb{Y}}$ and Bob answers with $y \in Y$;
- Victory condition: $\forall i \in [\text{ar}(R)] : x_i = x \implies y_i = y$.

Alice and Bob can win iff $\mathbb{X} \in \text{CSP}(\mathbb{Y})$!

CSP - example revisited



Referee to Alice: $(1, 2, 3) \in R_2$;

Referee to Bob: $3 \in \{1, 2, 3\}$;

Alice to Referee: (X, Y, X) ;

Bob to Referee: X

Alice and Bob share an entangled state during strategizing phase, and measure it according to their strategy.

Definition (Projection Valued Measurement)

Given a finite set S and Hilbert space $\mathbb{H}$, we define the set of all Projection Valued Measurements

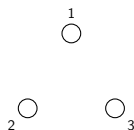
$\mathbf{PVM}_{\mathbb{H}}(S) = \{f : S \rightarrow \text{Proj}_{\mathbb{H}} : \sum_{s \in S} f(s) = I\}$, where $\text{Proj}_{\mathbb{H}}$ is the set of all (orthogonal) projectors^a on $\mathbb{H}$.

$$^a p \in \text{Proj}_{\mathbb{H}} \iff p^2 = p = p^*$$

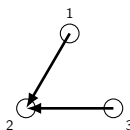
We assume that Alice and Bob share an entangled pair of qudits from a bipartite space $\mathbb{H}^{\otimes 2}$.

Quantum CSP - "example"

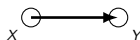
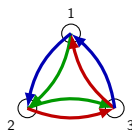
Domain



Binary relation



Ternary relation



Referee to Alice: $(3, 1, 2) \in R_2$;

Referee to Bob: $3 \in \{1, 2, 3\}$;

Alice measurement: (Y, X, Y) ;

Bob measurement: Y

Referee to Alice: $(3, 2) \in R_1$;

Referee to Bob: $3 \in \{1, 2, 3\}$;

Alice measurement: (X, Y) ;

Bob measurement: X

Quantum strategy for CSP

Alice and Bob strategize by selecting a set of:

- Alice's measurements¹: $\mathcal{A}^{(R, \mathbf{x})} \in \mathbf{PVM}_{\mathbb{H}}(R^{\mathbb{Y}})$ for $\mathbf{x} \in R^{\mathbb{X}}$;
- Bob's measurements: $\mathcal{B}^{(\mathbf{x})} \in \mathbf{PVM}_{\mathbb{H}}(Y)$ for $\mathbf{x} \in X$.

Probability of Alice and Bob winning

$$\sum_{\mathbf{y}, y: \forall i \ x_i = x \implies y_i = y} \langle \psi | A_{\mathbf{y}}^{(\mathbf{x})} \otimes B_y^{(\mathbf{x})} | \psi \rangle$$

The winning-with-certainty condition then becomes:

Winning condition for Quantum CSP

For all $R \in \sigma, \mathbf{x} \in R^{\mathbb{X}}, i \in [\text{ar}(R)]$, we have

$$B_y^{(x_i)} = \sum_{\mathbf{y} \in R^{\mathbb{Y}}; y_i = y} A_{\mathbf{y}}^{(\mathbf{x})}$$

¹We will skip R in the superscript from now on

A different perspective

Abramsky, Barbosa, de Silva, Zapata, 2017

Existence of a Quantum strategy on $\mathbb{H}$ qudits to a CSP with template $\mathbb{Y}$, is itself a CSP problem on a template denoted with $\mathbb{Y}^{\mathbb{H}}$, on the domain consisting of $\mathbf{PVM}_{\mathbb{H}}(Y)$.

Back to classical computers

We allow Alice and Bob to share a one-way classical channel with a fixed width.

BCKT, 2025

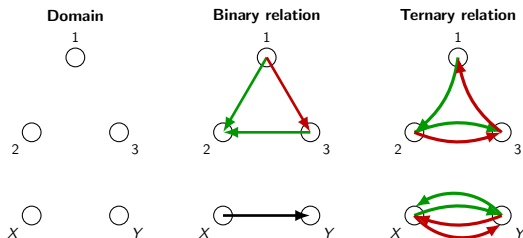
Existence of such a strategy is again itself a CSP on a template denoted with $\mathbb{Y}^{\rightarrow k}$ and $\mathbb{Y}^{\leftarrow k}$, depending on the direction of communication (Alice-to-Bob and Bob-to-Alice respectively).

