

# A Note on a Combinatorial Problem of Erdős and Hajnal

P. D. Seymour, 1974

Olgierd Zygmunt

June 10, 2026

# Property B

## Definition

A family of sets  $\mathcal{F}$  has **Property B** if there exists a set  $Z$  such that:

$$Z \cap X \neq \emptyset \quad \text{for every } X \in \mathcal{F},$$

but

$$X \not\subseteq Z \quad \text{for every } X \in \mathcal{F}.$$

## Property B: Example

$$\mathcal{F} = \{\{1, 4\}, \{2, 3\}, \{1, 3\}\}$$

Choose

$$Z = \{1, 2\}.$$

- ▶  $Z$  intersects every member of  $\mathcal{F}$ .
- ▶  $Z$  does not contain any of the sets from  $\mathcal{F}$ .
- ▶ Therefore this  $Z$  witnesses Property B.

# The Function $m(s)$

## Definition

$m(s)$  is the smallest integer  $k$  for which there exists a family  $\mathcal{F}$  of  $k$  sets such that:

$$|X| = s \quad \text{for every } X \in \mathcal{F},$$

and  $\mathcal{F}$  does **not** have Property B.

| $s$ | Known value / status |
|-----|----------------------|
| 1   | $m(1) = 1$           |
| 2   | $m(2) = 3$           |
| 3   | $m(3) = 7$           |
| 4   | unknown exactly      |

## Example: Why $(m(2)=3)$ ?

Consider the family

$$\mathcal{F} = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$$

### Claim

The family  $(\mathcal{F})$  does not have Property B.

# Historical Context

## Previous result

Abbott and Hanson constructed a family showing

$$m(4) \leq 24.$$

## Seymour's improvement

Seymour modifies their construction and obtains

$$m(4) \leq 23.$$

The whole paper is about saving **one set**.

# The Ground Set

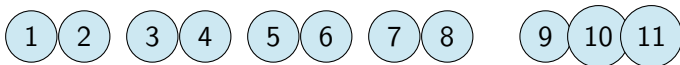
Seymour works inside

$$S = \{1, 2, \dots, 11\}.$$

Two auxiliary families

$$A = \{\{9, 10\}, \{10, 11\}, \{9, 11\}\},$$

$$B = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\}.$$



# The Family $C$

## Definition

$$C = \{X \cup Y : X \in A, Y \in B\}.$$

Each set in  $C$  has:

2 elements from  $\{9, 10, 11\}$     +    2 elements from one pair in  $B$ .

## Size

$$|C| = |A| \cdot |B| = 3 \cdot 4 = 12.$$

# The Family $D$

## Definition

$$D = \{ \{x_1, x_2, x_3, x_4\} : x_i \in \{2i - 1, 2i\}, i = 1, \dots, 4 \}.$$

So each member of  $D$  chooses exactly one element from each pair:

$$\{1, 2\}, \quad \{3, 4\}, \quad \{5, 6\}, \quad \{7, 8\}.$$

## Size

$$|D| = 2^4 = 16.$$

## Choosing $E \subseteq D$

We now choose a subfamily  $E \subseteq D$  satisfying two conditions.

### Condition (i)

If  $Y, Z \in D$  and

$$Y \cup Z = \{1, \dots, 8\},$$

then at least one of  $Y, Z$  belongs to  $E$ .

### Condition (ii)

Every 3-element subset of a member of  $D$  is contained in some member of  $E$ .

## Seymour's Choice of $E$

Instead of listing  $E$  directly, Seymour lists its complement in  $D$ :

$$D - E = \left\{ \begin{array}{l} \{1, 3, 5, 7\}, \{1, 3, 6, 8\}, \\ \{1, 4, 5, 8\}, \{2, 3, 5, 8\}, \\ \{2, 4, 6, 7\} \end{array} \right\}.$$

Since  $|D| = 16$  and  $|D - E| = 5$ ,

$$|E| = 11.$$

Final family

$$\mathcal{F} = C \cup E.$$

## Counting the Sets

$$|\mathcal{F}| = |C| + |E| = 12 + 11 = 23.$$

### Conclusion

If  $\mathcal{F}$  has no Property B, then

$$m(4) \leq 23.$$

The rest of the proof shows exactly that.

## Idea of the Proof

Assume, for contradiction, that  $\mathcal{F}$  has Property B.

Then there exists a set  $Z$  such that:

$$Z \cap X \neq \emptyset \quad \text{for every } X \in \mathcal{F},$$

but

$$X \not\subseteq Z \quad \text{for every } X \in \mathcal{F}.$$

### Key symmetry

If  $Z$  works, then  $S - Z$  also works.

Therefore, we may assume

$$|Z \cap \{9, 10, 11\}| \leq 1.$$

## Why This Forces a Choice from $D$

Since  $Z$  contains at most one element from  $\{9, 10, 11\}$ , it misses one member of  $A$ .

But  $Z$  must intersect every set in

$$C = \{X \cup Y : X \in A, Y \in B\}.$$

Therefore,  $Z$  must intersect every member of  $B$ .

### Consequence

There exists some  $Y \in D$  such that

$$Y \subseteq Z.$$

# The Contradiction

We reduce to the case

$$Z \subseteq \{1, \dots, 8\}.$$

There are two possibilities.

## Case 1: $Y = Z$

Then by the definition of property B:  $Z \notin E$ , so by condition (i), it belongs to  $D - E$  but then the complement of  $Z$  is disjoint from it and in  $\mathcal{F}$ .

## Case 2: $Y \subset Z$

Then  $|Z| \geq 5$ , so it contains both elements of some pair in  $B$ . Let's say it's  $\{1, 2\}$ . Let's consider  $Z - \{1, 2\}$ . By condition (ii) it is contained by some  $X \in E$ , so then  $X \subseteq Z$ . This is forbidden by Property B.

So  $\mathcal{F}$  has no Property B.

# What Happened Later?

## Seymour (1974)

The paper constructs a family of 23 four-element sets without Property B and proves  $m(4) \leq 23$ .

## Östergård (2014)

Using an exhaustive computer search, Östergård proved that no family of 22 four-element sets can avoid Property B, thus  $m(4) \geq 23$ .

## Final Result

Combining both bounds gives

$$m(4) = 23.$$

Thus Seymour's construction is optimal.

**Thank you for your attention**