For graphs:

$\mathbb{Y}^{\rightarrow k} = k\text{-fold co-normal product}$

$\mathbb{Y}^{\leftarrow k} = k\text{-fold join}$

Alice-to-Bob CSP - example



$k = 2$; They use mapping $1 \mapsto X, 2 \mapsto Y, 3 \mapsto X$ when message is "green", and $1 \mapsto X, 2 \mapsto X, 3 \mapsto Y$ if the message is "red".

Dequantizing quantum strategies

The main result of the publication:

BCKT, 2025

For each σ -structure $\mathbb{Y}$ and each Hilbert space $\mathbb{H}$ there exists some $k \in \mathbb{N}$ such that $\text{CSP}^{\mathbb{H}}(\mathbb{Y}) \subseteq \mathbb{Y}^{\rightarrow k}$ (similarly with $\mathbb{Y}^{\leftarrow k}$, with small caveats).

And furthermore it refinement to digraphs:

BCKT, 2025

For each digraph $\mathbb{Y}$ and each Hilbert space $\mathbb{H}$ of dimension d it holds that:

- 1 $k = d + \log_2 \chi(\mathbb{Y}) + o(d) + o(\log_2 \chi(\mathbb{Y}))$ in Alice-to-Bob case
- 2 $k = \chi(\mathbb{Y}) \cdot O(\exp(d))$ for Bob-to-Alice case

Proof outline

Chain of homomorphisms: $\mathbb{Y}^{\mathbb{H}} \rightarrow K_n^{\mathbb{H}} \rightarrow K_m \rightarrow \mathbb{Y}^{\rightarrow k}$

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Abramsky, Barbosa, de Silva, Zapata, 2017

If $\mathbb{Y} \rightarrow \mathbb{Y}'$, then $\mathbb{Y}^{\mathbb{H}} \rightarrow \mathbb{Y}'^{\mathbb{H}}$.

And obviously $\mathbb{Y} \rightarrow K_{\chi(\mathbb{Y})}$.

Step 2

Chain of homomorphisms: $\mathbb{Y}^{\mathbb{H}} \rightarrow K_n^{\mathbb{H}} \rightarrow K_m \rightarrow \mathbb{Y}^{\rightarrow k}$

Rogers, 1963; Prosanov, 2018

It is possible to color $S(\rho, d)$ with $(2\rho + o(1))^{d-1}$ colors in such a way that points at distance 1 have distinct colors.

By setting $\rho = \sqrt{2}/2$, we obtain *orthogonal coloring*.

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By setting $\rho = \sqrt{2}/2$, we obtain *orthogonal coloring*.

Ciardo, 2024

For any Hilbert space $\mathbb{H}$ of dimension d and any integer $n \geq 2$, it holds that $K_n^{\mathbb{H}} \rightarrow \mathcal{S}_{\mathbb{C}}(d, n) \rightarrow \mathcal{S}_{\mathbb{R}}(2d, n)$.

where $\mathcal{S}$ are certain infinite structures on the domain made of frames - matrices such that $\text{tr}(\mathbf{M}\mathbf{M}^*) = 1$. Connection is made by placing frames' renormalized rows on sphere. $m \approx \chi(\mathbb{Y})2^d$.

Step 3

Chain of homomorphisms: $\mathbb{Y}^{\mathbb{H}} \rightarrow K_n^{\mathbb{H}} \rightarrow K_m \rightarrow \mathbb{Y}^{\rightarrow k}$

Third step by blowing up a single directed edge. We take k so large, so that $\binom{k}{\lfloor k/2 \rfloor} > m$. For example,
 $k = \lceil \log_2 m + \log_2 \log_2 m + 2 \rceil$. $k \approx d + \log_2(\chi(\mathbb{Y}))$

Proof outline: $\mathbb{Y}^{\mathbb{H}} \rightarrow K_{n,\text{ptn}(\mathbb{Y})}^{\mathbb{H}} \rightarrow K_{m,\text{ptn}(\mathbb{Y})} \rightarrow \mathbb{Y}^{\rightarrow k}$

Requires careful handling of edge patterns, orthonormal bases approximations etc. - gives exponential upper bound.

That's all!

Thank you for your attention!